CONVERGENCE RESULTS OF SELF-ADAPTIVE PROJECTION ALGORITHMS FOR VARIATIONAL INEQUALITIES IN HILBERT SPACES

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In this paper, we investigate iterative methods for solving the pseudomonotone monotone variational inequality and the generalized variational inequality in Hilbert spaces. We propose an iterative algorithm by using self-adaptive method and projection method. Strong convergence result of the proposed algorithm is obtained under a weaker condition than sequential weak continuity imposed on pseudomonotone operators.

Keywords: pseudomonotone variational inequality, generalized variational inequality, self-adaptive method, projection.

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1. Introduction

The variational inequality problem (in short, VIP(C,ψ)) is to find a point $x^* \in C$ such that

$$\langle \psi(x^*), x - x^* \rangle \geq 0, \ \ \forall\, x \in C, \tag{1}$$

where C is a nonempty convex closed subset of a real Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$ and $\psi : C \to H$ is a nonlinear operator. The solution of VIP(C,ψ)) is denoted by Sol(C,ψ).

VIP(C,ψ)) studied by Stampacchia ([16]) unveiled theory and algorithms for the study of a large category of problems such as differential equations, optimization problems ([34, 44]), fixed point problems ([8, 17, 18, 19, 32]), mathematical programming problems ([10]), equilibrium problems ([30, 47]) and so on. Elaborate efforts were made to study VIP(C,ψ)) in different directions including existence theories, solution methods and applications in augmented reality, see, e.g., [1]-[47]. One of the most influential algorithms for solving VIP(C,ψ)) is projection algorithm ([1, 15, 28]) which defines a sequence $\{x^k\}$ by the following manner

$$x^{k+1} = \text{proj}_C[x^k - \varsigma\psi(x^k)], \ k \geq 0, \tag{2}$$

where proj_C means the orthogonal projection from H onto C and the constant ς is the step-size.

In general, the operator ψ in (2) should be strongly monotone and Lipschitz continuous. Note that the Lipschitz constant of ψ is very difficult to calculate. To relax these restrictions, several valuable methods have been presented for solving VIP(C,ψ), for example, Korpelevich's extragradient method ([11, 38, 46]), Tseng's method ([21, 22, 40]), forward-backward-forward method ([2]), subgradient-extragradient method ([5]), self-adaptive methods ([20, 27, 31, 41, 42]). Especially, Vuong [25] proved that Korpelevich's extragradient

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method has weak convergence under the conditions that ψ is sequentially weak-to-weak continuous and pseudo-monotone.

Let $\phi : C \to H$ and $\varphi : C \to C$ be two operators. Recall that the generalized variational inequality is to find a point $x^\dagger \in C$ such that

$$\langle \phi(x^\dagger), \varphi(x) - \varphi(x^\dagger) \rangle \geq 0, \ \forall x \in C. \tag{3}$$

The solution set of (3) is denoted by $\mathrm{Sol}(C, \phi, \varphi)$.

If $\varphi = I$, then the generalized variational inequality (3) reduces to $\mathrm{VIP}(C, \psi)$.

The general variational inequality (3) was introduced and studied in [13] and a wide class of linear and nonlinear problems including nonsymmetric and odd-order obstacle, unilateral and moving boundary value problems arising in pure and applied sciences can be studied in the unified framework of general variational inequalities, see [33, 36] and the references therein.

Our main purpose of this paper is to investigate the following problem of finding a point $u^\dagger$ such that

$$u^\dagger \in \mathrm{Sol}(C, \phi, \varphi) \text{ and } \varphi(u^\dagger) \in \mathrm{Sol}(C, \psi), \tag{4}$$

where ϕ is λ-inverse strongly φ-monotone and ψ is pseudomonotone.

We propose an iterative algorithm by using self-adaptive method and projection method for solving problem (4). Strong convergence result of the proposed algorithm is obtained under a weaker condition than sequential weak continuity imposed on the pseudomonotone operator ψ.

2. **Preliminaries**

Let C be a nonempty closed convex subset of a real Hilbert space H. Recall that an operator $f : C \to C$ is said to be L-Lipschitz ($L \geq 0$) if

$$\|f(u) - f(v)\| \leq L\|u - v\|, \ \forall u, v \in C.$$

If $L < 1$, then f is said to be L-contraction. If $L = 1$, then f is said to be nonexpansive.

Let $\varphi : C \to C$ and $A : C \to H$ be two operators. Recall that an operator $A : C \to C$ is said to be

- η-strongly monotone if there exists a constant $\eta > 0$ such that

$$\langle A(u) - A(v), u - v \rangle \geq \eta \|u - v\|^2, \ \forall u, v \in C. \tag{5}$$

- λ-inverse strongly φ-monotone if there exists a constant $\lambda > 0$ such that

$$\langle A(u) - A(v), \varphi(u) - \varphi(v) \rangle \geq \lambda \|A(u) - A(v)\|^2, \ \forall u, v \in C.$$

- pseudomonotone if

$$\langle A(v), u - v \rangle \geq 0 \Rightarrow \langle A(u), u - v \rangle \geq 0, \ \forall u, v \in C.$$

Remark 2.1. *Let $\varphi : C \to C$ be an η-strongly monotone operator. Let $\phi : C \to H$ be a λ-inverse strongly φ-monotone operator. Then, we have the following assertions:*

- *According to the definition of φ, we have*

$$\|\varphi(x) - \varphi(y)\| \geq \eta \|x - y\|, \ \forall x, y \in C. \tag{6}$$

- *For all $x, y \in C$ and $\sigma > 0$, we have*

$$\|(\varphi(x) - \sigma\phi(x)) - (\varphi(y) - \sigma\phi(y))\|^2 \leq \sigma(\sigma - 2\lambda)\|\phi(x) - \phi(y)\|^2 + \|\varphi(x) - \varphi(y)\|^2. \tag{7}$$

An operator $S : H \to 2^H$ is said to be monotone if and only if $\langle x - y, u - v \rangle \geq 0$ for all $x, y \in \mathrm{dom}(S)$, $u \in S(x)$, and $v \in S(y)$. A monotone operator S on H is said to be maximal if and only if its graph is not strictly contained in the graph of any other monotone operator on H.

For $\forall x^\dagger \in H$, there exists a unique point in C, denoted by $\text{proj}_C[x^\dagger]$ satisfying

$$\|x^\dagger - \text{proj}_C[x^\dagger]\| \le \|x - x^\dagger\|,\ \forall x \in C.$$

Moreover, $proj_C$ is firmly nonexpansive, that is,

$$\|\text{proj}_C[q^*] - \text{proj}_C[v^\dagger]\|^2 \le \langle \text{proj}_C[q^*] - \text{proj}_C[v^\dagger], q^* - v^\dagger\rangle,\ \forall q^*, v^\dagger \in H. \tag{8}$$

Further, proj_C has the following property ([39])

$$\langle q^* - \text{proj}_C[q^*], x^\dagger - \text{proj}_C[q^*]\rangle \le 0,\ \forall q^* \in H, x^\dagger \in C. \tag{9}$$

Lemma 2.1 ([6]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let ψ be a continuous and pseudomonotone operator on H. Then $\hat{p} \in Sol(C,\psi)$ if and only if $p^\dagger$ satisfies*

$$\langle \psi(u^\dagger), u^\dagger - \hat{p}\rangle \ge 0,\ \forall u^\dagger \in C.$$

Lemma 2.2 ([26]). *Let $\{a_k\} \subset [0,\infty)$, $\{b_k\} \subset (0,1)$ and $\{c_k\}$ be real number sequences. Suppose the following conditions are satisfied*

(i) $a_{k+1} \le (1-b_k)a_k + c_k, \forall k \ge 1$;

(ii) $\sum_{n=1}^{\infty} b_k = \infty$;

(iii) $\limsup\limits_{k\to\infty} \dfrac{c_k}{b_k} \le 0$ *or* $\sum_{n=1}^{\infty} |c_k| < \infty$.

Then $\lim_{k\to\infty} a_k = 0$.

Lemma 2.3 ([12]). *Let $\{\vartheta_k\}$ be a real number sequence. Assume there exists at least a subsequence $\{\vartheta_{k_i}\}$ of $\{\vartheta_k\}$ such that $\vartheta_{k_i} \le \vartheta_{k_i+1}$ for all $i \ge 0$. For every $k \ge K_0$, define an integer sequence $\{\tau(k)\}$ as $\tau(k) = \max\{i \le k : \vartheta_{k_i} < \vartheta_{k_i+1}\}$. Then $\tau(k) \to \infty$ as $k \to \infty$ and for all $k \ge K_0$, $\max\{\vartheta_{\psi(k)}, \vartheta_k\} \le \vartheta_{\tau(k)+1}$.*

3. **Main results**

In this section, we first present an iterative algorithm and its convergence analysis. Finally, we include several corollaries.

Let H be a real Hilbert space and C a nonempty closed convex subset of H. Let $f : C \to C$ be a ρ-contractive operator. Let $\varphi : C \to C$ be a weakly continuous and η-strongly monotone operator with $\text{Rang}(\varphi) = C$. Let $\phi : C \to H$ be a λ-inverse strongly φ-monotone operator. Let the operator ψ be pseudomonotone on H and L-Lipschitz continuous on C. Set $\Gamma := \{x | x \in \text{Sol}(C,\phi,\varphi) \text{ and } \varphi(x) \in \text{Sol}(C,\psi)\}$.

Let $\{\lambda_k\}$ and $\{\varsigma_k\}$ be two real number sequences in $[0,1]$ and $\{\sigma_k\}$ be a real number sequence in $(0,\infty)$. Let $\nu \in (0,1)$, $\varpi \in (0,1)$, $\sigma \in (0,1)$ and $\mu \in (0,2)$ be four constants. In what follows, we suppose that $\Gamma \ne \emptyset$.

Next, we present an iterative algorithm for solving problem (4).

Algorithm 3.1. *Let $x^0 \in C$ be an initial point. Set $k = 0$.*

Step 1. Let x^k be given. Calculate

$$z^k = \text{proj}_C[\lambda_k f(x^k) + (1-\lambda_k)(\varphi(x^k) - \sigma_k \phi(x^k))]. \tag{10}$$

Step 2. Find the smallest nonnegative integer $m = \min\{0,1,2,\cdots\}$ such that

$$u^k = \text{proj}_C[z^k - \nu\varpi^m \psi(z^k)], \tag{11}$$

and

$$\nu\varpi^m \|\psi(u^k) - \psi(z^k)\| \le \sigma\|u^k - z^k\|. \tag{12}$$

If $u^k = z^k$, then set $y^k = z^k$ and go to Step 3. Otherwise, calculate

$$y^k = \text{proj}_C\left[z^k + \mu(1-\sigma)\|u^k - z^k\|^2 \frac{\hat{u}^k}{\|\hat{u}^k\|^2}\right], \tag{13}$$

where $\hat{u}^k = u^k - z^k - \nu\varpi^m\psi(u^k)$. *Consequently, set* $\varpi_k = \varpi^m$.

Step 3. Calculate

$$\varphi(x^{k+1}) = (1-\varsigma_k)\varphi(x^k) + \varsigma_k y^k. \tag{14}$$

Step 4. Set $k := k+1$ *and return to step 1.*

Remark 3.1. *We have the following assertions*

(i) *The variational inequality*

$$\langle f(x) - \varphi(x), \varphi(y) - \varphi(x)\rangle \le 0,\ \forall y \in \Gamma, \tag{15}$$

has a unique solution denoted by q^*.

(ii) *There exists the smallest nonnegative integer* m *satisfying* (11) *and* (12).

(iii) $0 < \frac{\varpi\sigma}{\nu L} < \varpi_k \le 1 (\forall k \ge 0)$.

(iv) *If* $z^k = \mathrm{proj}_C[z^k - \nu\varpi_k\psi(z^k)]$, *then* $z^k \in Sol(C,\psi)$.

In order to prove convergence analysis of Algorithm 3.1, we add an extra condition (P): Let $\{t^k\}$ be any given sequence in H. If $t^k \rightharpoonup t^\dagger \in H$ and $\liminf_{k\to\infty} \|\psi(t^k)\| = 0$, then we get $\psi(t^\dagger) = 0$.

Remark 3.2. *If* ψ *is sequentially weak-to-weak continuous, then* ψ *satisfies the above condition (P).*

Theorem 3.1. *Suppose that the following conditions are satisfied:*

(c1): $\lim_{k\to\infty}\lambda_k = 0$ *and* $\sum_{k=1}^{\infty}\lambda_k = \infty$;

(c2): $0 < \liminf_{k\to\infty}\varsigma_k \le \limsup_{k\to\infty}\varsigma_k < 1$;

(c3): $0 < \nu < \eta < 2\lambda$ *and* $0 < \liminf_{k\to\infty}\sigma_k \le \limsup_{k\to\infty}\sigma_k < 2\lambda$.

Then the sequence $\{x^k\}$ *generated by Algorithm 3.1 converges strongly to* $q^* \in \Gamma$ *which solves VI* (15).

Proof. Since q^* solves VI (15), $q^* \in \mathrm{Sol}(C,\phi,\varphi)$ and $\varphi(q^*) \in \mathrm{Sol}(C,\psi)$. It follows that $\varphi(q^*) = \mathrm{proj}_C[\varphi(q^*) - \sigma_k\phi(q^*)]$ for all $k \ge 0$. Set $w^k = \varphi(x^k) - \sigma_k\phi(x^k) - (\varphi(q^*) - \sigma_k\phi(q^*))$ for all $k \ge 0$. Using (7), we obtain

$$\begin{aligned}\|w^k\|^2 &\le \|\varphi(x^k) - \varphi(q^*)\|^2 + \sigma_k(\sigma_k - 2\lambda)\|\phi(x^k) - \phi(q^*)\|^2 \\ &\le \|\varphi(x^k) - \varphi(q^*)\|^2,\end{aligned} \tag{16}$$

and

$$\begin{aligned}&\|\varphi(x^{k+1}) - \sigma_{k+1}\phi(x^{k+1}) - (\varphi(x^k) - \sigma_{k+1}\phi(x^k))\|^2 \\ &\quad \le \|\varphi(x^{k+1}) - \varphi(x^k)\|^2 + \sigma_{k+1}(\sigma_{k+1} - 2\lambda)\|\phi(x^{k+1}) - \phi(x^k)\|^2.\end{aligned} \tag{17}$$

By (6), (10) and (16), we have

$$\begin{aligned}\|z^k - \varphi(q^*)\| &= \|\mathrm{proj}_C[\lambda_k f(x^k) + (1-\lambda_k)(\varphi(x^k) - \sigma_k\phi(x^k))] - \mathrm{proj}_C[\varphi(q^*) - \sigma_k\phi(q^*)]\| \\ &\le \|\lambda_k(f(x^k) - \varphi(q^*) + \sigma_k\phi(q^*)) + (1-\lambda_k)w^k\| \\ &\le \lambda_k\|f(x^k) - f(q^*)\| + \lambda_k\|f(q^*) - \varphi(q^*) + \sigma_k\phi(q^*)\| + (1-\lambda_k)\|w^k\| \\ &\le \lambda_k\nu/\eta\|\varphi(x^k) - \varphi(q^*)\| + \lambda_k\|f(q^*) - \varphi(q^*) + \sigma_k\phi(q^*)\| \\ &\quad + (1-\lambda_k)\|\varphi(x^k) - \varphi(q^*)\| \\ &\le [1 - (1-\nu/\eta)\lambda_k]\|\varphi(x^k) - \varphi(q^*)\| + \lambda_k(\|f(q^*) - \varphi(q^*)\| + 2\lambda\|\phi(q^*)\|).\end{aligned} \tag{18}$$

Combining (16) and (18), we get

$$\begin{aligned}\|z^k - \varphi(q^*)\|^2 &\le \lambda_k\|f(x^k) - \varphi(q^*) + \sigma_k\phi(q^*)\|^2 + (1-\lambda_k)\|w^k\|^2 \\ &\le \lambda_k\|f(x^k) - \varphi(q^*) + \sigma_k\phi(q^*)\|^2 + (1-\lambda_k)[\|\varphi(x^k) - \varphi(q^*)\|^2 \\ &\quad + \sigma_k(\sigma_k - 2\lambda)\|\phi(x^k) - \phi(q^*)\|^2].\end{aligned} \tag{19}$$

From (13), we have

$$\begin{aligned}\|y^k-\varphi(q^*)\|^2 &= \Big\|\mathrm{proj}_C\Big[z^k+\mu(1-\sigma)\|u^k-z^k\|^2\frac{\hat{u}^k}{\|\hat{u}^k\|^2}\Big]-\mathrm{proj}_C[\varphi(q^*)]\Big\|^2\\ &\le \Big\|z^k-\varphi(q^*)+\mu(1-\sigma)\|u^k-z^k\|^2\frac{\hat{u}^k}{\|\hat{u}^k\|^2}\|^2\\ &= \|z^k-\varphi(q^*)\|^2+2\mu(1-\sigma)\frac{\|u^k-z^k\|^2}{\|\hat{u}^k\|^2}\langle\hat{u}^k,z^k-\varphi(q^*)\rangle\\ &\quad+\mu^2(1-\sigma)^2\frac{\|u^k-z^k\|^4}{\|\hat{u}^k\|^2}.\end{aligned}\tag{20}$$

Now, we estimate $\langle\hat{u}^k,z^k-\varphi(q^*)\rangle$. First, observe that

$$\begin{aligned}\langle\hat{u}^k,z^k-\varphi(q^*)\rangle &= \langle u^k-z^k-\nu\varpi_k\psi(u^k),z^k-\varphi(q^*)\rangle\\ &= \langle u^k-z^k+\nu\varpi_k\psi(z^k),z^k-\varphi(q^*)\rangle-\nu\varpi_k\langle\psi(z^k),z^k-\varphi(q^*)\rangle\\ &\quad-\nu\varpi_k\langle\psi(u^k),z^k-u^k\rangle-\nu\varpi_k\langle\psi(u^k),u^k-\varphi(q^*)\rangle\\ &= \nu\varpi_k\langle\psi(u^k),\varphi(q^*)-u^k\rangle+\nu\varpi_k\langle\psi(z^k),\varphi(q^*)-z^k\rangle\\ &\quad+\langle u^k-z^k+\nu\varpi_k(\psi(z^k)-\psi(u^k)),z^k-u^k\rangle\\ &\quad+\langle u^k-z^k+\nu\varpi_k\psi(z^k),u^k-\varphi(q^*)\rangle.\end{aligned}\tag{21}$$

Owing to $\varphi(q^*)\in Sol(C,\psi)$, we have $\langle\psi(\varphi(q^*)),\varphi(q^*)-z^k\rangle\le 0$ and $\langle\psi(\varphi(q^*)),\varphi(q^*)-u^k\rangle\le 0$. Utilizing the pseudomonotonicity of ψ, we deduce

$$\langle\psi(z^k),\varphi(q^*)-z^k\rangle\le 0,\tag{22}$$

and

$$\langle\psi(u^k),\varphi(q^*)-u^k\rangle\le 0.\tag{23}$$

Applying inequality (9) to (11), we achieve

$$\langle u^k-z^k+\nu\varpi_k\psi(z^k),u^k-\varphi(q^*)\rangle\le 0.\tag{24}$$

In the light of (21)-(24), we derive

$$\begin{aligned}\langle\hat{u}^k,z^k-\varphi(q^*)\rangle &\le \langle u^k-z^k+\nu\varpi_k(\psi(z^k)-\psi(u^k)),z^k-u^k\rangle\\ &\le -\|u^k-z^k\|^2+\nu\varpi_k\|\psi(z^k)-\psi(u^k)\|\|z^k-u^k\|.\end{aligned}\tag{25}$$

On the basis of (12) and (25), we get

$$\begin{aligned}\langle\hat{u}^k,z^k-\varphi(q^*)\rangle &\le -\|u^k-z^k\|^2+\sigma\|u^k-z^k\|^2\\ &= -(1-\sigma)\|u^k-z^k\|^2.\end{aligned}\tag{26}$$

This together with (20) implies that

$$\begin{aligned}\|y^k-\varphi(q^*)\|^2 &\le \|z^k-\varphi(q^*)\|^2-2\mu(1-\sigma)^2\frac{\|u^k-z^k\|^4}{\|\hat{u}^k\|^2}+\mu^2(1-\sigma)^2\frac{\|u^k-z^k\|^4}{\|\hat{u}^k\|^2}\\ &= \|z^k-\varphi(q^*)\|^2-(2-\mu)\mu(1-\sigma)^2\frac{\|u^k-z^k\|^4}{\|\hat{u}^k\|^2}\\ &\le \|z^k-\varphi(q^*)\|^2.\end{aligned}\tag{27}$$

According to (14), (18) and (27), we obtain

$$\begin{aligned}\|\varphi(x^{k+1})-\varphi(q^*)\| &\le (1-\varsigma_k)\|\varphi(x^k)-\varphi(q^*)\|+\varsigma_k\|y^k-\varphi(q^*)\| \\ &\le (1-\varsigma_k)\|\varphi(x^k)-\varphi(q^*)\|+\varsigma_k\|z^k-\varphi(q^*)\| \\ &\le (1-\varsigma_k)\|\varphi(x^k)-\varphi(q^*)\|+\varsigma_k[1-(1-\nu/\eta)\lambda_k] \\ &\quad \times \|\varphi(x^k)-\varphi(q^*)\|+\varsigma_k\lambda_k(\|f(q^*)-\varphi(q^*)\|+2\lambda\|\phi(q^*)\|) \\ &= [1-(1-\nu/\eta)\varsigma_k\lambda_k]\|\varphi(x^k)-\varphi(q^*)\|+(1-\nu/\eta)\varsigma_k\lambda_k \\ &\quad \times \frac{\|f(q^*)-\varphi(q^*)\|+2\lambda\|\phi(q^*)\|}{1-\nu/\eta}.\end{aligned} \tag{28}$$

It results in that

$$\|\varphi(x^k)-\varphi(q^*)\| \le \max\left\{\|\varphi(x^0)-\varphi(q^*)\|, \frac{\|f(q^*)-\varphi(q^*)\|+2\lambda\|\phi(q^*)\|}{1-\nu/\eta}\right\}.$$

Thus, the sequence $\{\varphi(x^k)\}$ is bounded and the sequences $\{z^k\}$ and $\{y^k\}$ are also bounded due to (18) and (27). Since $\|x^k-q^*\| \le \frac{1}{\eta}\|\varphi(x_n)-\varphi(q^*)\|$, $\{x^k\}$ is bounded.

By (14), we have $\langle \varphi(x^{k+1})-\varphi(x^k), \varphi(x^k)-\varphi(q^*)\rangle = \varsigma_k\langle y^k-\varphi(x^k), \varphi(x^k)-\varphi(q^*)\rangle$. It follows that

$$\begin{aligned}&\|\varphi(x^{k+1})-\varphi(q^*)\|^2-\|\varphi(x^k)-\varphi(q^*)\|^2 \\ &\quad = \varsigma_k[\|y^k-\varphi(q^*)\|^2-\|\varphi(x^k)-\varphi(q^*)\|^2-\|y^k-\varphi(x^k)\|^2]+\varsigma_k^2\|y^k-\varphi(x^k)\|^2 \\ &\quad = \varsigma_k[\|y^k-\varphi(q^*)\|^2-\|\varphi(x^k)-\varphi(q^*)\|^2]-\varsigma_k(1-\varsigma_k)\|y^k-\varphi(x^k)\|^2.\end{aligned} \tag{29}$$

Thanks to (27) and (29), we obtain

$$\begin{aligned}\|\varphi(x^{k+1})-\varphi(q^*)\|^2 &\le (1-\varsigma_k)\|\varphi(x^k)-\varphi(q^*)\|^2+\varsigma_k\|z^k-\varphi(q^*)\|^2 \\ &\quad -\varsigma_k(1-\varsigma_k)\|y^k-\varphi(x^k)\|^2.\end{aligned} \tag{30}$$

By virtue of (18), we get

$$\begin{aligned}\|z^k-\varphi(q^*)\|^2 &\le [1-(1-\nu/\eta)\lambda_k]\|\varphi(x^k)-\varphi(q^*)\|^2 \\ &\quad +(1-\nu/\eta)\lambda_k\left(\frac{\|f(q^*)-\varphi(q^*)\|+2\lambda\|\phi(q^*)\|}{1-\nu/\eta}\right)^2.\end{aligned} \tag{31}$$

Now, we analyze two cases: the sequence $\{\|\varphi(x^k)-\varphi(q^*)\|\}$ is either monotone decreasing at infinity (Case 1) or not (Case 2).

For Case 1, there exists a large enough positive integer K such that $\{\|\varphi(x^k)-\varphi(q^*)\|\}$ is monotone decreasing when $k \ge K$. In this case, $\lim_{k\to\infty}\|\varphi(x^k)-\varphi(q^*)\|$ exists. On account of (30) and (31), we have

$$\begin{aligned}\varsigma_k(1-\varsigma_k)\|y^k-\varphi(x^k)\|^2 &\le (1-\varsigma_k)\|\varphi(x^k)-\varphi(q^*)\|^2+\varsigma_k\|z^k-\varphi(q^*)\|^2 \\ &\quad -\|\varphi(x^{k+1})-\varphi(q^*)\|^2 \\ &\le \|\varphi(x^k)-\varphi(q^*)\|^2-\|\varphi(x^{k+1})-\varphi(q^*)\|^2 \\ &\quad +(1-\nu/\eta)\lambda_k\left(\frac{\|f(q^*)-\varphi(q^*)\|+2\lambda\|\phi(q^*)\|}{1-\nu/\eta}\right)^2 \\ &\to 0.\end{aligned}$$

It follows that

$$\lim_{k\to\infty}\|y^k-\varphi(x^k)\| = 0. \tag{32}$$

Furthermore,

$$\lim_{k\to\infty}\|\varphi(x^{k+1})-\varphi(x^k)\| = \lim_{k\to\infty}\varsigma_k\|y^k-\varphi(x^k)\| = 0, \tag{33}$$

which together with the strong monotonicity of φ implies that

$$\lim_{k\to\infty}\|x^{k+1}-x^k\| = 0.$$

Taking into account (14), (19) and (27), we deduce

$$\begin{aligned}
\|\varphi(x^{k+1})-\varphi(q^*)\|^2 &\le (1-\varsigma_k)\|\varphi(x^k)-\varphi(q^*)\|^2+\varsigma_k\|y^k-\varphi(q^*)\|^2\\
&\le (1-\varsigma_k)\|\varphi(x^k)-\varphi(q^*)\|^2+\varsigma_k\|z^k-\varphi(q^*)\|^2\\
&\quad -\varsigma_k(2-\mu)\mu(1-\sigma)^2\frac{\|u^k-z^k\|^4}{\|\hat{u}^k\|^2}\\
&\le (1-\varsigma_k)\|\varphi(x^k)-\varphi(q^*)\|^2+\varsigma_k(1-\lambda_k)\|\varphi(x^k)-\varphi(q^*)\|^2\\
&\quad +\varsigma_k(1-\lambda_k)\sigma_k(\sigma_k-2\lambda)\|\phi(x^k)-\phi(q^*)\|^2\\
&\quad +\varsigma_k\lambda_k\|f(x^k)-\varphi(q^*)+\sigma_k\phi(q^*)\|^2\\
&\quad -\varsigma_k(2-\mu)\mu(1-\sigma)^2\frac{\|u^k-z^k\|^4}{\|\hat{u}^k\|^2}\\
&\le \|\varphi(x^k)-\varphi(q^*)\|^2+\varsigma_k\lambda_k\|f(x^k)-\varphi(q^*)+\sigma_k\phi(q^*)\|^2\\
&\quad -\varsigma_k(2-\mu)\mu(1-\sigma)^2\frac{\|u^k-z^k\|^4}{\|\hat{u}^k\|^2}\\
&\quad +\varsigma_k(1-\lambda_k)\sigma_k(\sigma_k-2\lambda)\|\phi(x^k)-\phi(q^*)\|^2.
\end{aligned} \tag{34}$$

It results in that

$$\begin{aligned}
&\varsigma_k(2-\mu)\mu(1-\sigma)^2\frac{\|u^k-z^k\|^4}{\|\hat{u}^k\|^2}+\varsigma_k(1-\lambda_k)\sigma_k(2\lambda-\sigma_k)\|\phi(x^k)-\phi(q^*)\|^2\\
&\le \|\varphi(x^k)-\varphi(q^*)\|^2-\|\varphi(x_{n+1})-\varphi(q^*)\|^2+\varsigma_k\lambda_k\|f(x^k)-\varphi(q^*)+\sigma_k\phi(q^*)\|^2\\
&\to 0.
\end{aligned}$$

Accordingly,

$$\lim_{k\to\infty}\frac{\|u^k-z^k\|^2}{\|\hat{u}^k\|} = 0, \tag{35}$$

and

$$\lim_{k\to\infty}\|\phi(x^k)-\phi(q^*)\| = 0. \tag{36}$$

As a result of (8), (10) and (16), we have

$$\begin{aligned}
\|z^k - \varphi(q^*)\|^2 &= \|\mathrm{proj}_C[\lambda_k f(x^k) + (1-\lambda_k)(\varphi(x^k) - \sigma_k\phi(x^k))] - \mathrm{proj}_C[\varphi(q^*) - \sigma_k\phi(q^*)]\|^2 \\
&\le \langle \lambda_k f(x^k) + (1-\lambda_k)(\varphi(x^k) - \sigma_k\phi(x^k)) - \varphi(q^*) + \sigma_k\phi(q^*), z^k - \varphi(q^*)\rangle \\
&= \lambda_k\langle f(x^k) - \varphi(q^*) + \sigma_k\phi(q^*), z^k - \varphi(q^*)\rangle + (1-\lambda_k)\langle w^k, z^k - \varphi(q^*)\rangle \\
&\le \lambda_k\langle f(x^k) - \varphi(q^*) + \sigma_k\phi(q^*), z^k - \varphi(q^*)\rangle + \frac{1}{2}\Big\{\|z^k - \varphi(q^*)\|^2 \\
&\quad + \|w^k\|^2 - \|\varphi(x^k) - z^k - \sigma_k(\phi(x^k) - \phi(q^*))\|^2\Big\} \\
&\le \lambda_k\|f(x^k) - \varphi(q^*) + \sigma_k\phi(q^*)\|\|z^k - \varphi(q^*)\| \\
&\quad + \frac{1}{2}\Big\{\|\varphi(x^k) - \varphi(q^*)\|^2 + \|z^k - \varphi(q^*)\|^2 - \sigma_k^2\|\phi(x^k) - \phi(q^*)\|^2 \\
&\quad - \|\varphi(x^k) - z^k\|^2 + 2\sigma_k\langle\varphi(x^k) - z^k, \phi(x^k) - \phi(q^*)\rangle\Big\}.
\end{aligned}$$

It yields

$$\begin{aligned}
\|z^k - \varphi(q^*)\|^2 \le\ & \|\varphi(x^k) - \varphi(q^*)\|^2 - \|\varphi(x^k) - z^k\|^2 \\
&+ 2\sigma_k\|\varphi(x^k) - z^k\|\|\phi(x^k) - \phi(q^*)\| \\
&+ 2\lambda_k\|f(x^k) - \varphi(q^*) + \sigma_k\phi(q^*)\|\|z^k - \varphi(q^*)\|.
\end{aligned} \tag{37}$$

By (34) and (37), we obtain

$$\begin{aligned}
\|\varphi(x^{k+1}) - \varphi(q^*)\|^2 \le\ & \|\varphi(x^k) - \varphi(q^*)\|^2 + 2\sigma_k\|\varphi(x^k) - z^k\|\|\phi(x^k) - \phi(q^*)\| \\
&+ 2\lambda_k\|f(x^k) - \varphi(q^*) + \sigma_k\phi(q^*)\|\|z^k - \varphi(q^*)\| \\
&- \varsigma_k\|\varphi(x^k) - z^k\|^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
\varsigma_k\|\varphi(x^k) - z^k\|^2 \le\ & \|\varphi(x^k) - \varphi(q^*)\|^2 - \|\varphi(x^{k+1}) - \varphi(q^*)\|^2 \\
&+ 2\sigma_k\|\varphi(x^k) - z^k\|\|\phi(x^k) - \phi(q^*)\| \\
&+ 2\lambda_k\|f(x^k) - \varphi(q^*) + \sigma_k\phi(q^*)\|\|z^k - \varphi(q^*)\|.
\end{aligned} \tag{38}$$

In the light of (36) and (38), we deduce

$$\lim_{k\to\infty}\|\varphi(x^k) - z^k\| = 0. \tag{39}$$

From (11), we have

$$\begin{aligned}
\|u^k - \varphi(q^*)\| &\le \|proj_C[z^k - \nu\varpi_k\psi(z^k)] - \varphi(q^*)\| \\
&\le \|z^k - \varphi(q^*)\| + \nu\varpi_k\|\psi(z^k)\|.
\end{aligned}$$

Thus, the sequences $\{u^k\}$ and $\{\hat{u}^k\}$ are bounded. Consequently, from (35), we get

$$\lim_{k\to\infty}\|u^k - z^k\| = 0. \tag{40}$$

In view of (12) and (40), we deduce

$$\lim_{k\to\infty}\|\psi(u^k) - \psi(z^k)\| = 0. \tag{41}$$

As a result of (13), we have the following estimate

$$\|y^k - z^k\| \le \mu(1-\sigma)\frac{\|u^k - z^k\|^2}{\|\hat{u}^k\|}$$

This together with (35) implies that

$$\lim_{k\to\infty}\|y^k - z^k\| = 0. \tag{42}$$

Since the sequences $\{x^k\}$ and $\{z^k\}$ are bounded, we can select a common subsequence $\{k_i\}$ of $\{k\}$ such that $x^{k_i} \rightharpoonup p^\dagger$ and

$$\limsup_{k\to\infty}\langle f(q^*) - \varphi(q^*), z^k - \varphi(q^*)\rangle = \lim_{i\to\infty}\langle f(q^*) - \varphi(q^*), z^{k_i} - \varphi(q^*)\rangle. \tag{43}$$

Then, $\varphi(x^{k_i}) \rightharpoonup \varphi(p^\dagger)$ because of the weak continuity of φ, $z^{k_i} \rightharpoonup \varphi(p^\dagger)$ by (39) and $y^{k_i} \rightharpoonup \varphi(p^\dagger)$ due to (42).

Next, we show $p^\dagger \in \mathrm{Sol}(C, \phi, \varphi)$. Define an operator A by the following form

$$A(\tilde{v}) = \begin{cases} \phi(\tilde{v}) + N_C(\tilde{v}), & \tilde{v} \in C, \\ \emptyset, & \tilde{v} \notin C. \end{cases}$$

Then, A is maximal φ-monotone. Let $(\tilde{v}, u) \in \mathrm{G}(A)$. It follows that $u - \phi(\tilde{v}) \in N_C(\tilde{v})$ and $\langle \varphi(\tilde{v}) - \varphi(x^{k_i}), u - \phi(\tilde{v})\rangle \geq 0$. Since $\langle \varphi(\tilde{v}) - z^{k_i}, z^{k_i} - [\lambda_{k_i} f(x^{k_i}) + (1-\lambda_{k_i})(\varphi(x^{k_i}) - \sigma_{k_i}\phi(x^{k_i}))]\rangle \geq 0$, we have

$$\left\langle \varphi(\tilde{v}) - z^{k_i}, \frac{z^{k_i} - \varphi(x^{k_i})}{\sigma_{k_i}} + \phi(x^{k_i})\right\rangle + \frac{\lambda_{k_i}}{\sigma_{k_i}}\langle \varphi(\tilde{v}) - z^{k_i}, \varphi(x^{k_i}) - \sigma_{k_i}\phi(x^{k_i}) - f(x^{k_i})\rangle \geq 0.$$

It leads to

$$\begin{aligned}
\langle \varphi(\tilde{v}) - \varphi(x^{k_i}), u\rangle &\geq \langle \varphi(\tilde{v}) - \varphi(x^{k_i}), \phi(\tilde{v})\rangle \\
&\geq \langle \varphi(\tilde{v}) - \varphi(x^{k_i}), \phi(\tilde{v})\rangle - \left\langle \varphi(\tilde{v}) - z^{k_i}, \frac{z^{k_i} - \varphi(x^{k_i})}{\sigma_{k_i}} + \phi(x^{k_i})\right\rangle \\
&\quad - \frac{\lambda_{k_i}}{\sigma_{k_i}}\langle \varphi(\tilde{v}) - z^{k_i}, \varphi(x^{k_i}) - \sigma_{k_i}\phi(x^{k_i}) - f(x^{k_i})\rangle \\
&= \langle \varphi(\tilde{v}) - \varphi(x^{k_i}), \phi(\tilde{v}) - \phi(x^{k_i})\rangle + \langle z^{k_i} - \varphi(x^{k_i}), \phi(x^{k_i})\rangle \\
&\quad - \frac{\lambda_{k_i}}{\sigma_{k_i}}\langle \varphi(\tilde{v}) - z^{k_i}, \varphi(x^{k_i}) - \sigma_{k_i}\phi(x^{k_i}) - f(x^{k_i})\rangle \\
&\quad - \left\langle \varphi(\tilde{v}) - z^{k_i}, \frac{z^k - \varphi(x^{k_i})}{\sigma_{k_i}}\right\rangle \\
&\geq \langle z^{k_i} - \varphi(x^{k_i}), \phi(x^{k_i})\rangle - \left\langle \varphi(\tilde{v}) - z^{k_i}, \frac{z^{k_i} - \varphi(x^{k_i})}{\sigma_{k_i}}\right\rangle \\
&\quad - \frac{\lambda_{k_i}}{\sigma_{k_i}}\langle \varphi(\tilde{v}) - z^{k_i}, \varphi(x^{k_i}) - \sigma_{k_i}\phi(x^{k_i}) - f(x^{k_i})\rangle.
\end{aligned} \tag{44}$$

We have $\|z^{k_i} - \varphi(x^{k_i})\| \to 0$ by (39), $\lambda_{k_i} \to 0$ by (c1) and $\varphi(x^{k_i}) \rightharpoonup \varphi(p^\dagger)$. Letting $i \to \infty$ in (44), we conclude that $\langle \varphi(\tilde{v}) - \varphi(p^\dagger), u\rangle \geq 0$. Hence, $p^\dagger \in A^{-1}(0)$. Therefore, $p^\dagger \in \mathrm{Sol}(C, \phi, \varphi)$.

Next, we show $\varphi(p^\dagger) \in \mathrm{Sol}(C, \psi)$. In view of (11), we have

$$\langle u^{k_i} - z^{k_i} + \nu\varpi_{k_i}\psi(z^{k_i}), u^\dagger - u^{k_i}\rangle \geq 0,\ \forall u^\dagger \in C.$$

It results in that

$$\begin{aligned}
\langle \psi(z^{k_i}), u^\dagger - z^{k_i}\rangle &\geq \langle \psi(z^{k_i}), u^{k_i} - z^{k_i}\rangle \\
&\quad + \frac{1}{\nu\varpi_{k_i}}\langle u^{k_i} - z^{k_i}, u^{k_i} - u^\dagger\rangle,\ \forall u^\dagger \in C.
\end{aligned} \tag{45}$$

According to (40) and (45), we receive

$$\liminf_{i\to\infty}\langle \psi(z^{k_i}), u^\dagger - z^{k_i}\rangle \geq 0,\ \forall u^\dagger \in C. \tag{46}$$

There are two possibilities, i.e., possibility 1: $\liminf_{i\to\infty}\|\psi(z^{k_i})\| = 0$ and possibility 2: $\liminf_{i\to\infty}\|\psi(z^{k_i})\| > 0$.

For possibility 1, since $z^{k_i} \rightharpoonup \varphi(p^\dagger)$ and ψ satisfying condition (P), we deduce that $\psi(\varphi(p^\dagger)) = 0$. Consequently, $\varphi(p^\dagger) \in Sol(C, \psi)$. Next, we consider possibility 2. In terms of (46), we can choose a positive real numbers sequence $\{\epsilon_j\}$ satisfying $\lim_{j\to\infty} \epsilon_j = 0$. For each ϵ_j, there exists the smallest positive integer n_j such that

$$\langle \frac{\psi(z^{k_{i_j}})}{\|\psi(z^{k_{i_j}})\|}, u^\dagger - z^{k_{i_j}} \rangle + \epsilon_j \geq 0,\ \forall j \geq n_j. \tag{47}$$

Set $g(z^{k_{i_j}}) = \frac{\psi(z^{k_{i_j}})}{\|\psi(z^{k_{i_j}})\|^2}$ for $j \geq n_j$. Then $\langle \psi(z^{k_{i_j}}), g(z^{k_{i_j}})\rangle = 1$. By virtue of (47), we have

$$\langle \psi(z^{k_{i_j}}), u^\dagger + \epsilon_j \|\psi(z^{k_{i_j}})\| g(z^{k_{i_j}}) - z^{k_{i_j}} \rangle \geq 0,$$

which implies, together with the pseudomonotonicity of ψ on H, that

$$\langle \psi(u^\dagger + \epsilon_j \|\psi(z^{k_{i_j}})\| g(z^{k_{i_j}})), u^\dagger + \epsilon_j \|\psi(z^{k_{i_j}})\| g(z^{k_{i_j}}) - z^{k_{i_j}} \rangle \geq 0. \tag{48}$$

Note that $\lim_{j\to\infty} \epsilon_j \|\psi(z^{k_{i_j}})\| \|g(z^{k_{i_j}})\| = \lim_{j\to\infty} \epsilon_j = 0$. Thus, taking the limit as $j \to \infty$ in (48), we obtain

$$\langle \psi(u^\dagger), u^\dagger - \varphi(p^\dagger)\rangle \geq 0, \forall u^\dagger \in C. \tag{49}$$

By Lemma 2.1 and (49), we deduce that $\varphi(p^\dagger) \in \mathrm{Sol}(C, \psi)$. Therefore, $p^\dagger \in \mathrm{Sol}(C, \phi, \varphi) \cap \varphi^{-1}(\mathrm{Sol}(C, \psi)) = \Gamma$.

From (43), we obtain

$$\begin{aligned}\limsup_{k\to\infty} \langle f(q^*) - \varphi(q^*), z^k - \varphi(q^*)\rangle &= \lim_{i\to\infty} \langle f(q^*) - \varphi(q^*), z^{k_i} - \varphi(q^*)\rangle \\ &= \langle f(q^*) - \varphi(q^*), \varphi(p^\dagger) - \varphi(q^*)\rangle \leq 0.\end{aligned} \tag{50}$$

By (9) and (10), we have

$$\begin{aligned}\|z^k - \varphi(q^*)\|^2 &= \|\mathrm{proj}_C[\lambda_k f(x^k) + (1-\lambda_k)(\varphi(x^k) - \sigma_k \phi(x^k))] \\ &\quad - \mathrm{proj}_C[\varphi(q^*) - (1-\lambda_k)\sigma_k \phi(q^*)]\|^2 \\ &\leq \langle \lambda_k (f(x^k) - \varphi(q^*)) + (1-\lambda_k) w^k, z^k - \varphi(q^*)\rangle \\ &= \lambda_k \langle f(x^k) - f(q^*), z^k - \varphi(q^*)\rangle + \lambda_k \langle f(q^*) - \varphi(q^*), z^k - \varphi(q^*)\rangle \\ &\quad + (1-\lambda_k)\langle w^k, z^k - \varphi(q^*)\rangle \\ &\leq [1 - (1-\nu/\eta)\lambda_k] \|\varphi(x^k) - \varphi(q^*)\| \|z^k - \varphi(q^*)\| \\ &\quad + \lambda_k \langle f(q^*) - \varphi(q^*), z^k - \varphi(q^*)\rangle \\ &\leq \frac{1 - (1-\nu/\eta)\lambda_k}{2} \|\varphi(x^k) - \varphi(q^*)\|^2 + \frac{1}{2}\|z^k - \varphi(q^*)\|^2 \\ &\quad + \lambda_k \langle f(q^*) - \varphi(q^*), z^k - \varphi(q^*)\rangle.\end{aligned}$$

It follows that

$$\|z^k - \varphi(q^*)\|^2 \leq [1 - (1-\nu/\eta)\lambda_k] \|\varphi(x^k) - \varphi(q^*)\|^2 + 2\lambda_k \langle f(q^*) - \varphi(q^*), z^k - \varphi(q^*)\rangle.$$

Therefore,

$$\begin{aligned}\|\varphi(x^{k+1}) - \varphi(q^*)\|^2 &\leq (1 - \varsigma_k)\|\varphi(x^k) - \varphi(q^*)\|^2 + \varsigma_k \|z^k - \varphi(q^*)\|^2 \\ &\leq [1 - (1-\nu/\eta)\varsigma_k \lambda_k] \|\varphi(x^k) - \varphi(q^*)\|^2 \\ &\quad + 2\varsigma_k \lambda_k \langle f(q^*) - \varphi(q^*), z^k - \varphi(q^*)\rangle.\end{aligned} \tag{51}$$

By Lemma 2.2 and (51), we conclude that $\varphi(x^k) \to \varphi(q^*)$ and $x^k \to q^*$.

In Case 2, for any integer K, there exists integer $n > K$ such that $\|\varphi(x^n) - \varphi(q^*)\| \leq \|\varphi(x^{n+1}) - \varphi(q^*)\|$. Let $\vartheta_k = \{\|\varphi(x^k) - \varphi(q^*)\|^2\}$. Then, we have $\vartheta_n \leq \vartheta_{n+1}$. For all $k \geq n$, define an integer sequence $\{\tau(k)\}$ as follows $\tau(k) = \max\{i \in \mathbb{N} | n \leq i \leq k, \vartheta_i \leq \vartheta_{i+1}\}$. It is easy to check that $\tau(k)$ is non-decreasing and satisfies $\lim_{k\to\infty}\tau(k)=\infty$ and $\vartheta_{\tau(k)} \leq \vartheta_{\tau(k)+1}, \forall k \geq n$.

Similarly, we can show

$$\limsup_{k\to\infty}\langle f(q^*)-\varphi(q^*), z^{\tau(k)}-\varphi(q^*)\rangle \leq 0 \tag{52}$$

and

$$\vartheta_{\tau(k)+1} \leq [1-(1-\nu/\eta)\lambda_{\tau(k)}\varsigma_{\tau(k)}]\vartheta_{\tau(k)} + 2\lambda_{\tau(k)}\varsigma_{\tau(k)}\langle f(q^*)-\varphi(q^*), z^{\tau(k)}-\varphi(q^*)\rangle. \tag{53}$$

Since $\vartheta_{\tau(k)} \leq \vartheta_{\tau(k)+1}$, from (53), we have

$$\vartheta_{\tau(k)} \leq \frac{2}{1-\nu/\eta}\langle f(q^*)-\varphi(q^*), z^{\tau(k)}-\varphi(q^*)\rangle. \tag{54}$$

Taking into account (52) and (54), we derive $\limsup_{k\to\infty}\vartheta_{\tau(k)} \leq 0$ and hence

$$\lim_{k\to\infty}\vartheta_{\tau(k)} = 0. \tag{55}$$

Based on (52) and (53), we can deduce $\limsup_{k\to\infty}\vartheta_{\tau(k)+1} \leq \limsup_{k\to\infty}\vartheta_{\tau(k)}$. This together with (55) implies that $\lim_{k\to\infty}\vartheta_{\tau(k)+1}=0$. By Lemma 2.3, we obtain $0 \leq \vartheta_k \leq \max\{\vartheta_{\tau(k)}, \vartheta_{\tau(k)+1}\}$. Therefore, $\vartheta_k \to 0$. That is, $\varphi(x^k) \to \varphi(q^*)$ and thus $x^k \to q^*$. This completes the proof. □

Algorithm 3.2. *Let $x^0 \in C$ be an initial point. Set $k=0$.*

Step 1. Let x^k be given. Calculate

$$z^k = \text{proj}_C[\lambda_k f(x^k) + (1-\lambda_k)(x^k - \sigma_k\phi(x^k))].$$

Step 2. Find the smallest nonnegative integer $m=\min\{0,1,2,\cdots\}$ such that

$$u^k = \text{proj}_C[z^k - \nu\varpi^m\psi(z^k)],$$

and

$$\nu\varpi^m\|\psi(u^k)-\psi(z^k)\| \leq \sigma\|u^k - z^k\|.$$

If $u^k = z^k$, then set $y^k = z^k$ and go to Step 3. Otherwise, calculate

$$y^k = \text{proj}_C\left[z^k + \mu(1-\sigma)\|u^k-z^k\|^2\frac{\hat{u}^k}{\|\hat{u}^k\|^2}\right],$$

where $\hat{u}^k = u^k - z^k - \nu\varpi^m\psi(u^k)$. Consequently, set $\varpi_k = \varpi^m$.

Step 3. Calculate

$$x^{k+1} = (1-\varsigma_k)x^k + \varsigma_k y^k.$$

Step 4. Set $k := k+1$ and return to step 1.

Corollary 3.1. *Let H be a real Hilbert space and C a nonempty closed convex subset of H. Let $f: C\to C$ be a ρ-contractive operator. Let $\phi: C \to H$ be a λ-inverse strongly monotone operator. Let the operator ψ be pseudomonotone on H and L-Lipschitz continuous on C. Suppose that the conditions (c1)-(c3) are satisfied. Suppose that $\Gamma_1 := \{x | x \in \text{Sol}(C,\phi) \cap \text{Sol}(C,\psi)\} \neq \emptyset$. Then the sequence $\{x^k\}$ generated by Algorithm 3.2 converges strongly to $q^* \in \Gamma_1$.*

4. **Conclusion**

In this paper, we survey iterative algorithm for solving the pseudomonotone monotone variational inequality (1) and the generalized variational inequality (3) in Hilbert spaces. We propose an iterative algorithm for solving problem (4) by using self-adaptive method and projection method. Strong convergence result of the proposed algorithm is obtained under a weaker condition than sequential weak continuity imposed on pseudomonotone operators.

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