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ON 2-ABSORBING SUBMODULE ELEMENTS IN LE-MODULES AND ITS GENERALIZATIONS

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In this paper, we introduce the concept of 2-absorbing submodule elements in an le-module M as follows: a proper submodule element q in M is said to be 2-absorbing for any $r, s \in R$ and $m \in M$ if $rsm \leq q$, then either $rs \in (q : e)$ or $rm \leq q$ or $sm \leq q$. Moreover, we define some generalizations of the new concept such as weakly 2-absorbing, n-absorbing, weakly n-absorbing, (n, k)-absorbing, weakly (n, k)-absorbing submodule elements in le-modules. After presenting a main example for le-modules, we study some counter examples for the generalizations. In addition to giving some characterizations for the new concepts, we investigate the relationship between prime (primary) submodule elements and them.

Keywords: complete lattices, le-modules, submodule elements, 2-absorbing (weakly 2-absorbing) submodule elements

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1. Introduction

As a generalization of the commutative ideal theory, there are a great many publications addressing multiplicative lattices, see [2], [7], [12]. In 1970, the authors introduced a new concept called lattice modules, see [9]. They defined a lattice module similar to a module over a ring. A lattice module M over a multiplicative lattice L is a complete lattice together a multiplication between elements of L and M, which satisfies similar properties of a module over a ring, see [9]. If we desire to make it more concrete with an example, consider L as the lattice of every ideals in a ring which M is the lattice of every submodules in any module over the ring. In this case, M is a lattice module over L. Beside to Noetherian lattice modules, there exist a large number of papers on lattice modules, see [1], [3], [8]-[9]. Furthermore, the concept of prime ideals, prime submodules, and prime elements have a significant place in abstract algebra since they are used in understanding the structure of rings, modules, lattices, and lattice modules, see [3], [6]-[8], [13]-[14].

In 2018, the authors introduced a new algebraic construction, called "le-module over a commutative ring", by the help of a different approach to the "abstract submodule theory", see [10]. In the paper, they had two main goals: First one was to make it achievable to separate submodules from a typical subset in any module, thus they generalized the properties of special kinds of submodules. The other one was to build a channel to study the characteristics of rings more straight than the lattice module theory. To introduce the concept of le-modules, the authors defined, in [5], an le-semigroup $(M, +, \leq, e)$ as a complete lattice with the greatest element e, which is also a commutative monoid with the zero element 0_M that holds the property $m + (\bigvee_{i \in I} m_i) = \bigvee_{i \in I} (m + m_i)$, for all $m, m_i \in M, i \in I$.

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Afterwards, they introduced the concept of le-modules as followings: Assume that $(M,+,\leq,e)$ is an le-semigroup with 0_M and R is a commutative ring. We say M is an le-module over R if there exists a mapping $R\times M\to M$, which holds

(1) $r(m_1+m_2)=rm_1+rm_2$,
(2) $(r_1+r_2)m\leq r_1m+r_2m$,
(3) $(r_1r_2)m=r_1(r_2m)$,
(4) $1_Rm=m$ and $0_Rm=r0_M=0_M$,
(5) For every $r,r_1,r_2\in R$ and $m,m_1,m_2,m_j\in M$, $r(\bigvee_{j\in\triangle} m_j)=(\bigvee_{j\in\triangle} rm_j)$.

Note that from (5), we have $(5)'$: $m_1\leq m_2\Rightarrow rm_1\leq rm_2$, for all $r\in R$ and $m_1,m_2\in M$.

An element n of an le-module M is said to be a *submodule element* if $n+n, rn\leq n$, for all $r\in R$. We denote the set of all submodule elements of M by $Sub(M)$. We set $(n:a)=\bigvee\{x\in M: ax\leq n\}$, where $a\in R$ and $n\in Sub(M)$. It is easy to see that $(n:a)$ is in $Sub(M)$ and also, $n\leq(n:a)$ and $a(n:a)\leq n$. For all $x\in M$ and $a\in R$, we know $x\leq(n:a)$ necessary and sufficient condition $ax\leq n$. For $k\in Sub(M)$ and $n\in M$, the set $(k:n)=\{a\in R: an\leq k\}$ is an ideal of R. For any two submodule elements of M such that $n\leq k$, we have $(n:e)\subseteq(k:e)$. A submodule element n of M is called compact if for a family of submodule elements $\{n_\lambda\}_{\lambda\in\Lambda}$ if $n\leq\sum_{\lambda\in\Lambda} n_\lambda$ implies $n\leq n_{\lambda_1}+n_{\lambda_2}+\cdots+n_{\lambda_k}$ for some subset $\{\lambda_1,\lambda_2,\dots,\lambda_k\}$. Note that a sum of finite number of compact submodule elements is a compact submodule element. We denote the set of all compact submodule elements of M as M_*. Futhermore, if each $n\in Sub(M)$ is a sum of compact submodule elements in M, it is called a compactly generated le-module, or briefly CG-le-module. Also, an le-module M is called a faithful le-module when $(0_M:e)=0_R$. For more information, we refer the reader to [5], [10], [11].

In Section 2, we introduce the concept of 2-absorbing and weakly 2-absorbing submodule elements in an le-module M. One can easily see that the class of 2-absorbing submodule elements is a subclass of weakly 2-absorbing submodule elements. However, we show every weakly 2-absorbing submodule elements is not a 2-absorbing submodule element, see Example 2.2. After proving some main properties of the new concepts, in Theorem 2.6 (resp., in Theorem 2.7), we characterized 2-absorbing (resp., weakly 2-absorbing) submodule elements of a CG-le-module M. In Section 3, for a positive integer n, we introduce the n-absorbing (weakly n-absorbing) submodule elements in an le-module M and immediately conclude the concept is a generalization of 2-absorbing (weakly 2-absorbing) submodule elements. In Example 3.1, we show every weakly n-absorbing submodule elements is not an n-absorbing submodule element. Also, we define the notion of $\omega(p)$, which analyzes in some sense how far p is from being a prime submodule element for a submodule element p in M. In the last section, for any two positive integers n and k such that $k<n$, we present the concept of (n,k)-absorbing and weakly (n,k)-absorbing submodule elements in an le-module M. It is easy to see that the class of weakly (n,k)-absorbing submodule elements contains the class of (n,k)-absorbing submodule elements. On the other hand, the other containment doesn't hold, see Example 4.1. Moreover, the relation between the class of (n,k)-absorbing (also, weakly (n,k)-absorbing) submodule elements and the other classes defined above is invastigated, see Theorem 4.2, 4.3, 4.4. As final results, we characterize the (n,k)-absorbing (resp., weakly (n,k)-absorbing) submodule elements in Theorem 4.5 (resp., Theorem 4.6).

2. *2*-absorbing and weakly *2*-absorbing submodule elements in le-modules

Throughout our study ${}_RM$ (briefly, M) represents an le-module M over R, where R is a commutative ring with 1_R.

Definition 2.1. *Let p be a proper submodule element of M. Then*

(1) *p is called a* ***2-absorbing submodule element*** *for any $a, b \in R$ and $n \in M$ if $abn \leq p$, then $ab \in (p : e)$ or $an \leq p$ or $bn \leq p$.*
(2) *p is called a* ***weakly 2-absorbing submodule element*** *for any $a, b \in R$ and $n \in M$ if $0_M \neq abn \leq p$, then $ab \in (p : e)$ or $an \leq p$ or $bn \leq p$.*

Example 2.1. *Let R be a ring and $M = \{S \subseteq R : 0 \in S\}$. Then $(M, \subseteq)$ is a complete lattice with the greatest element $e = R$. Now, define $X + Y = \{x + y : x \in X, y \in Y\}$ and $rX = \{rx : x \in X\}$ for each $r \in R$; $X, Y \in M$. Thus, $(M, +, \subseteq, e)$ is an le-module with the zero element $0_M = \{0\}$. Also, it is clear that N is an R-submodule of M if and only if $N \in M$ is a submodule element of M. In particular, let $R = \mathbb{Z}_5$ and $M = \{S \subseteq R : 0 \in S\}$. The complete lattice $(M, \subseteq)$ with the following Hasse Diagram is an le-module according to above addition and scalar multiplication, see Figure 1.*

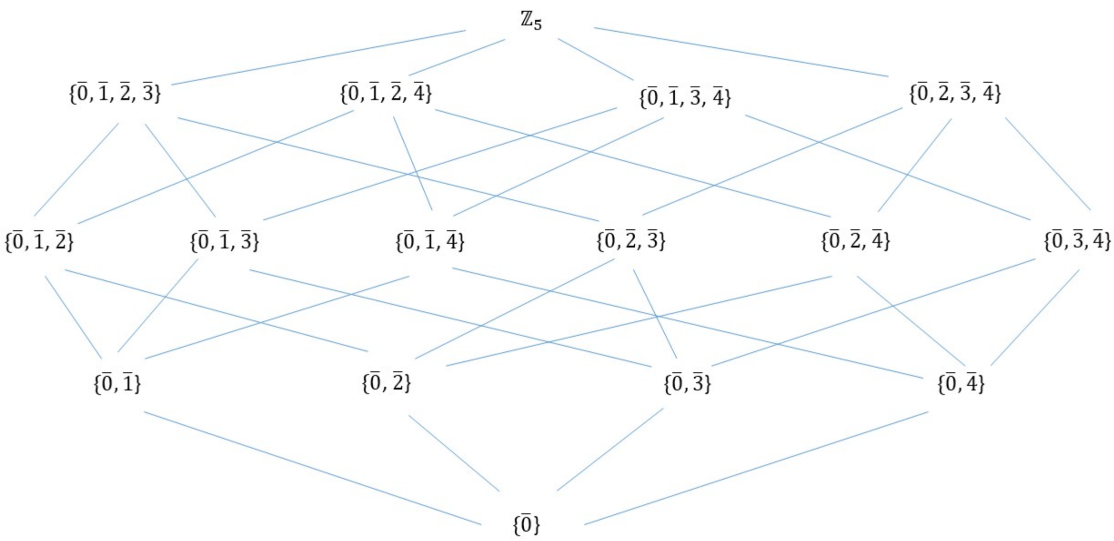

FIGURE 1. The Hasse Diagram of the Complete Lattice $(\mathbb{Z}_5, \subseteq)$.

Remark 2.1. *It is easiy to see that every 2-absorbing submodule element is a weakly 2-absorbing submodule element. However, we can't claim the converse, to see this, investigate the following example:*

Example 2.2. *Let $R = \mathbb{Z}_{30}$ and $M = \{X \subseteq R : \overline{0} \in X\}$. Now, consider $n = \{\overline{0}\}$. Notice that $(n : e) = (\overline{0})$. It is a weakly 2-absorbing submodule element but not 2-absorbing submodule element. Indeed, $\overline{2} \cdot \overline{3} \cdot \{\overline{0}, \overline{5}\} \leq n$, but $\overline{2} \cdot \overline{3} \notin (n : e)$ and $\overline{2} \cdot \{\overline{0}, \overline{5}\} \not\leq n$, $\overline{3} \cdot \{\overline{0}, \overline{5}\} \not\leq n$.*

Theorem 2.1. *Let n be a proper submodule element of M. If n is a 2-absorbing submodule element, $(n : r)$ is a 2-absorbing submodule element for all $r \in R$.*

Proof. Take $a, b \in R$ and $m \in M$ such that $abm \leq (n : r)$, that is, $abrm \leq n$. As n is a 2-absorbing submodule element, either $ab \in (n : e)$ or $arm \leq n$ or $brm \leq n$. Then $abe \leq n \leq (n : r)$, so $ab \in ((n : r) : e)$. Consequently, $ab \in ((n : r) : e)$ or $am \leq (n : r)$ or $bm \leq (n : r)$, as needed. □

Theorem 2.2. *Let n be a proper submodule element of M. If n is a 2-absorbing submodule element, then $(n : x)$ is a 2-absorbing ideal of R for all $x \in M$.*

Proof. Choose $a, b, c \in R$ such that $abc \in (n : x)$, i.e., $abcx \leq n$. Since n is a 2-absorbing submodule element, $ab \in (n : e)$ or $acx \leq n$ or $bcx \leq n$. Then $abx \leq abe \leq n$, so $ab \in (n : x)$. As a consequent, $ab \in (n : x)$ or $ac \in (n : x)$ or $bc \in (n : x)$. □

Corollary 2.1. *Let n be a proper submodule element of M. If n is a 2-absorbing submodule element, then $(n : e)$ is a 2-absorbing ideal of R.*

Theorem 2.3. *Let M be a faithful le-module and n be a proper submodule element of M. If n is a weakly 2-absorbing submodule element, then $(n : e)$ is a weakly 2-absorbing ideal of R.*

Proof. Take $a, b, c \in R$ such that $0_R \neq abc \in (n : e)$, that is, $abce \leq n$. Now, we have two cases: **Case 1:** Let $0_M = abce$. Then $abc \in (0_M : e) = 0_R$, which is a contradiction. **Case 2:** Let $0_M \neq abce$. As n is a weakly 2-absorbing submodule element, either $ab \in (n : e)$ or $ace \leq n$ or $bce \leq n$. Consequently, $ab \in (n : e)$ or $ac \in (n : e)$ or $bc \in (n : e)$, as required. □

Theorem 2.4. *Let M be a CG-le-module and n be a proper submodule element of M. If n is a weakly 2-absorbing submodule element such that $(n : e)^2 n \neq 0_M$, n is a 2-absorbing submodule element.*

Proof. Suppose that $(n : e)^2 n \neq 0_M$. Let choose $r, s \in R$ and $q \in M$ such that $rsq \leq n$. Then we have two cases: **Case 1:** Let $0_M \neq rsq$. Since n is a weakly 2-absorbing submodule element, $rs \in (n : e)$ or $rq \leq n$ or $sq \leq n$. It is done. **Case 2:** Suppose $rsq = 0_M$. By our hypothesis $(n : e)^2 n \neq 0_M$, there are $a, b \in (n : e)$ and a compact submodule element $h \in M$ such that $h \leq n$ and $0_M \neq abh \leq (n : e)^2 n$. Then

$$
\begin{aligned}
0_M \neq abh &\leq (r + a)(s + b)(q + h) = (rs + rb + as + ab)(q + h) \\
&= (rs + rb + as + ab)q + (rs + rb + as + ab)h \\
&\leq rsq + rbq + asq + abq + rsh + rbh + ash + abh \leq n + n + ... + n = n
\end{aligned}
$$

Thus, since n is a weakly 2-absorbing submodule element, $(r + a)(s + b) \in (n : e)$ or $(r + a)(q + h) \leq n$ or $(s + b)(q + h) \leq n$. Since $a, b \in (n : e)$, we have that either $rs \in (n : e)$ or $rq \leq n$ or $sq \leq n$. □

Definition 2.2. *A proper element n of M is called a nilpotent submodule element of M if $(n : e)^k n = 0_M$ for some positive integer k.*

Corollary 2.2. *Let M be a CG-le-module and n be a proper submodule element of M. If n is a weakly 2-absorbing submodule element which not 2-absorbing, then n is a nilpotent submodule element.*

Proof. By Theorem 2.4, it must be $(n : e)^2 n = 0_M$, thus it is done. □

Corollary 2.3. *Let M be a CG-le-module and n be a proper submodule element of M. If n is a weakly 2-absorbing submodule element but not 2-absorbing, then $(n : e)^k n = 0_M$, for all $k \geq 2$.*

Proof. By Theorem 2.4, $(n : e)^2 n = 0_M$. Also, we know $(n : e)^3 \subseteq (n : e)^2$. Then one can easily see that $(n : e)^3 n \leq (n : e)^2 n = 0_M$, that is, $(n : e)^3 n = 0_M$. By iteration, $(n : e)^k n = 0_M$, for all $k \geq 2$. □

Theorem 2.5. *Suppose that M is a CG-le-module and $\{n_i\}_{i \in \triangle}$ is a (ascending or descending) chain of submodule elements of M. Then*

(1) *For every $i \in \triangle$, if n_i is a 2-absorbing submodule element of M, then $\bigwedge_{i\in\triangle} n_i$ is a 2-absorbing submodule element of M.*

(2) *For every $i \in \triangle$, if n_i is a weakly 2-absorbing submodule element of M, then $\bigwedge_{i\in\triangle} n_i$ is a weakly 2-absorbing submodule element of M.*

Proof. (1): If $\{n_i\}_{i\in\triangle}$ is a ascending chain, $\bigwedge_{i\in\triangle} n_i = n_1$, so it is done. Suppose that $... \leq n_i \leq ... \leq n_2 \leq n_1$ is an descending chain of 2-absorbing submodule elements of M. Now, it is clear that $\bigwedge_{i\in\triangle} n_i$ is proper since $\bigwedge_{i\in\triangle} n_i \neq e$. Take $r, s \in R$ and $n \in M$, $rsn \leq \bigwedge_{i\in\triangle} n_i$. Suppose $rn \nleq \bigwedge_{i\in\triangle} n_i$ and $sn \nleq \bigwedge_{i\in\triangle} n_i$. Then there exist n_k and n_m such that $rn \nleq n_k$ and $sn \nleq n_m$. Without losing the generality, assume that $n_k \leq n_m$. Then we have $sn \nleq n_k$. Since n_k is a 2-absorbing submodule element and $rsn \leq n_k$, we get $rs \in (n_k : e)$. Also, note that for each $n_i \leq n_k$, we have $rn, sn \nleq n_i$, which implies that $rs \in (n_i : e)$. Since for each $n_k \leq n_i$, we have $(n_k : e) \subseteq (n_i : e)$, so $rs \in (n_i : e)$. Thus $rs \in \bigcap_{i\in\triangle} (n_i : e)$. By Proposition 2.2 in [5], we have $\bigcap_{i\in\triangle} (n_i : e) = (\bigwedge_{i\in\triangle} n_i : e)$. Consequently, $\bigwedge_{i\in\triangle} n_i$ is a 2-absorbing submodule element of M.

(2): Similar to previous proof. □

Now, we give a characterization for the concept of 2-absorbing submodule elements of M.

Theorem 2.6. *Suppose that M is a CG-le-module and p is a proper submodule element in M. The next items are equivalent:*

(1) *p is a 2-absorbing submodule element in M.*

(2) *For any $a, b \in R$ and $n \in M$ such that $p \leq n$, if $abn \leq p$, $ab \in (p : e)$ or $an \leq p$ or $bn \leq p$.*

(3) *For any $a, b \in R$ with $ab \notin (p : e)$, $(p : a) = (p : ab)$ or $(p : ab) = (p : b)$.*

(4) *For any $a, b \in R$, $h \in M_*$, if $abh \leq p$, $ab \in (p : e)$ or $ah \leq p$ or $bh \leq p$.*

Proof. (1) $\Rightarrow$ (2) : It is clear. (2) $\Rightarrow$ (3) : Suppose that $a, b \in R$ such that $ab \notin (p : e)$. Take a submodule element $k \in M$ such that $k \leq (p : ab)$. Then $abk \leq p$. Consider $u = k + p$. It is clear that $ab(k + p) = abk + abp \leq p$. Thus, we have $p \leq u$, $abu \leq p$ and $ab \notin (p : e)$. By our hypothesis (2), we conclude $au \leq p$ or $bu \leq p$. This means that $ak \leq p$ or $bk \leq p$, that is, $k \leq (p : a)$ or $k \leq (p : b)$. Then we say $(p : ab) \leq (p : a)$ or $(p : ab) \leq (p : b)$, since M is a CG-le-module. On the other hand, it is clear that $(p : a) \leq (p : ab)$ and $(p : b) \leq (p : ab)$. Consequently, $(p : ab) = (p : a)$ or $(p : ab) = (p : b)$.

(3) $\Rightarrow$ (4) : Let $a, b \in R$, $h \in M_*$ such that $abh \leq p$, that is, $h \leq (p : ab)$. Assume that $ab \notin (p : e)$. By item (3), we get $(p : ab) = (p : a)$ or $(p : ab) = (p : b)$. This implies that $h \leq (p : a)$ or $h \leq (p : b)$. Hence $ah \leq p$ or $bh \leq p$, as desired.

(4) $\Rightarrow$ (1) : Choose $a, b \in R$ and $n \in M$, $abn \leq p$ but $an \nleq p$ and $bn \nleq p$. Since M is a CG-le-module, there are compact submodule elements h and h' such that $h \leq n, ah \nleq p, h' \leq n, bh' \nleq p$. Also, we know that $h + h' \in M_*$. Then we have $ab(h + h') = abh + abh' \leq abn + abn \leq p + p = p$. Hence, by our assumption, it must be either $ab \in (p : e)$ or $a(h + h') \leq p$ or $b(h + h') \leq p$. The second and third one imply a contradiction with $ah \nleq p$ and $bh' \nleq p$, recpectively. As a conclusion, $ab \in (p : e)$. □

Similar to Theorem 2.6, we can characterize the concept of weakly 2-absorbing submodule elements of M.

Theorem 2.7. *Suppose that M is a CG-le-module and p is a proper submodule element in M. Then the next items are equivalent:*

(1) *p is a weakly 2-absorbing submodule element.*
(2) *For any $a, b \in R$ and $n \in M$ with $p \leq n$, if $0_M \neq abn \leq p$, $ab \in (p : e)$ or $an \leq p$ or $bn \leq p$.*
(3) *For any $a, b \in R$ with $ab \notin (p : e)$, $(p : ab) = (0_M : ab)$ or $(p : ab) = (p : a)$ or $(p : ab) = (p : b)$.*
(4) *For any elements $a, b \in R$, $h \in M_*$, if $0_M \neq abh \leq p$, $ab \in (p : e)$ or $ah \leq p$ or $bh \leq p$.*

Proof. (1) $\Rightarrow$ (2) : Obvious. (2) $\Rightarrow$ (3) : Let $a, b \in R$ and $n \in M$ such that $p \leq n$ and $0_M \neq abn \leq p$. Assume $ab \notin (p : e)$. Take a submodule element $k \in M$ such that $k \leq (p : ab)$. Then $abk \leq p$. Now, we have two cases: **Case 1:** If $0_M = abk$, then $k \leq (0_M : ab)$. Also, we know that $(0_M : ab) \leq (p : ab)$. Thus $(p : ab) = (0_M : ab)$. **Case 2:** If $0_M \neq abk$, then we have $0_M \neq ab(k+p) = abk+abp \leq p$. Consider $u = k+p$. Thus we have $p \leq u$, $0_M \neq abu \leq p$ and $ab \notin (p : e)$. By our hypothesis (2), we conclude $au \leq p$ or $bu \leq p$. This means that $ak \leq p$ or $bk \leq p$, that is, $k \leq (p : a)$ or $k \leq (p : b)$. Then we say $(p : ab) \leq (p : a)$ or $(p : ab) \leq (p : b)$. On the other hand, it is clear that $(p : a) \leq (p : ab)$ and $(p : b) \leq (p : ab)$. Consequently, $(p : ab) = (p : a)$ or $(p : ab) = (p : b)$. (3) $\Rightarrow$ (4) : Similar to (3) $\Rightarrow$ (4) in previous theorem. (4) $\Rightarrow$ (1) : Take $a, b \in R$ and $n \in M$, $0_M \neq abn \leq p$ but $an \nleq p$ and $bn \nleq p$. As M is a CG-le-module, there is a compact submodule element h such that $h \leq n, ah \nleq p$, and $0_M \neq abh$. Similarly, there is a compact submodule element h' such that $h' \leq n, bh' \nleq p$ and $0_M \neq abh'$. Moreover, we know that $h + h' \in M_*$. Then we get $0_M \neq ab(h + h') = abh + abh' \leq abn + abn \leq p + p = p$. Thus, by our assumption, either $ab \in (p : e)$ or $a(h + h') \leq p$ or $b(h + h') \leq p$. Both of the last two options contradicts with $ah \nleq p$ and $bh' \nleq p$, recpectively. As a result, $ab \in (p : e)$. □

3. *n*-absorbing and weakly *n*-absorbing submodule elements in le-modules

In this section, we would like to generalize 2-absorbing (resp., weakly 2-absorbing) submodule elements to n-absorbing (resp., weakly n-absorbing) submodule elements for a positive integer n.

Definition 3.1. *Let p be a proper submodule element of M. For a positive integer n,*
(1) *p is called an **n-absorbing submodule element** for every $r_1, r_2, ..., r_n \in R$ and $q \in M$ if $r_1 r_2 ... r_n q \leq p$, either $r_1 r_2 ... r_n \in (p : e)$ or $\widehat{r_i} q \leq p$ for some i, where $\widehat{r_i}$ is the element $r_1 r_2 ... r_{i-1} r_{i+1} ... r_n$ and $1 \leq i \leq n$.*
(2) *p is called a **weakly n-absorbing submodule element** for every $r_1, r_2, ..., r_n \in R$ and $q \in M$ if $0_M \neq r_1 r_2 ... r_n q \leq p$, either $r_1 r_2 ... r_n \in (p : e)$ or $\widehat{r_i} q \leq p$ for some i, where $\widehat{r_i}$ is the element $r_1 r_2 ... r_{i-1} r_{i+1} ... r_n$ and $1 \leq i \leq n$.*

Remark 3.1. *The concept of weakly n-absorbing submodule elements is a generalization of n-absorbing submodule elements. However, every weakly n-absorbing submodule elements is not an n-absorbing submodule element, see the next example:*

Example 3.1. *Let $R = \mathbb{Z}_{210}$ and $M = \{X \subseteq R : \overline{0} \in X\}$. Now, consider $n = \{\overline{0}\}$. It is a weakly 3-absorbing submodule element but not 3-absorbing submodule element. Indeed, $\overline{2} \cdot \overline{3} \cdot \overline{7} \cdot \{\overline{0}, \overline{5}\} \leq n$, but $\overline{2} \cdot \overline{3} \cdot \overline{7} \notin (n : e) = (\overline{0})$ and $\overline{2} \cdot \overline{3} \cdot \{\overline{0}, \overline{5}\} \nleq n$, $\overline{2} \cdot \overline{7} \cdot \{\overline{0}, \overline{5}\} \nleq n$, $\overline{3} \cdot \overline{7} \cdot \{\overline{0}, \overline{5}\} \nleq n$.*

Theorem 3.1. *Let $n \in \mathbb{Z}^+$ and p be a proper submodule element of M. If p is an n-absorbing submodule element, then $(p : e)$ is an n-absorbing ideal of R.*

Proof. Take $r_1, r_2, ..., r_n, r_{n+1} \in R$ such that $r_1 r_2 ... r_n r_{n+1} \in (p : e)$, that is, $r_1 r_2 ... r_n r_{n+1} e \leq p$. Let $\widehat{r_i}$ be the element $r_1 r_2 ... r_{i-1} r_{i+1} ... r_n$ and $1 \leq i \leq n$. Assume that $\widehat{r_i}\ r_{n+1} \notin (p : e)$ for all i. Then $r_1 r_2 ... r_n (r_{n+1} e) \leq p$ and $\widehat{r_i}\ (r_{n+1} e) \nleq p$ imply that $r_1 r_2 ... r_n \in (p : e)$ because p is an n-absorbing submodule element. Thus it is done. □

Theorem 3.2. *Let $n, m \in \mathbb{Z}^+$ such that $n < m$ and p be a proper submodule element of M. If p is an n-absorbing submodule element, then p is an m-absorbing submodule element of M.*

Proof. Choose $r_1, r_2, ..., r_n, ..., r_m \in R$ and $q \in M$ such that $r_1 r_2 ... r_n(r_{n+1} ... r_m q) \leq p$. Then since p is an n-absorbing submodule element, $r_1 r_2 ... r_n \in (p : e)$ or $\widehat{r_i}(r_{n+1} ... r_m q) \leq p$ for some i, where $\widehat{r_i}$ is the element $r_1 r_2 ... r_{i-1} r_{i+1} ... r_n$ and $1 \leq i \leq n$. This means that $r_1 r_2 ... r_n r_{n+1} ... r_m \in (p : e)$ or $\widehat{r_i}(r_{n+1} ... r_m) q \leq p$. Thus p is an m-absorbing submodule element of M. □

Recall that a proper submodule element q of an le-module M is called a prime submodule element for any $r \in R$ and $n \in M$ if $rn \leq q$, then $n \leq q$ or $r \in (q : e)$, see [5]. Moreover, it is proven that $(n : e)$ is a prime ideal of R in Corollary 3.7 of [5].

Now, for an n-absorbing submodule element p of M, we would like to introduce a new notion, denoted by $\omega(p)$, which analyzes in some sense how far p is from being a prime submodule element.

Definition 3.2. *Let $n \in \mathbb{Z}^+$ and p be an n-absorbing submodule element of M. Then $\omega(p) :=$min$\{n \in \mathbb{Z}^+ : p$ is an n-absorbing submodule element of $M\}$, otherwise $\omega(p) := \infty$. Also, $\omega(e) := 0$.*

Remark 3.2. *For any $p \in Sub(M)$,*

(1) *p is prime $\Leftrightarrow \omega(p) = 1$.*
(2) *$\omega(p) = 0 \Leftrightarrow p = e$.*

Note that a proper submodule element q of an le-module M is called a primary submodule element for any $r \in R$ and $n \in M$ if $rn \leq q$, then $n \leq q$ or $r^k \in (q : e)$ for some $k \in \mathbb{N}$, see [5]. Also, we know that if q is a primary submodule element of M, then $Rad(q) := Rad((q : e))$ is a prime ideal of R, see Proposition 3.3 in [5]. In this case, if $Rad(q) = P$, then we say q is a P-primary submodule element.

Now, we invastigate the relationship between the primary submodule elements and the n-absorbing submodule elements of M.

Theorem 3.3. *Let $n \in \mathbb{Z}^+$. If q is a P-primary submodule element of M such that $P^n e \leq q$, then q is an n-absorbing submodule element of M, where $Rad(q) = P$ is a prime ideal of R. Furthermore $\omega(q) \leq n$.*

Proof. Take $r_1, r_2, ..., r_n \in R$ and $x \in M$ such that $r_1 r_2 ... r_n x \leq q$ and $\widehat{r_i} x \nleq q$ for all $1 \leq i \leq n$, where $\widehat{r_i}$ is the element $r_1 r_2 ... r_{i-1} r_{i+1} ... r_n$. Consider $r_i(\widehat{r_i} x) \leq q$. Then as $\widehat{r_i} x \nleq q$ and q is a P-primary submodule element, we get $r_i \in Rad(q) = Rad((q : e)) = P$, for all $1 \leq i \leq n$. This means that $r_1 r_2 ... r_n \in P^n$. Then, as $P^n e \leq q$, we say $r_1 r_2 ... r_n \in (q : e)$. Thus q is an n-absorbing submodule element of M. Finally, $\omega(q) \leq n$ is obvious. □

As a final result for this section, we present the relationship between the primary submodule elements and the 2-absorbing submodule elements of M.

Corollary 3.1. *Let q be a P-primary submodule element of M, where the ideal $Rad(q) = P$ is prime. Then q is a 2-absorbing submodule element of $M \Leftrightarrow P^2 e \leq q$.*

Proof. Suppose that q is a P-primary submodule element of M. ($\Rightarrow$:) Let q be a 2-absorbing. By Corollary 2.1, $(q : e)$ is a 2-absorbing ideal in R. By Theorem 2.4 in [4], $[Rad((q : e))]^2 \subseteq (q : e)$. As $Rad(q) = Rad((q : e)) = P$, we get $P^2 \subseteq (q : e)$, that is, $P^2 e \leq q$. ($\Leftarrow$:) By previous Thorem. □

4. *(n,k)*-absorbing and weakly *(n,k)*-absorbing submodule elements in le-modules

In this part of our study, we would like to generalize the concept of n-absorbing (weakly n-absorbing) submodule elements to (n,k)-absorbing (weakly (n,k)-absorbing) submodule elements for any two positive integers n and k such that $k < n$.

Definition 4.1. *Let p be a proper submodule element in M. Then for any two positive integers n and k such that $k < n$,*

(1) *p is called an* ***(n, k)-absorbing submodule element*** *for all $r_1, r_2, ..., r_n \in R$ and $q \in M$ if $r_1 r_2 ... r_n q \leq p$, then either there are k of the r_i's whose product is in $(p:e)$ or there are $(k-1)$ of the r_i's whose product with q is less than or equal to p.*
(2) *p is called a* ***weakly (n, k)-absorbing submodule element*** *for all $r_1, r_2, ..., r_n \in R$ and $q \in M$ if $0_M \neq r_1 r_2 ... r_n q \leq p$, then either there are k of the r_i's whose product is in $(p:e)$ or there are $(k-1)$ of the r_i's whose product with q is less than or equal to p.*

Remark 4.1. *It is easy to see that the class of weakly (n,k)-absorbing submodule elements contains the class of (n,k)-absorbing submodule elements. On the other hand, the converse of the containment doesn't hold, see the following example:*

Example 4.1. *Let $R = \mathbb{Z}_{2310}$ and $M = \{X \subseteq R : \overline{0} \in X\}$. Now, consider $n = \{\overline{0}\}$. It is a weakly (4, 3)-absorbing submodule element but not (4, 3)-absorbing submodule element. Firstly, note that $(n:e) = (\overline{0})$. It is clear $\overline{2} \cdot \overline{3} \cdot \overline{7} \cdot \overline{11} \cdot \{\overline{0}, \overline{5}\} \leq n$. However, $\overline{2} \cdot \overline{3} \cdot \overline{7} \notin (n:e), \overline{2} \cdot \overline{3} \cdot \overline{11} \notin (n:e), \overline{2} \cdot \overline{7} \cdot \overline{11} \notin (n:e), \overline{3} \cdot \overline{7} \cdot \overline{11} \notin (n:e)$, and $\overline{2} \cdot \overline{3} \cdot \{\overline{0}, \overline{5}\} \nleq n$, $\overline{2} \cdot \overline{7} \cdot \{\overline{0}, \overline{5}\} \nleq n$, $\overline{2} \cdot \overline{11} \cdot \{\overline{0}, \overline{5}\} \nleq n$, $\overline{3} \cdot \overline{7} \cdot \{\overline{0}, \overline{5}\} \nleq n$, $\overline{3} \cdot \overline{11} \cdot \{\overline{0}, \overline{5}\} \nleq n$, $\overline{7} \cdot \overline{11} \cdot \{\overline{0}, \overline{5}\} \nleq n$.*

In the next theorem, we examine the relationship between the concept of prime submodule elements and the concept of $(2,1)$-absorbing submodule elements.

Theorem 4.1. *Let n be a proper submodule element of M. If n is a prime submodule element, then n is a $(2,1)$-absorbing submodule element of M.*

Proof. Let $r_1, r_2 \in R$ and $q \in M$, $r_1 r_2 q \leq n$. We will show that either $r_1 \in (n:e)$ or $r_2 \in (n:e)$ or $q \leq n$. Since n is prime, $r_1 r_2 \in (n:e)$ or $q \leq n$. Also, by Corollary 3.7 in [5] , we know that $(n:e)$ is prime. Thus, $q \leq n$ or $r_1 \in (n:e)$ or $r_2 \in (n:e)$. □

To obtain the relationship between weakly prime submodule elements and weakly $(2,1)$-absorbing submodule elements of an le-module, firstly we need the followings:

Definition 4.2. *A proper submodule element q in M is called a* ***weakly prime submodule element*** *for any $a \in R$ and $m \in M$ if $0_M \neq am \leq q$, then $a \in (q:e)$ or $m \leq q$.*

Lemma 4.1. *Let M be a faithful le-module and n be a proper submodule element of M. If n is a weakly prime submodule element, then $(n:e)$ is a weakly prime ideal of R.*

Proof. Take $a, b \in R$ such that $0_R \neq ab \in (n:e)$, that is, $abe \leq n$. Now, we have two cases: **Case 1:** $0_M = abe$. Then $ab \in (0_M : e) = 0_R$, a contradiction. **Case 2:** $0_M \neq abe$. As $abe \leq n$ and n is weakly prime, either $a \in (n:e)$ or $be \leq n$, that is, $b \in (n:e)$. □

Theorem 4.2. *Suppose that M is a faithful le-module and n is a proper submodule element of M. Then if n is a weakly prime submodule element, n is a weakly $(2,1)$-absorbing submodule element of M.*

Proof. Choose $r_1, r_2 \in R$ and $q \in M$, $0_M \neq r_1 r_2 q \leq n$. We will show that either $r_1 \in (n:e)$ or $r_2 \in (n:e)$ or $q \leq n$. As n is weakly prime, either $r_1 r_2 \in (n:e)$ or $q \leq n$. Moreover, $0_R \neq r_1 r_2$, otherwise we get a contradiction with $0_M \neq r_1 r_2 q$. On the other hand, by

Lemma 4.1, we know $(n:e)$ is a weakly prime ideal. Thus, $0_R \neq r_1r_2 \in (n:e)$ implies that $r_1 \in (n:e)$ or $r_2 \in (n:e)$. Consequently, we obtain that $r_1 \in (n:e)$ or $r_2 \in (n:e)$ or $q \leq n$. □

In the next two theorems, we present the relationship between 2-absorbing (resp., weakly prime) submodule elements and $(3,2)$-absorbing (resp., weakly $(3,2)$-absorbing) submodule elements of an le-module.

Theorem 4.3. *Let n be a proper submodule element of M. If n is a 2-absorbing submodule element, then n is a $(3,2)$-absorbing submodule element of M.*

Proof. Let $r_1, r_2, r_3 \in R$ and $q \in M$, $r_1r_2r_3q \leq n$. Since n is a 2-absorbing submodule element, $r_1r_2 \in (n:e)$ or $r_1r_3q \leq n$ or $r_2r_3q \leq n$. Again, using the concept of 2-absorbing submodule elements, $r_1r_2 \in (n:e)$ or $r_1r_3 \in (n:e)$ or $r_2r_3 \in (n:e)$ or $r_1q \leq n$ or $r_2q \leq n$ or $r_3q \leq n$. Thus it is done. □

Theorem 4.4. *Let n be a proper submodule element of M. If n is a weakly 2-absorbing submodule element, then n is a weakly $(3,2)$-absorbing submodule element of M.*

Proof. Choose $r_1, r_2, r_3 \in R$ and $q \in M$, $0_M \neq r_1r_2r_3q \leq n$. Since n is a weakly 2-absorbing submodule element, $r_1r_2 \in (n:e)$ or $r_1r_3q \leq n$ or $r_2r_3q \leq n$. Also, it is clear that $0_M \neq r_1r_3q$ and $0_M \neq r_2r_3q$. Then again since n is a weakly 2-absorbing submodule elements, we have either $r_1r_2 \in (n:e)$ or $r_1r_3 \in (n:e)$ or $r_2r_3 \in (n:e)$ or $r_1q \leq n$ or $r_2q \leq n$ or $r_3q \leq n$. It is done. □

As final results, we characterize (n,k)-absorbing and weakly (n,k)-absorbing submodule elements in Theorem 4.5 and Theorem 4.6, respectively.

Theorem 4.5. *Suppose that p is a proper submodule element in M. For arbitrary two positive integers n and k such that $k < n$,*

(1) *If p is an (n,k)-absorbing submodule element, p is a $(k+1,k)$-absorbing submodule element.*
(2) *If p is an (n,k)-absorbing submodule element, then p is a (n,k')-absorbing submodule element, for any positive integer $k' > k$.*

Proof. (1): Suppose that p is an (n,k)-absorbing for any $k < n$. Then since $k < k+1$, we can consider $k+1$ as n. Thus by our assumption, p is $(k+1,k)$-absorbing.

(2): Let p be (n,k)-absorbing and k' be a positive integer such that $k' > k$. Choose $r_1, r_2, ..., r_n \in R$ and $q \in M$, $r_1r_2...r_nq \leq p$. Then by our hypothesis, we have either $a_1a_2...a_k \in (p:e)$ or $b_1b_2...b_{k-1}q \leq p$, where the a_i's and b_i's are some of the r_i's is obtained on renaming. Let choose a among the r_i's but other than the a_i's. Thus, it is clear that $aa_1a_2...a_k \in (p:e)$. Let choose b among the r_i's but other than the b_i's. Then $bb_1b_2...b_{k-1}q \leq p$. As a consequence, continuing the same way, p is an (n,k')-absorbing submodule element, for every positive integer $k' > k$. □

Theorem 4.6. *Suppose that p is a proper submodule element in M. For arbitrary two positive integers n and k such that $k < n$,*

(1) *If p is a weakly (n,k)-absorbing submodule element, p is a weakly $(k+1,k)$-absorbing submodule element.*
(2) *If p is a weakly (n,k)-absorbing submodule element, p is a weakly (n,k')-absorbing submodule element, for any positive integer $k' > k$.*

Proof. Similar to the previous proof. □

REFERENCES

[1] *Al-Khouja E. A.*, Maximal elements and prime elements in lattice modules, Damascus Univ. J. of Basic Sciences, 2003, **19,** 9–21.

[2] *Anderson D. D.*, Multiplicative lattices, Dissertation, University of Chicago, 1974.

[3] *Aslankarayigit Ugurlu E., Callialp F. and Tekir U.,* Prime, weakly prime and almost prime elements in multiplication lattice modules, Open Mathematics, 2016, **14,** 673-680.

[4] *Badawi A.*, On 2-absorbing ideals of commutative rings, Bulletin of the Australian Mathematical Society, 2007, **75(3),** 417-429.

[5] *Bhuniya A. K. and Kumbhakar M.*, Uniqueness of primary decompositions in Laskerian le-modules, Acta Math. Hunga, 2019, **158 (1),** 202-215.

[6] *Bouba E. M., Mahdou N. and Tamekkante M.*, Duplication of a module along an ideal, Acta Math. Hungar, 2018, **154,** 29–42.

[7] *Callialp F., Jayaram C. and Tekir U.*, Weakly Prime Elements in Multiplicative Lattices, Communıcations in Algebra, 2012, **40(8),** 2825-2840.

[8] *Callialp F., Tekir U. and Aslankarayigit E.*, On multiplication lattice modules, Hacettepe Journal of Mathematics and Statistics, 2014, **43(4),** 571-578.

[9] *Johnson E. W. and Johnson J. A.*, Lattice modules over principal element domains, Comm. Algebra, 2003, **31,** 3505–3518.

[10] *Kumbhakar M. and Bhuniya A. K.*, On irreducible pseudo-prime spectrum of topological le-modules, Quasigroups and Related Systems, 2018, **2 (26),** 251-262.

[11] *Kumbhakar M. and Bhuniya A. K.*, On the prime spectrum of an le-module, Journal of Algebra and Its Applications, 2020; https://doi.org/10.1142/S0219498821502200 (in press).

[12] *Majcherek D. J.,* Multiplicative lattice versions of some results from Noetherian commutative rings, Dissertation, University of California, Riverside, 2014.

[13] *Şengelen Sevim, E. and Koç, S.,* On n-Prime Ideals, University Politehnica of Bucharest Scientific Bulletin-Series A-Applied Mathematics and Physics, 2018, **80(2)**, 179-190.

[14] *Manjarekar C. S., Bingi A. V.*, Absorbing elements in lattice modules, International Electronic Journal of Algebra, 2016, **19(19)**, 58-76.