FRACTIONAL HERMITE-HADAMARD INEQUALITIES FOR SOME CLASSES OF DIFFERENTIABLE PREINVEX FUNCTIONS

Muhammad Aslam NOOR[1], Khalida Inayat NOOR,[2] Marcela V. MIHAI[3], Muhammad Uzair AWAN[4]

The main objective of this article is to establish some new inequalities of Hermite-Hadamard type via Riemann-Liouville fractional integrals. We establish a new fractional integral identity for differentiable function, then using this identity as auxiliary result we derive some fractional Hermite-Hadamard type inequalities for differentiable s-preinvex functions and for differentiable s-Godunova-Levin preinvex functions.

Keywords: Hermite-Hadamard's inequalities, s-preinvex functions, s-Godunova-Levin preinvex functions, Riemann-Liouville fractional integrals

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1. Introduction

The relationship between theory of convex functions and theory of inequalities has inspired many researchers to investigate these theories. A very interesting result in this regard is due to Hermite and Hadamard independently that is Hermite-Hadamard's inequality. This remarkable result of Hermite and Hadamard can be viewed as necessary and sufficient condition for a function to be convex. This famous result reads as follows:

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function with $a < b$ and $a, b \in I$. Then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}. \quad (1.1)$$

For some useful details on Hermite-Hadamard type inequalities, see [2,4,5,7,9-13,15-20].

Recently many researchers have extended the classical concept of convex

[1] Department of Mathematics, COMSATS Institute of Information Technology, Park Road, Islamabad, Pakistan
[2] Department of Mathematics, COMSATS Institute of Information Technology, Park Road, Islamabad, Pakistan
[3] Department of Mathematics, University of Craiova, Street A. I. Cuza 13, Craiova, RO-200585, Romania
[4] Department of Mathematics, COMSATS Institute of Information Technology, Park Road, Islamabad, Pakistan , Corresponding Author, e-mail: awan.uzair@gmail.com

functions. As a result many new and interesting generalizations of classical convex functions can be found in the literature, see [3-6,15,17,18,19,21]. An important extension of convex functions was the introduction of preinvex functions [21]. For some useful and interesting investigations on preinvex functions, see [1, 2, 9, 10, 11, 14, 15, 18, 19, 22]. Noor [19] and Noor [18] extended the class of preinvex functions and introduced the concepts of s -preinvex functions and s -Godunova-Levin preinvex functions respectively.

In this paper, we derive a new fractional integral identity for differntiable functions. Using this we obtain our main results that are fractional Hermite-Hadamard type inequalities for differentiable s-preinvex functions and differentiable s-Godunova-Levin functions. Some special cases are also deduced. This is the main motivation of this paper.

2 Preliminary Results

Let K be a nonempty closed set in $\mathbb{R}^n$. Let $f: K \to \mathbb{R}$ be a continuous function and let $\eta(.,.): K \times K \to \mathbb{R}^n$ be a continuous bifunction. First of all, we recall some known results and concepts.

Definition 2.1 [21]. *A set K is said to be invex set with respect to $\eta(.,.)$, if*

$$u + t\eta(v,u) \in K, \quad \forall u, v \in K, t \in [0,1]. \tag{2.1}$$

The invex set K is also called η-connected set.

Remark 2.2 [1]. *We would like to mention that Definition 2.1 of an invex set has a clear geometric interpretation. This definition essentially says that there is a path starting from a point u which is contained in K. We do not require that the point v should be one of the end points of the path. This observation plays an important role in our analysis. Note that, if we demand that v should be an end point of the path for every pair of points $u, v \in K$, then $\eta(v,u) = v - u$, and consequently invexity reduces to convexity. Thus, it is true that every convex set is also an invex set with respect to $\eta(v,u) = v - u$, but the converse is not necessarily true, see $[14, 22]$ and the references therein. For the sake of simplicity, we always assume that $K = [u, u + \eta(v,u)]$, unless otherwise specified.*

Definition 2.3 [21] *A function f is said to be preinvex with respect to arbitrary bifunction $\eta(.,.)$, if*

$$f(u + t\eta(v,u)) \le (1-t) f(u) + t f(v), \quad \forall u, v \in K, t \in [0,1]. \tag{2.2}$$

The function f is said to be preconcave if and only if $-f$ is preinvex.

For $\eta(v,u) = v - u$ in (2.2) the preinvex functions becomes convex functions in the classical sense.

Note that every convex function is a preinvex function. However it is known [21] that preinvex functions may not be convex functions.

Definition 2.4 [19]. *A function $f : K \to \mathbb{R}$ is said to be s-preinvex of second kind with respect to $\eta(.,.)$, if*

$$f(u + t\eta(v,u)) \le (1-t)^s f(u) + t^s f(v), \quad u, v \in K, t \in [0,1], s \in (0,1).$$

Note that for $s = 1$ the definition of s-preinvex functions reduces to the definition of preinvex functions. And for $\eta(b,a) = b - a$, then we have the definition of s-Breckner convex functions.

Definition 2.5 [18]. *A function $f : K \to \mathbb{R}$ is said to be s-Godunova-Levin preinvex of second kind with respect to $\eta(.,.)$, if*

$$f(u + t\eta(v,u)) \le \frac{f(u)}{(1-t)^s} + \frac{f(v)}{t^s}, \quad \forall u, v \in K, t \in (0,1), s \in [0,1]. \qquad (2.3)$$

It is obvious that for $s = 0$, s-Godunova-Levin preinvex functions of second kind reduces to the definition of P-preinvex functions [18] and for $s = 1$ it reduces to the definition of Godunova-Levin preinvex functions [18]. When $\eta(v,u) = v - u$ then we have definition of s-Godunova-Levin functions of second kind [4, 5].

Definition 2.6 [8]. *Let $f \in L_1[a,b]$. Then Riemann-Liouville integrals $J^{\alpha}_{a^+} f$ and $J^{\alpha}_{b^-} f$ of order $\alpha > 0$ with $a \ge 0$ are defined by*

$$J^{\alpha}_{a^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) \mathrm{d}t, \quad x > a,$$

and

$$J^{\alpha}_{b^-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) \mathrm{d}t, \quad x < b,$$

where

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t} x^{\alpha-1} \mathrm{d}x,$$

is the well known Gamma function.

We now give the definition of hypergeometric series which will be used in the obtaining some integrals.

Definition 2.7 [8]. *For the real or complex numbers a, b, c, other than $0, -1, -2, \dots$, the hypergeometric series is defined by*

$${}_2F_1[a,b,c;z] = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \dots = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{z^m}{m!}.$$

Here $(\phi)_m$ is the Pochhammer symbol, which is defined by

$$(\phi)_m = \begin{cases} 1 & m = 0, \\ \phi(\phi+1)\ldots(\phi+m-1), & m > 0, \end{cases}$$

which has the integral form:

$${}_2F_1[a,b;c;z] = \frac{1}{\mathrm{B}(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,\mathrm{d}t$$

where $|z|<1, c>b>0$ and

$$\mathrm{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t,$$

is Euler function Beta with

$$\mathrm{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Now we prove the following auxiliary result which plays a key role in proving our main results.

Lemma 2.8. *Let* $f : K \to \mathbb{R}$ *be differentiable function such that* $f' \in L[a, a+\eta(b,a)]$. *Then for* $\alpha > 0$, *we have*

$$\begin{aligned} &\Phi(\alpha;a,x,b)(f) \\ &= \frac{\eta^{\alpha+1}(x,a)}{2\eta(b,a)}\{\int_0^1 t^\alpha f'(a+\frac{1+t}{2}\eta(x,a))\mathrm{d}t - \int_0^1 t^\alpha f'(a+\frac{1-t}{2}\eta(x,a))\mathrm{d}t\} \\ &- \frac{\eta^{\alpha+1}(b,x)}{2\eta(b,a)}\{\int_0^1 t^\alpha f'(x+\frac{1-t}{2}\eta(b,x))\mathrm{d}t - \int_0^1 t^\alpha f'(x+\frac{1+t}{2}\eta(b,x))\mathrm{d}t\}, \end{aligned}$$

where

$$\begin{aligned} &\Phi(\alpha;a,x,b)(f) \\ &= \frac{\eta^\alpha(x,a)f(a+\eta(x,a))+\eta^\alpha(b,x)f(x+\eta(b,x))}{\eta(b,a)} + \frac{\eta^\alpha(x,a)f(a)+\eta^\alpha(b,x)f(x)}{\eta(b,a)} \\ &- \frac{2^\alpha\Gamma(\alpha+1)}{\eta(b,a)}[J^\alpha_{[a+\eta(x,a)]^-} f(a+\frac{1}{2}\eta(x,a)) + J^\alpha_{a^+} f(a+\frac{1}{2}\eta(x,a)) \\ &+ J^\alpha_{x^+} f(x+\frac{1}{2}\eta(b,x)) + J^\alpha_{[x+\eta(b,x)]^-} f(x+\frac{1}{2}\eta(b,x))]. \end{aligned}$$

Proof. It suffices to show that

$$\int_0^1 \frac{t^\alpha}{2} f'(a+\frac{1+t}{2}\eta(x,a))\mathrm{d}t$$

$$=\frac{1}{\eta(x,a)}f(a+\eta(x,a))-\frac{\Gamma(\alpha+1)}{\eta(x,a)}\frac{1}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1}f(a+\frac{1+t}{2}\eta(x,a))\mathrm{d}t$$

$$=\frac{1}{\eta(x,a)}f(a+\eta(x,a))-\frac{2^{\alpha}\Gamma(\alpha+1)}{\eta^{\alpha+1}(x,a)}\frac{1}{\Gamma(\alpha)}\int_{a+\frac{1}{2}\eta(x,a)}^{a+\eta(x,a)}(u-a-\frac{1}{2}\eta(x,a))^{\alpha-1}f(u)\mathrm{d}u$$

$$=\frac{1}{\eta(x,a)}f(a+\eta(x,a))-\frac{2^{\alpha}\Gamma(\alpha+1)}{\eta^{\alpha+1}(x,a)}J^{\alpha}_{[a+\eta(x,a)]^-}f(a+\frac{1}{2}\eta(x,a)). \quad (2.4)$$

Similarly

$$\int_0^1\frac{t^{\alpha}}{2}f'(a+\frac{1-t}{2}\eta(x,a))\mathrm{d}t=-\frac{1}{\eta(x,a)}f(a)+\frac{2^{\alpha}\Gamma(\alpha+1)}{\eta^{\alpha+1}(x,a)}J^{\alpha}_{a^+}f(a+\frac{1}{2}\eta(x,a)), \quad (2.5)$$

$$\int_0^1\frac{t^{\alpha}}{2}f'(x+\frac{1-t}{2}\eta(b,x))\mathrm{d}t=-\frac{1}{\eta(b,x)}f(x)+\frac{2^{\alpha}\Gamma(\alpha+1)}{\eta^{\alpha+1}(b,x)}J^{\alpha}_{x^+}f(x+\frac{1}{2}\eta(b,x)), \quad (2.6)$$

and

$$\int_0^1\frac{t^{\alpha}}{2}f'(x+\frac{1+t}{2}\eta(b,x))\mathrm{d}t=\frac{1}{\eta(b,x)}f(x+\eta(b,x))-\frac{2^{\alpha}\Gamma(\alpha+1)}{\eta^{\alpha+1}(b,x)}J^{\alpha}_{[x+\eta(b,x)]^-}f(x+\frac{1}{2}\eta(b,x)), \quad (2.7)$$

After suitable rearrangements the proof is complete.

Remark 2.9. *We would like to remark that for* $\eta(b,a)=b-a$ *Lemma 2.8 reduces to Lemma 1 [13].*

3 Main Results

In this section, we derive our main results.

Theorem 3.1. *Let* $f:K\to\mathbb{R}$ *be differentiable function such that* $f'\in L[a,a+\eta(b,a)]$. *If* $|f'|$ *is is* s*-preinvex function of second kind, then, for* $\alpha>0$, *we have*

$$|\Phi(\alpha;a,x,b)(f)|$$
$$\leq\frac{\vartheta_1+\vartheta_2}{2^{s+1}\eta(b,a)}[\eta^{\alpha+1}(x,a)|f'(a)|+\{\eta^{\alpha+1}(x,a)+\eta^{\alpha+1}(b,x)\}|f'(x)|+\eta^{\alpha+1}(b,x)|f'(b)|],$$

where

$$\vartheta_1 = \int_0^1 t^{\alpha}(1-t)^s dt = \frac{\Gamma(1+s)\Gamma(1+\alpha)}{\Gamma(2+s+\alpha)} \quad (3.1)$$

$$\vartheta_2 = \int_0^1 t^{\alpha}(1+t)^s dt = (1+\alpha)\cdot {}_2F_1[-s,1+\alpha,2+\alpha,-1]. \quad (3.2)$$

Proof. Using Lemma 2.8, taking modulus and the fact that $|f'|$ is s-preinvex function of second kind, we have:

$$|\Phi(\alpha;a,x,b)(f)|$$

$$\leq \frac{\eta^{\alpha+1}(x,a)}{2\eta(b,a)}\{\int_0^1 t^{\alpha}|f'(a+\frac{1+t}{2}\eta(x,a))|\mathrm{d}t + \int_0^1 t^{\alpha}|f'(a+\frac{1-t}{2}\eta(x,a))|\mathrm{d}t\}$$

$$+\frac{\eta^{\alpha+1}(b,x)}{2\eta(b,a)}\{\int_0^1 t^{\alpha}|f'(x+\frac{1-t}{2}\eta(b,x))|\mathrm{d}t + \int_0^1 t^{\alpha}|f'(x+\frac{1+t}{2}\eta(b,x))|\mathrm{d}t\}$$

$$\leq \frac{\eta^{\alpha+1}(x,a)}{2\eta(b,a)}\{\int_0^1 t^{\alpha}[\left(\frac{1-t}{2}\right)^s|f'(a)|+\left(\frac{1+t}{2}\right)^s|f'(x)|]\mathrm{d}t$$

$$+\int_0^1 t^{\alpha}[\left(\frac{1+t}{2}\right)^s|f'(a)|+\left(\frac{1-t}{2}\right)^s|f'(x)|]\mathrm{d}t\}$$

$$+\frac{\eta^{\alpha+1}(b,x)}{2\eta(b,a)}\{\int_0^1 t^{\alpha}[\left(\frac{1+t}{2}\right)^s|f'(x)|+\left(\frac{1-t}{2}\right)^s|f'(b)|]\mathrm{d}t$$

$$+\int_0^1 t^{\alpha}[\left(\frac{1-t}{2}\right)^s|f'(x)|+\left(\frac{1+t}{2}\right)^s|f'(b)|]\mathrm{d}t\}$$

$$= \frac{\vartheta_1+\vartheta_2}{2^{s+1}\eta(b,a)}[\eta^{\alpha+1}(x,a)|f'(a)|+\{\eta^{\alpha+1}(x,a)+\eta^{\alpha+1}(b,x)\}|f'(x)|+\eta^{\alpha+1}(b,x)|f'(b)|].$$

This completes the proof.

Theorem 3.2. *Let* $f: K \to \mathbb{R}$ *be differentiable function such that* $f' \in L[a, a+\eta(b,a)]$. *If* $|f'|$ *is is* s*-Godunova-Levin preinvex function of second kind, then, for* $\alpha > 0$, *we have*

$$|\Phi(\alpha;a,x,b)(f)|$$

$$\leq \frac{\varphi_1+\varphi_2}{2^{1-s}\eta(b,a)}[\eta^{\alpha+1}(x,a)|f'(a)|+\{\eta^{\alpha+1}(x,a)+\eta^{\alpha+1}(b,x)\}|f'(x)|+\eta^{\alpha+1}(b,x)|f'(b)|],$$

where

$$\varphi_1 = \int_0^1 t^\alpha (1-t)^{-s}\, dt = \frac{\Gamma(1-s)\Gamma(1+\alpha)}{\Gamma(2-s+\alpha)} \tag{3.3}$$

$$\varphi_2 = \int_0^1 t^\alpha (1+t)^{-s}\, dt = (1+\alpha)\cdot {}_2F_1[s, 1+\alpha, 2+\alpha, -1]. \tag{3.4}$$

Theorem 3.3. *Let* $f : K \to \mathbb{R}$ *be differentiable function such that* $f' \in L[a, a+\eta(b,a)]$ *and* $\alpha > 0$. *If* $|f'|^q$ *is* s*-preinvex function of second kind where* $0 < s < 1$ *and* $\frac{1}{p}+\frac{1}{q}=1, q>1$, *then*

$$|\Phi(\alpha; a, x, b)(f)|$$
$$\leq \frac{1}{2^{1+\frac{s}{q}}\eta(b,a)}\left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}\{\eta^{\alpha+1}(x,a)\{(\mu_1|f'(a)|^q+\mu_2|f'(x)|^q)^{\frac{1}{q}}$$
$$+(\mu_2|f'(a)|^q+\mu_1|f'(x)|^q)^{\frac{1}{q}}\}+\eta^{\alpha+1}(b,x)\{(\mu_2|f'(x)|^q+\mu_1|f'(b)|^q)^{\frac{1}{q}}$$
$$+(\mu_1|f'(x)|^q+\mu_2|f'(b)|^q)^{\frac{1}{q}}\}\},$$

where

$$\mu_1 = \int_0^1 (1-t)^s\, dt = \frac{1}{1+s}$$
$$\mu_2 = \int_0^1 (1+t)^s\, dt = \frac{2^{1+s}-1}{1+s}.$$

Proof. Using Lemma 2.8, Hölder's inequality and the fact that $|f'|^q$ is s-preinvex function of second kind

$$|\Phi(\alpha; a, x, b)(f)|$$

$$\leq \frac{\eta^{\alpha+1}(x,a)}{2\eta(b,a)}\{\int_0^1 t^\alpha\,|f'(a+\frac{1+t}{2}\eta(x,a))|\,\mathrm{d}t+\int_0^1 t^\alpha\,|f'(a+\frac{1-t}{2}\eta(x,a))|\,\mathrm{d}t\}$$

$$+\frac{\eta^{\alpha+1}(b,x)}{2\eta(b,a)}\{\int_0^1 t^\alpha\,|f'(x+\frac{1-t}{2}\eta(b,x))|\,\mathrm{d}t+\int_0^1 t^\alpha\,|f'(x+\frac{1+t}{2}\eta(b,x))|\,\mathrm{d}t\}$$

$$\leq \frac{\eta^{\alpha+1}(x,a)}{2\eta(b,a)}\left(\int_0^1 (t^\alpha)^p\,\mathrm{d}t\right)^{\frac{1}{p}}[(\int_0^1 |f'(a+\frac{1+t}{2}\eta(x,a))|^q\,\mathrm{d}t)^{\frac{1}{q}}$$

$$+(\int_0^1 |f'(a+\frac{1-t}{2}\eta(x,a))|^q \,\mathrm{d}t)^{\frac{1}{q}}]$$

$$+\frac{\eta^{\alpha+1}(b,x)}{2\eta(b,a)}(\int_0^1 (t^\alpha)^p \mathrm{d}t)^{\frac{1}{p}}[(\int_0^1 |f'(x+\frac{1-t}{2}\eta(b,x))|^q \,\mathrm{d}t)^{\frac{1}{q}}$$

$$+(\int_0^1 |f'(x+\frac{1+t}{2}\eta(b,x))|^q \,\mathrm{d}t)^{\frac{1}{q}}]$$

$$\leq \frac{\eta^{\alpha+1}(x,a)}{2^{1+\frac{s}{q}}\eta(b,a)}(\frac{1}{p\alpha+1})^{\frac{1}{p}}\{(\int_0^1[(1-t)^s|f'(a)|^q+(1+t)^s|f'(x)|^q]\mathrm{d}t)^{\frac{1}{q}}$$

$$+(\int_0^1[(1+t)^s|f'(a)|^q+(1-t)^s|f'(x)|^q]\mathrm{d}t)^{\frac{1}{q}}\}$$

$$+\frac{\eta^{\alpha+1}(b,x)}{2^{1+\frac{s}{q}}\eta(b,a)}(\frac{1}{p\alpha+1})^{\frac{1}{p}}\{(\int_0^1[(1+t)^s|f'(x)|^q+(1-t)^s|f'(b)|^q]\mathrm{d}t)^{\frac{1}{q}}$$

$$+(\int_0^1[(1-t)^s|f'(x)|^q+(1+t)^s|f'(b)|^q]\mathrm{d}t)^{\frac{1}{q}}\}$$

$$=\frac{1}{2^{1+\frac{s}{q}}\eta(b,a)}(\frac{1}{p\alpha+1})^{\frac{1}{p}}\{\eta^{\alpha+1}(x,a)\{(\mu_1|f'(a)|^q+\mu_2|f'(x)|^q)^{\frac{1}{q}}$$

$$+(\mu_2|f'(a)|^q+\mu_1|f'(x)|^q)^{\frac{1}{q}}\}+\eta^{\alpha+1}(b,x)\{(\mu_2|f'(x)|^q+\mu_1|f'(b)|^q)^{\frac{1}{q}}$$

$$+(\mu_1|f'(x)|^q+\mu_2|f'(b)|^q)^{\frac{1}{q}}\}\}.$$

This completes the proof.

Theorem 3.4. *Let* $f:K\to\mathbb{R}$ *be differentiable function such that* $f'\in L[a,a+\eta(b,a)]$ *and* $\alpha>0$. *If* $|f'|^q$ *is* s*-Godunova-Levin preinvex function of second kind where* $0<s<1$ *and* $\frac{1}{p}+\frac{1}{q}=1, q>1$, *then*

$$|\Phi(\alpha;a,x,b)(f)|$$

$$\leq \frac{1}{2^{1-\frac{s}{q}}\eta(b,a)}(\frac{1}{p\alpha+1})^{\frac{1}{p}}\{\eta^{\alpha+1}(x,a)\{(\lambda_1|f'(a)|^q+\lambda_2|f'(x)|^q)^{\frac{1}{q}}$$

$$+(\lambda_2|f'(a)|^q+\lambda_1|f'(x)|^q)^{\frac{1}{q}}\}+\eta^{\alpha+1}(b,x)\{(\lambda_2|f'(x)|^q+\lambda_1|f'(b)|^q)^{\frac{1}{q}}$$
$$+(\lambda_1|f'(x)|^q+\lambda_2|f'(b)|^q)^{\frac{1}{q}}\}\},$$

where

$$\lambda_1=\int_0^1(1-t)^{-s}dt=\frac{1}{1-s}$$
$$\lambda_2=\int_0^1(1+t)^{-s}dt=\frac{2^{-s}(-2+2^s)}{s-1}.$$

Theorem 3.5. *Let* $f:K\to\mathbb{R}$ *be differentiable function such that* $f'\in L[a,a+\eta(b,a)]$ *and* $\alpha>0$. *If* $|f'|^q$ *is* s*-preinvex function of second kind, where* $0<s<1$ *and* $q>1$, *then*

$$|\Phi(\alpha;a,x,b)(f)|$$
$$\leq\frac{1}{2^{1+\frac{s}{q}}\eta(b,a)}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\{\eta(x,a)^{\alpha+1}\{(\vartheta_1|f'(a)|^q+\vartheta_2|f'(x)|^q)^{\frac{1}{q}}$$

$$+(\vartheta_2|f'(a)|^q+\vartheta_1|f'(x)|^q)^{\frac{1}{q}}\}+\eta(b,x)^{\alpha+1}\{(\vartheta_2|f'(x)|^q+\vartheta_1|f'(b)|^q)^{\frac{1}{q}}$$
$$+(\vartheta_1|f'(x)|^q+\vartheta_2|f'(b)|^q)^{\frac{1}{q}}\}\},$$

where ϑ_1 and ϑ_2 are given by (3.1) and (3.2).

Proof. Using Lemma 2.8, Power's mean inequality and the fact that $|f'|^q$ is s-preinvex function of second kind

$$|\Phi(\alpha;a,x,b)(f)|$$

$$\leq\frac{\eta^{\alpha+1}(x,a)}{2\eta(b,a)}\{\int_0^1t^\alpha|f'(a+\frac{1+t}{2}\eta(x,a))|\,\mathrm{d}t+\int_0^1t^\alpha|f'(a+\frac{1-t}{2}\eta(x,a))|\,\mathrm{d}t\}$$
$$+\frac{\eta^{\alpha+1}(b,x)}{2\eta(b,a)}\{\int_0^1t^\alpha|f'(x+\frac{1-t}{2}\eta(b,x))|\,\mathrm{d}t+\int_0^1t^\alpha|f'(x+\frac{1+t}{2}\eta(b,x))|\,\mathrm{d}t\}$$
$$\leq\frac{\eta^{\alpha+1}(x,a)}{2\eta(b,a)}\left(\int_0^1t^\alpha\mathrm{d}t\right)^{1-\frac{1}{q}}[(\int_0^1t^\alpha|f'(a+\frac{1+t}{2}\eta(x,a))|^q\,\mathrm{d}t)^{\frac{1}{q}}$$

$$+(\int_0^1 t^\alpha |f'(a+\frac{1-t}{2}\eta(x,a))|^q\,\mathrm{d}t)^{\frac{1}{q}}]$$

$$+\frac{\eta^{\alpha+1}(b,x)}{2\eta(b,a)}(\int_0^1 t^\alpha \mathrm{d}t)^{1-\frac{1}{q}}[(\int_0^1 t^\alpha |f'(x+\frac{1-t}{2}\eta(b,x))|^q\,\mathrm{d}t)^{\frac{1}{q}}$$

$$+(\int_0^1 t^\alpha |f'(x+\frac{1+t}{2}\eta(b,x))|^q\,\mathrm{d}t)^{\frac{1}{q}}]$$

$$\leq \frac{1}{2^{1+\frac{s}{q}}\eta(b,a)}(\frac{1}{\alpha+1})^{1-\frac{1}{q}}\{\eta(x,a)^{\alpha+1}\{(\vartheta_1|f'(a)|^q+\vartheta_2|f'(x)|^q)^{\frac{1}{q}}$$

$$+(\vartheta_2|f'(a)|^q+\vartheta_1|f'(x)|^q)^{\frac{1}{q}}\}+\eta(b,x)^{\alpha+1}\{(\vartheta_2|f'(x)|^q+\vartheta_1|f'(b)|^q)^{\frac{1}{q}}$$

$$+(\vartheta_1|f'(x)|^q+\vartheta_2|f'(b)|^q)^{\frac{1}{q}}\}\}.$$

This completes the proof.

Theorem 3.6. *Let* $f: K \to \mathbb{R}$ *be differentiable function such that* $f' \in L[a, a+\eta(b,a)]$ *and* $\alpha > 0$. *If* $|f'|^q$ *is* s*-Godunova-Levin preinvex function of second kind, where* $0 < s < 1$ *and* $q > 1$, *then*

$$|\Phi(\alpha;a,x,b)(f)|$$

$$\leq \frac{1}{2^{1+\frac{s}{q}}\eta(b,a)}(\frac{1}{\alpha+1})^{1-\frac{1}{q}}\{\eta(x,a)^{\alpha+1}\{(\varphi_1|f'(a)|^q+\varphi_2|f'(x)|^q)^{\frac{1}{q}}$$

$$+(\varphi_2|f'(a)|^q+\varphi_1|f'(x)|^q)^{\frac{1}{q}}\}+\eta(b,x)^{\alpha+1}\{(\varphi_2|f'(x)|^q+\varphi_1|f'(b)|^q)^{\frac{1}{q}}$$

$$+(\varphi_1|f'(x)|^q+\varphi_2|f'(b)|^q)^{\frac{1}{q}}\}\},$$

where φ_1 and φ_2 are given by (3.3) and (3.4).

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REFERENCES

[1] *T. Antczak*, Mean value in invexity analysis, Nonl. Anal. 60(2005), 1473-1484.

[2] *A. Barani, A. G. Ghazanfari and S. S. Dragomir*, Hermite-Hadamard inequality for functions whose derivatives absolute values are preinvex, J. Inequal. Appl. 2012, 2012:247.

[3] *G. Cristescu and L. Lupsa*: Non-connected Convexities and Applications. Kluwer Academic Publishers, Dordrecht, Holland, 2002.

[4] *S. S. Dragomir*, Inequalities of Hermite-Hadamard type for h-convex functions on linear spaces, preprint, (2014).

[5] *S. S. Dragomir*, n-points inequalities of Hermite-Hadamard type for h-convex functions on linear spaces, preprint, (2014).

[6] *E. K. Godunova and V. I. Levin*, Inequalities for functions of a broad class that contains convex, monotone and some other forms of functions. (Russian) Numerical mathematics and mathematical physics (Russian), 138-142, 166, Moskov. Gos. Ped. Inst., Moscow, 1985.

[7] *S. K. Khattri*, Three proofs of the inequality $e < \left(1+\frac{1}{n}\right)^{n+0.5}$, Amer. Math. Monthly, 117(3), 273-277 (2010).

[8] *A. Kilbas , H. M. Srivastava, J. J. Trujillo:* Theory and applications of fractional differential equations, Elsevier B.V., Amsterdam, Netherlands, (2006).

[9] *M. A. Latif*, Some inequalities for differentiable prequasiinvex functions with applications, Konuralp J. Math., 1(2), 17-29, 2013.

[10] *M. A. Latif and S. S. Dragomir*, Some Hermite-Hadamard type inequalities for functions whose partial derivatives in absloute value are preinvex on the co-oordinates, Facta Universitatis (NIS) Ser. Math. Inform. 28(3), 257-270, (2013).

[11] *M. A. Latif, S. S. Dragomir, E. Momoniat*, Some weighted integral inequalities for differentiable preinvex and prequasiinvex functions, RGMIA, (2014).

[12] *M. V. Mihai*, New Hermite-Hadamard type inequalities obtained via Riemann-Liouville fractional calculus, Annal. Univ. Oradea, Fasc. Math., Tom XX (2013), Issue No. 2, 127-132.

[13] *M. V. Mihai, F. C. Mitroi*, Hermite-Hadamard type inequalities obtained via Riemann-Liouville fractional calculus, Acta Mathematica Universitatis Comenianae, Vol. LXXXIII, 2 (2014), pp. 209-215.

[14] *S. R. Mohan, S. K. Neogy*, On invex sets and preinvex functions, J. Math. Anal. Appl. 189(1995), 901-908.

[15] *M. A. Noor*, On Hadamard integral inequalities involving two log-preinvex functions. J. Inequal. Pure Appl. Math. 8, 1-6 (2007).

[16] *M. A. Noor, G. Cristescu, M. U. Awan*, Generalized fractional Hermite-Hadamard inequalities for twice differentiable s-convex functions, Filomat, inpress.

[17] *M. A. Noor, K. I. Noor, M. U. Awan, S. Khan*, Fractional Hermite-Hadamard inequalities for some new classes of Godunova-Levin functions, Appl. Math. Inf. Sci. 8(6), 2865-2872, (2014).

[18] *M. A. Noor, K. I. Noor, M. U. Awan, S. Khan*, Hermite-Hadamard inequalities for s-Godunova-Levin preinvex functions, J. Adv. Math. Stud. 7(2), 12-19, (2014).

[19] *M. A. Noor, K. I. Noor, M. U. Awan, J. Li*, On Hermite-Hadamard Inequalities for h-preinvex functions, Filomat, inpress.

[20] *M. Z. Sarikaya, E. Set, H. Yaldiz and N. Basak*: Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities, Mathematical and Computer Modelling 57, 2403-2407, 2013.

[21] *T. Weir*, *B. Mond*, Preinvex functions in multiobjective optimization, J. Math. Anal. Appl. 136(1988), 29-38.

[22] *X. M. Yang*, *X. Q. Yang*, *K. L. Teo*, Generalized invexity and generalized invariant monotonicity, J. Optim. Theory. Appl., 117(2003), 607-625.