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MATHEMATICAL PROGRAMS INVOLVING E-CONVEX FUNCTIONS

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In this paper, we have shown that generalized M-stationary condition is sufficient for global optimality under the assumptions of E-convexity and mathematical programming problems with equilibrium constraints. Further, we formulate and study Wolfe type and Mond-Weir type dual models for the MPEC and we establish weak and strong duality theorems relating to the MPEC and the two dual models.

Keywords: E-convexity, Stationary point, Wolfe type dual, Mond-Weir type dual, Mathematical program with equilibrium constraints

1. Introduction

The present study focuses on mathematical programs with equilibrium constraints (MPEC), which is a special class among the nonlinear optimization problems. This designation is established because its feasible set has complementarity conditions that do not usually appear in standard optimization problems that is the reason why MPEC are also known as mathematical programs with complementarity constraints (MPCC). In general, this distinguishing characteristic produces a challenging problem for most of the algorithms developed for solving continuous optimization problems, notably because many constraint qualifications do not hold at the desired solution for example, the Mangasarian Fromovitz constraint qualification (MFCQ) is not valid at any feasible point of an MPEC [25].

The MPEC are strongly connected with the bilevel optimization problem [4]. Although these problems are not equivalent [1,6], one of the most usual ways of dealing with a bilevel problem is to consider it as a special MPEC instance derived from the Karush-Kuhn-Tucker (KKT) conditions applied to the lower level problem. Since bilevel optimization is related with important real-world problems (e.g., see [2, 5]), and many other applications may be viewed as minimization problems with complementarity constraints [7,13]. For more literature on MPEC problems see [8, 10–12] and references therein.

Usually, generalized convex functions have been introduced in order to weaken the convexity requirements as much as possible to obtain results related to optimization theory [17, 19–24]. One of the significant generalization of convex function is E-convex function which was first introduced by Youness [27]. Subsequently, necessary and sufficient optimality conditions for a class of E-convex programming problems were discussed by Youness [28]. Modified form of the Kuhn-Tucker and Fritz-John problems i.e., E-Kuhn-Tucker and E-Fritz-John problems were also presented by Youness [28]. Later, Megahed *et al.* [16], presented the concept of an E-differentiable convex function which transforms a non-differentiable convex function to a differentiable function, then a solution of mathematical programming with a non-differentiable function could be found by applying the Kuhn-Tucker and Fritz-John

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conditions due to Mangasarian [14]. For more literature on E-convex functions one can see [3, 9, 15] and references therein.

The organization of this paper is as follows: in Section 2, we give some preliminary, definitions and results which will be used in the sequel. In Section 3, we show that M-stationary condition is sufficient for global optimality under some MPEC and E-convexity assumptions. In Section 4, we formulate Wolfe and Mond-Weir type dual models for the MPEC and establish weak and strong duality theorems relating to the MPEC and the two dual models under E-convexity assumptions. In Section 5, we conclude the results of this paper.

2. Preliminaries

In this section, we give some preliminaries and definitions which will be used in the paper. Throughout the paper $\mathbb{R}^n$ denotes the n-dimensional Euclidean space.

Definition 2.1. *[27] A set $M \subseteq \mathbb{R}^n$ is said to be E-convex iff there is a map $E : \mathbb{R}^n \to \mathbb{R}^n$ such that $(1-\lambda)E(x) + \lambda E(y) \in M$, for each $x, y \in M$, and $\lambda \in [0, 1]$.*

Example 2.1. *Consider the set $S_1 = \{(x, y) \in \mathbb{R}^2 | y \leq x, 0 \leq x \leq 1\}$. Let $E(x, y) = (\sqrt{x}, y)$ then it is clear that S_1 is E-convex (since S_1 is convex). It is easy to check that $E(S_1)$ is E-convex by taking the map $E(x, y) = (\sqrt{x}, y)$, while $E(S_1)$ is not convex, where $E(S_1) = \{(x, y) \in \mathbb{R}^2 | y \leq x^2, 0 \leq x \leq 1\}$.*

Definition 2.2. *[27] A function $F : \mathbb{R}^n \to \mathbb{R}$ is said to be E-convex on $M \subseteq \mathbb{R}^n$ iff there is a map $E : \mathbb{R}^n \to \mathbb{R}^n$ such that M is an E-convex set and*

$$F(\lambda E(x) + (1-\lambda)E(y)) \leq \lambda F(E(x)) + (1-\lambda)F(E(y)).$$

Obviously, if f is a real-valued differentiable function on an E-convex set $M \subseteq \mathbb{R}^n$, we can define a differentiable E-convex function in the following.

Definition 2.3. *F is E-convex on M at $\tilde{x}$ if for each $x \in M$, we have*

$$F(E(x)) - F(E(\tilde{x})) \geq \langle \nabla F(E(\tilde{x})), E(x) - E(\tilde{x}) \rangle.$$

Definition 2.4. *F is E-pseudoconvex on M at $\tilde{x}$ if for each $x \in M$, we have*

$$\langle \nabla F(E(\tilde{x})), E(x) - E(\tilde{x}) \rangle \geq 0 \Rightarrow F(E(x)) \geq F(E(\tilde{x})).$$

Definition 2.5. *F is E-quasiconvex on M at $\tilde{x}$ if for each $x \in M$, we have*

$$F(E(x)) \leq F(E(\tilde{x})) \Rightarrow \langle \nabla F(E(\tilde{x})), E(x) - E(\tilde{x}) \rangle \leq 0.$$

We provide following examples in support of the definitions of E-convex functions and generalized E-convex functions respectively.

Example 2.2. *Consider the function $F : \mathbb{R} \to \mathbb{R}$ is given by $F(x) = x$ and $E : \mathbb{R} \to \mathbb{R}$ is given by $E(x) = x^2$ then $\mathbb{R}$ is an E-convex set and F is an E-convex function at $\tilde{x} = 0$.*

Example 2.3. *Consider the function $F : \mathbb{R} \to \mathbb{R}$ is given by $F(x) = cosx$ and $E : \mathbb{R} \to \mathbb{R}$ is given by $E(x) = x$. Then $\mathbb{R}$ is an E-convex set and F is an E-pseudoconvex function at $\tilde{x} = \pi$.*

Example 2.4. *Consider the function $F : \mathbb{R} \to \mathbb{R}$ is given by $F(x) = -\sin x$. If $E : \mathbb{R} \to \mathbb{R}$ is given by $E(x) = -x$ then $\mathbb{R}$ is E-convex set and F is E-quasiconvex function at $\tilde{x} = \frac{\pi}{2}$.*

We consider the following MPEC problem over the E-convex sets in the following form:

$$\begin{aligned} \text{MPEC} \quad & \min \quad F(E(x)) \\ & \text{subject to}: \quad g(E(x)) \leq 0, \quad h(E(x)) = 0, \\ & G(E(x)) \geq 0, \ H(E(x)) \geq 0, \ \langle G(E(x)), H(E(x)) \rangle = 0, \end{aligned}$$

where $E : \mathbb{R}^n \to \mathbb{R}^n, F : \mathbb{R}^n \to \mathbb{R},\ g : \mathbb{R}^n \to \mathbb{R}^k,\ h : \mathbb{R}^n \to \mathbb{R}^p,\ G : \mathbb{R}^n \to \mathbb{R}^l$ and $H : \mathbb{R}^n \to \mathbb{R}^l$ are continuously differentiable functions on $\mathbb{R}^n$.

The feasible set of the problem MPEC is denoted by X and defined by

$$X := \{x \in \mathbb{R}^n : g(E(x)) \leq 0,\ h(E(x)) = 0,\ G(E(x)) \geq 0,\ H(E(x)) \geq 0,\ \langle G(E(x)), H(E(x)) \rangle = 0\}.$$

Given a feasible vector $\tilde{x} \in X$ for the problem (MPEC), we define the following index sets:

$$\begin{aligned} I_g &:= I_g(E(\tilde{x})) := \{i = 1, 2, \ldots, k : g_i(E(\tilde{x})) = 0\}, \\ \delta &:= \delta(E(\tilde{x})) := \{i = 1, 2, \ldots, l : G_i(E(\tilde{x})) = 0, H_i(E(\tilde{x})) > 0\}, \\ \beta &:= \beta(E(\tilde{x})) := \{i = 1, 2, \ldots, l : G_i(E(\tilde{x})) = 0, H_i(E(\tilde{x})) = 0\}, \\ \kappa &:= \kappa(E(\tilde{x})) := \{i = 1, 2, \ldots, l : G_i(E(\tilde{x})) > 0, H_i(E(\tilde{x})) = 0\}. \end{aligned}$$

Here the set β is known as degenerate set and if β is empty, the vector $\tilde{x}$ is said to satisfy the strict complementarity condition.

Based on the definition of Mordukhovich stationary point [18], we are defining the following concept of E-M-stationary point.

Definition 2.6. *A feasible point $\tilde{x}$ of MPEC is said to be E-Mordukhovich stationary point if, $\exists\ \alpha = (\alpha^g, \alpha^h, \alpha^G, \alpha^H) \in \mathbb{R}^{k+p+2l}$, such that following conditions hold:*

$$0 = \nabla F(E(\tilde{x})) + \sum_{i \in I_g} \alpha_i^g \nabla g_i(E(\tilde{x})) + \sum_{i=1}^{p} \alpha_i^h \nabla h_i(E(\tilde{x})) - \sum_{i=1}^{l} [\alpha_i^G \nabla G_i(E(\tilde{x})) + \alpha_i^H \nabla H_i(E(\tilde{x}))], \tag{1}$$

$$\alpha_{I_g}^g \geq 0, \quad \alpha_\kappa^G = 0, \quad \alpha_\delta^H = 0, \tag{2}$$

$$\forall i \in \beta,\ \textit{either}\ \alpha_i^G > 0, \quad \alpha_i^H > 0 \quad \textit{or} \quad \alpha_i^G \alpha_i^H = 0.$$

Based on the definition of No Nonzero Abnormal Multiplier Constraint Qualification [26], we define the following concept of E-No Nonzero Abnormal Multiplier Constraint Qualification (ENNAMCQ).

Definition 2.7. *Let $E(\tilde{x})$ be a feasible point of MPEC and all functions are continuously differentiable at $\tilde{x}$. We say that the E-No Nonzero Abnormal Multiplier Constraint Qualification (ENNAMCQ) is satisfied at $\tilde{x}$, if there is no nonzero vector $\alpha = (\alpha^g, \alpha^h, \alpha^G, \alpha^H) \in \mathbb{R}^{k+p+2l}$, such that*

$$0 = \sum_{i \in I_g} \alpha_i^g \nabla g_i(E(\tilde{x})) + \sum_{i=1}^{p} \alpha_i^h \nabla h_i(E(\tilde{x})) - \sum_{i=1}^{l} [\alpha_i^G \nabla G_i(E(\tilde{x})) + \alpha_i^H \nabla H_i(E(\tilde{x}))],$$

$$\alpha_{I_g}^g \geq 0, \quad \alpha_\kappa^G = 0, \quad \alpha_\delta^H = 0,$$

$$\forall i \in \beta,\ \textit{either}\ \alpha_i^G > 0, \quad \alpha_i^H > 0 \quad \textit{or} \quad \alpha_i^G \alpha_i^H = 0.$$

In the next section, it can be seen that M-stationary condition turns into a sufficient optimality condition for a certain MPEC problem involving E-convexity assumptions.

Note Throughout the paper {} will denote an empty set.

3. **Sufficient E-M-stationary condition**

Theorem 3.1. *Let $\tilde{x}$ be a feasible point of MPEC and E-M-stationary condition holds at $\tilde{x}$, i.e., $\exists\ \alpha = (\alpha^g, \alpha^h, \alpha^G, \alpha^H) \in \mathbb{R}^{k+p+2l}$, such that*

$$0 = \nabla F(E(\tilde{x})) + \sum_{i \in I_g} \alpha_i^g \nabla g_i(E(\tilde{x})) + \sum_{i=1}^{p} \alpha_i^h \nabla h_i(E(\tilde{x})) - \sum_{i=1}^{l} [\alpha_i^G \nabla G_i(E(\tilde{x})) + \alpha_i^H \nabla H_i(E(\tilde{x}))],$$

$$\alpha_{I_g}^g \geq 0, \quad \alpha_\kappa^G = 0, \quad \alpha_\delta^H = 0, \tag{3}$$

$$\forall i \in \beta, \text{ either } \alpha_i^G > 0, \quad \alpha_i^H > 0 \quad \text{or} \quad \alpha_i^G \alpha_i^H = 0.$$

Let

$$j^+ := \{i : \alpha_i^h > 0\}, \quad j^- := \{i : \alpha_i^h < 0\},$$
$$\beta^+ := \{i \in \beta : \alpha_i^G > 0,\ \alpha_i^H > 0\},$$
$$\beta_G^+ := \{i \in \beta : \alpha_i^G = 0, \alpha_i^H > 0\},\ \beta_G^- := \{i \in \beta : \alpha_i^G = 0, \alpha_i^H < 0\},$$
$$\beta_H^+ := \{i \in \beta : \alpha_i^H = 0, \alpha_i^G > 0\},\ \beta_H^- := \{i \in \beta : \alpha_i^H = 0, \alpha_i^G < 0\},$$
$$\delta^+ := \{i \in \delta : \alpha_i^G > 0\}, \quad \delta^- := \{i \in \delta : \alpha_i^G < 0\},$$
$$\kappa^+ := \{i \in \kappa : \alpha_i^H > 0\}, \quad \kappa^- := \{i \in \kappa : \alpha_i^H < 0\},$$

and assume that F is E- pseudoconvex at $\tilde{x}$, g_i $(i \in I_g)$, h_i $(i \in j^+)$, $-h_i$ $(i \in j^-)$, G_i $(i \in \delta^- \cup \beta_H^-)$, $-G_i$ $(i \in \delta^+ \cup \beta_H^+ \cup \beta^+)$, H_i $(i \in \kappa^- \cup \beta_G^-)$ and $-H_i$ $(i \in \kappa^+ \cup \beta_G^+ \cup \beta^+)$ are E-quasiconvex at $\tilde{x}$. If $\delta^- \cup \kappa^- \cup \beta_G^- \cup \beta_H^- = \{\}$ then $\tilde{x}$ is a global optimal solution of MPEC.

Proof. Assume that x be any feasible point of MPEC, i.e., for any $i \in I_g$,

$$g_i(E(x)) \leq 0 = g_i(E(\tilde{x})).$$

Using E-quasiconvexity of g_i at $\tilde{x}$, it follows that,

$$\langle \nabla g_i(E(\tilde{x})), E(x) - E(\tilde{x}) \rangle \leq 0, \quad \forall i \in I_g. \tag{4}$$

Similarly, we have

$$\langle \nabla h_i(E(\tilde{x})), E(x) - E(\tilde{x}) \rangle \leq 0, \quad \forall i \in j^+, \tag{5}$$

$$-\langle \nabla h_i(E(\tilde{x})), E(x) - E(\tilde{x}) \rangle \leq 0, \quad \forall i \in j^-. \tag{6}$$

Since, for any feasible point $x, -G(E(x)) \leq 0, -H(E(x)) \leq 0$, one also have

$$-\langle \nabla G_i(E(\tilde{x})), E(x) - E(\tilde{x}) \rangle \leq 0, \quad \forall i \in \delta^+ \cup \beta_H^+ \cup \beta^+, \tag{7}$$

$$-\langle \nabla H_i(E(\tilde{x})), E(x) - E(\tilde{x}) \rangle \leq 0, \quad \forall i \in \kappa^+ \cup \beta_G^+ \cup \beta^+. \tag{8}$$

First, we take $\delta^- \cup \kappa^- \cup \beta_G^- \cup \beta_H^- = \{\}$, multiplying (4)-(8) by $\alpha_i^g \geq 0$ $(i \in I_g)$, $\alpha_i^h > 0$ $(i \in j^+), -\alpha_i^h > 0$ $(i \in j^-)$, $\alpha_i^G > 0$ $(i \in \delta^+ \cup \beta_H^+ \cup \beta^+)$, $\alpha_i^H > 0$ $(i \in \kappa^+ \cup \beta_G^+ \cup \beta^+)$ respectively and adding (4)-(8), we obtain

$$\left\langle \sum_{i \in I_g} \alpha_i^g \nabla g_i(E(\tilde{x})) + \sum_{i=1}^{p} \alpha_i^h \nabla h_i(E(\tilde{x})) - \sum_{i=1}^{l} [\alpha_i^G \nabla G_i(E(\tilde{x})) + \alpha_i^H \nabla H_i(E(\tilde{x}))], E(x) - E(\tilde{x}) \right\rangle \leq 0.$$

Using equation (3), the above inequality follows that

$$\langle \nabla F(E(\tilde{x})), E(x) - E(\tilde{x}) \rangle \geq 0.$$

Applying, E-pseudoconvexity of F at $\tilde{x}$, we get

$$F(E(x)) \geq F(E(\tilde{x}))$$

for all feasible point x. Hence $\tilde{x}$ is a global optimal solution of MPEC. □

In the next section, we formulate a Wolfe type dual problem and a Mond-Weir type dual problem for the MPEC under the E-convexity assumptions.

4. **Duality**

WDMPEC

$$\max_{x,\alpha} \left\{ F(E(x)) + \sum_{i\in I_g} \alpha_i^g g_i(E(x)) + \sum_{i=1}^{p} \alpha_i^h h_i(E(x)) - \sum_{i=1}^{l} [\alpha_i^G G_i(E(x)) + \alpha_i^H H_i(E(x))] \right\}$$

subject to:

$$0 = \nabla F(E(x)) + \sum_{i\in I_g} \alpha_i^g \nabla g_i(E(x)) + \sum_{i=1}^{p} \alpha_i^h \nabla h_i(E(x)) - \sum_{i=1}^{l} [\alpha_i^G \nabla G_i(E(x)) + \alpha_i^H \nabla H_i(E(x))], \tag{9}$$

$$\alpha_{I_g}^g \geq 0, \quad \alpha_\kappa^G = 0, \quad \alpha_\delta^H = 0,$$

$$\forall i \in \beta, \text{ either } \alpha_i^G > 0, \quad \alpha_i^H > 0 \text{ or } \alpha_i^G \alpha_i^H = 0,$$

$where, \alpha = (\alpha^g, \alpha^h, \alpha^G, \alpha^H) \in \mathbb{R}^{k+p+2l}$.

Theorem 4.1. *(Weak Duality) Let $\tilde{y}$ be feasible for MPEC, (x, α) be feasible for WDMPEC and index sets $I_g, \delta, \beta, \kappa$ defined accordingly. Suppose that F, g_i $(i \in I_g)$, h_i $(i \in j^+)$, $-h_i$ $(i \in j^-)$, G_i $(i \in \delta^- \cup \beta_H^-)$, $-G_i$ $(i \in \delta^+ \cup \beta_H^+ \cup \beta^+)$, H_i $(i \in \kappa^- \cup \beta_G^-)$, and $-H_i$ $(i \in \kappa^+ \cup \beta_G^+ \cup \beta^+)$ are E-convex functions at x. If $\delta^- \cup \kappa^- \cup \beta_G^- \cup \beta_H^- = \{\}$. Then, for any y feasible for the MPEC, we have*

$$F(E(y)) \geq F(E(x)) + \sum_{i\in I_g} \alpha_i^g g_i(E(x)) + \sum_{i=1}^{p} \alpha_i^h h_i(E(x)) - \sum_{i=1}^{l} [\alpha_i^G G_i(E(x)) + \alpha_i^H H_i(E(x))].$$

Proof. Let us consider that, y be any feasible point for MPEC. Then, we get

$$g_i(E(y)) \leq 0, \quad \forall i \in I_g,$$

and

$$h_i(E(y)) = 0, \quad i = 1, 2, \ldots, p.$$

Since, F is E-convex at x, then

$$F(E(y)) - F(E(x)) \geq \langle \nabla F(E(x)), E(y) - E(x) \rangle. \tag{10}$$

Similarly, we get

$$g_i(E(y)) - g_i(E(x)) \geq \langle \nabla g_i(E(x)), E(y) - E(x) \rangle, \qquad \forall i \in I_g, \tag{11}$$

$$h_i(E(y)) - h_i(E(x)) \geq \langle \nabla h_i(E(x)), E(y) - E(x) \rangle, \qquad \forall i \in j^+, \tag{12}$$

$$-h_i(E(y)) + h_i(E(x)) \geq -\langle \nabla h_i(E(x)), E(y) - E(x) \rangle, \qquad \forall i \in j^-, \tag{13}$$

$$-G_i(E(y)) + G_i(E(x)) \geq -\langle \nabla G_i(E(x)), E(y) - E(x) \rangle, \quad \forall i \in \delta^+ \cup \beta_H^+ \cup \beta^+, \tag{14}$$

$$-H_i(E(y)) + H_i(E(x)) \geq -\langle \nabla H_i(E(x)), E(y) - E(x) \rangle, \quad \forall i \in \kappa^+ \cup \beta_G^+ \cup \beta^+. \tag{15}$$

If $\delta^- \cup \kappa^- \cup \beta_G^- \cup \beta_H^- = \{\}$, multiplying (11)-(15) by $\alpha_i^g \geq 0$ $(i \in I_g)$, $\alpha_i^h > 0$ $(i \in j^+)$, $-\alpha_i^h > 0$ $(i \in j^-)$, $\alpha_i^G > 0$ $(i \in \delta^+ \cup \beta_H^+ \cup \beta^+)$, $\alpha_i^H > 0$ $(i \in \kappa^+ \cup \beta_G^+ \cup \beta^+)$, respectively and adding (10)-(15), it follows that

$$F(E(y)) - F(E(x)) + \sum_{i\in I_g} \alpha_i^g g_i(E(y)) - \sum_{i\in I_g} \alpha_i^g g_i(E(x)) + \sum_{i=1}^{p} \alpha_i^h h_i(E(y)) - \sum_{i=1}^{p} \alpha_i^h h_i(E(x))$$
$$- \sum_{i=1}^{l} \alpha_i^G G_i(E(y)) + \sum_{i=1}^{l} \alpha_i^G G_i(E(x)) - \sum_{i=1}^{l} \alpha_i^H H_i(E(y)) + \sum_{i=1}^{l} \alpha_i^H H_i(E(x))$$
$$\geq \Big\langle \nabla F(E(x)) + \sum_{i\in I_g} \alpha_i^g \nabla g_i(E(x)) + \sum_{i=1}^{p} \alpha_i^h \nabla h_i(E(x))$$
$$- \sum_{i=1}^{l} [\alpha_i^G \nabla G_i(E(x)) + \alpha_i^H \nabla H_i(E(x))], E(y) - E(x) \Big\rangle.$$

Using (9), we get

$$F(E(y)) - F(E(x)) + \sum_{i\in I_g} \alpha_i^g g_i(E(y)) - \sum_{i\in I_g} \alpha_i^g g_i(E(x))$$
$$+ \sum_{i=1}^{p} \alpha_i^h h_i(E(y)) - \sum_{i=1}^{p} \alpha_i^h h_i(E(x)) - \sum_{i=1}^{l} \alpha_i^G G_i(E(y))$$
$$+ \sum_{i=1}^{l} \alpha_i^G G_i(E(x)) - \sum_{i=1}^{l} \alpha_i^H H_i(E(y)) + \sum_{i=1}^{l} \alpha_i^H H_i(E(x)) \geq 0.$$

Using the feasibility of y for MPEC, that is $g_i(E(y)) \leq 0,\ h_i(E(y)) = 0,\ G_i(E(y)) \geq 0,\ H_i(E(y)) \geq 0$, we obtain

$$F(E(y)) - F(E(x)) - \sum_{i\in I_g} \alpha_i^g g_i(E(x)) - \sum_{i=1}^{p} \alpha_i^h h_i(E(x)) + \sum_{i=1}^{l} [\alpha_i^G G_i(E(x)) + \alpha_i^H H_i(E(x))] \geq 0.$$

Hence,

$$F(E(y)) \geq F(E(x)) + \sum_{i\in I_g} \alpha_i^g g_i(E(x)) + \sum_{i=1}^{p} \alpha_i^h h_i(E(x)) - \sum_{i=1}^{l} [\alpha_i^G G_i(E(x)) + \alpha_i^H H_i(E(x))],$$

and the proof is complete. □

Theorem 4.2. *(Strong Duality) If $\tilde{y}$ is a global optimal solution of MPEC, such that ENNAMCQ is satisfied at $\tilde{y}$ and index sets $I_g, \delta, \beta, \kappa$ defined accordingly. Let F, g_i $(i \in I_g)$, h_i $(i \in j^+)$, $-h_i$ $(i \in j^-)$, G_i $(i \in \delta^- \cup \beta_H^-)$, $-G_i$ $(i \in \delta^+ \cup \beta_H^+ \cup \beta^+)$, H_i $(i \in \kappa^- \cup \beta_G^-)$ and $-H_i$ $(i \in \kappa^+ \cup \beta_G^+ \cup \beta^+)$ fulfill the assumptions of the Theorem 4.1. Then, there exists $\tilde{\alpha}$, such that $(\tilde{y}, \tilde{\alpha})$ is a global optimal solution of WDMPEC and corresponding objective values of MPEC and WDMPEC are equal.*

Proof. Since, $\tilde{y}$ is a global optimal solution of MPEC and ENNAMCQ is satisfied at $\tilde{y}$, therefore, there exist $\tilde{\alpha} = (\tilde{\alpha}^g, \tilde{\alpha}^h, \tilde{\alpha}^G, \tilde{\alpha}^H) \in \mathbb{R}^{k+p+2l}$, such that E-M- stationarity conditions are satisfied for MPEC, i.e.,

$$0 = \nabla F(E(\tilde{y})) + \sum_{i\in I_g} \tilde{\alpha}_i^g \nabla g_i(E(\tilde{y})) + \sum_{i=1}^{p} \tilde{\alpha}_i^h \nabla h_i(E(\tilde{y})) - \sum_{i=1}^{l} [\tilde{\alpha}_i^G \nabla G_i(E(\tilde{y})) + \tilde{\alpha}_i^H \nabla H_i(E(\tilde{y}))],$$
$$\tilde{\alpha}_{I_g}^g \geq 0,\quad \tilde{\alpha}_\kappa^G = 0,\quad \tilde{\alpha}_\delta^H = 0, \tag{16}$$
$$\forall i \in \beta,\quad \text{either}\ \ \tilde{\alpha}_i^G > 0,\quad \tilde{\alpha}_i^H > 0,\quad \text{or}\ \tilde{\alpha}_i^G \tilde{\alpha}_i^H = 0.$$

Therefore, $(\tilde{y}, \tilde{\alpha})$ is feasible for WDMPEC. By Theorem 4.1, we get

$$F(E(\tilde{y})) \geq F(E(x)) + \sum_{i\in I_g} \alpha_i^g g_i(E(x)) + \sum_{i=1}^{p} \alpha_i^h h_i(E(x)) - \sum_{i=1}^{l}[\alpha_i^G G_i(E(x)) + \alpha_i^H H_i(E(x))], \tag{17}$$

for any feasible solution (x, α) for WDMPEC. Now, using the feasibility condition of MPEC and WDMPEC, i.e, for $i \in I_g(E(\tilde{y})),\ g_i(E(\tilde{y})) = 0$, also $h_i(E(\tilde{y})) = 0, G_i(E(\tilde{y})) = 0, \forall i \in \delta \cup \beta$ and $H_i(E(\tilde{y})) = 0, \forall i \in \beta \cup \kappa$, then, it follows that

$$F(E(\tilde{y})) = F(E(\tilde{y})) + \sum_{i\in I_g} \tilde{\alpha}_i^g g_i(E(\tilde{y})) + \sum_{i=1}^{p} \tilde{\alpha}_i^h h_i(E(\tilde{y})) - \sum_{i=1}^{l}[\tilde{\alpha}_i^G G_i(E(\tilde{y})) + \tilde{\alpha}_i^H H_i(E(\tilde{y}))]. \tag{18}$$

Using (17) and (18), we get

$$\begin{aligned} F(E(\tilde{y})) &+ \sum_{i\in I_g} \tilde{\alpha}_i^g g_i(E(\tilde{y})) + \sum_{i=1}^{p} \tilde{\alpha}_i^h h_i(E(\tilde{y})) - \sum_{i=1}^{l}[\tilde{\alpha}_i^G G_i(E(\tilde{y})) + \tilde{\alpha}_i^H H_i(E(\tilde{y}))] \\ &\geq F(E(x)) + \sum_{i\in I_g} \alpha_i^g g_i(E(x)) + \sum_{i=1}^{p} \alpha_i^h h_i(E(x)) - \sum_{i=1}^{l}[\alpha_i^G G_i(E(x)) + \alpha_i^H H_i(E(x))]. \end{aligned}$$

Therefore, $(\tilde{y}, \tilde{\alpha})$ is a global optimal solution for WDMPEC. Moreover the corresponding objective values of MPEC and WDMPEC are equal. □

Now, we establish, the duality relation between the MPEC and the following Mond-Weir type dual.

$$\text{MWDMPEC} \qquad \max_{x,\alpha} F(E(x))$$

subject to:

$$0 = \nabla F(E(x)) + \sum_{i\in I_g} \alpha_i^g \nabla g_i(E(x)) + \sum_{i=1}^{p} \alpha_i^h \nabla h_i(E(x)) - \sum_{i=1}^{l}[\alpha_i^G \nabla G_i(E(x)) + \alpha_i^H \nabla H_i(E(x))], \tag{19}$$

$$\sum_{i\in I_g} \alpha_i^g g_i(E(x)) \geq 0, \quad \sum_{i=1}^{p} \alpha_i^h h_i(E(x)) \geq 0,$$

$$\sum_{i=1}^{l} \alpha_i^G G_i(E(x)) \leq 0, \quad \sum_{i=1}^{l} \alpha_i^H H_i(E(x)) \leq 0,$$

$$\alpha_{I_g}^g \geq 0, \quad \alpha_\kappa^G = 0, \quad \alpha_\delta^H = 0,$$

$$\forall i \in \beta, \text{ either } \alpha_i^G > 0, \quad \alpha_i^H > 0 \text{ or } \alpha_i^G \alpha_i^H = 0,$$

where, $\alpha = (\alpha^g, \alpha^h, \alpha^G, \alpha^H) \in \mathbb{R}^{k+p+2l}$.

Theorem 4.3. *(Weak Duality) Let $\tilde{y}$ be feasible for MPEC, (x, α) be feasible for MWDMPEC and the index sets $I_g, \delta, \beta, \kappa$ defined accordingly. Suppose that F, g_i $(i \in I_g)$, h_i $(i \in j^+)$, $-h_i$ $(i \in j^-)$, G_i $(i \in \delta^- \cup \beta_H^-)$, $-G_i$ $(i \in \delta^+ \cup \beta_H^+ \cup \beta^+)$, H_i $(i \in \kappa^- \cup \beta_G^-)$, $-H_i$ $(i \in \kappa^+ \cup \beta_G^+ \cup \beta^+)$ are E-convex functions at x. If $\delta^- \cup \kappa^- \cup \beta_G^- \cup \beta_H^- = \{\}$, then, for any y feasible for the MPEC, we have*

$$F(E(y)) \geq F(E(x)).$$

Proof. Let us consider that, y be any feasible point for MPEC. Then, we have

$$g_i(E(y)) \leq 0, \quad \forall i \in I_g,$$

and

$$h_i(E(y)) = 0, \quad i = 1, 2, \ldots, p.$$

Since, F is E-convex at x, we have

$$F(E(y)) - F(E(x)) \geq \langle \nabla F(E(x)), E(y) - E(x) \rangle. \tag{20}$$

Similarly, we have

$$g_i(E(y)) - g_i(E(x)) \geq \langle \nabla g_i(E(x)), E(y) - E(x) \rangle, \qquad \forall i \in I_g, \tag{21}$$

$$h_i(E(y)) - h_i(E(x)) \geq \langle \nabla h_i(E(x)), E(y) - E(x) \rangle, \qquad \forall i \in j^+, \tag{22}$$

$$-h_i(E(y)) + h_i(E(x)) \geq -\langle \nabla h_i(E(x)), E(y) - E(x) \rangle, \qquad \forall i \in j^-, \tag{23}$$

$$-G_i(E(y)) + G_i(E(x)) \geq -\langle \nabla G_i(E(x)), E(y) - E(x) \rangle, \quad \forall i \in \delta^+ \cup \beta_H^+ \cup \beta^+, \tag{24}$$

$$-H_i(E(y)) + H_i(E(x)) \geq -\langle \nabla H_i(E(x)), E(y) - E(x) \rangle, \quad \forall i \in \kappa^+ \cup \beta_G^+ \cup \beta^+. \tag{25}$$

If $\delta^- \cup \kappa^- \cup \beta_G^- \cup \beta_H^- = \{\}$, multiplying(21)-(25) by $\alpha_i^g \geq 0$ $(i \in I_g)$, $\alpha_i^h > 0$ $(i \in j^+)$, $-\alpha_i^h > 0$ $(i \in j^-)$, $\alpha_i^G > 0$ $(i \in \delta^+ \cup \beta_H^+ \cup \beta^+)$, $\alpha_i^H > 0$ $(i \in \kappa^+ \cup \beta_G^+ \cup \beta^+)$, respectively and adding (20)-(25), we obtain

$$\begin{aligned}
F(E(y)) - F(E(x)) &+ \sum_{i \in I_g} \alpha_i^g g_i(E(y)) - \sum_{i \in I_g} \alpha_i^g g_i(E(x)) + \sum_{i=1}^{p} \alpha_i^h h_i(E(y)) - \sum_{i=1}^{p} \alpha_i^h h_i(E(x)) \\
&- \sum_{i=1}^{l} \alpha_i^G G_i(E(y)) + \sum_{i=1}^{l} \alpha_i^G G_i(E(x)) - \sum_{i=1}^{l} \alpha_i^H H_i(E(y)) + \sum_{i=1}^{l} \alpha_i^H H_i(E(x)) \\
&\geq \Bigg\langle \nabla F(E(x)) + \sum_{i \in I_g} \alpha_i^g \nabla g_i(E(x)) + \sum_{i=1}^{p} \alpha_i^h \nabla h_i(E(x)) \\
&- \sum_{i=1}^{l} [\alpha_i^G \nabla G_i(E(x)) + \alpha_i^H \nabla H_i(E(x))], E(y) - E(x) \Bigg\rangle.
\end{aligned}$$

Using (19), it follows that

$$\begin{aligned}
F(E(y)) - F(E(x)) &+ \sum_{i \in I_g} \alpha_i^g g_i(E(y)) - \sum_{i \in I_g} \alpha_i^g g_i(E(x)) \\
&+ \sum_{i=1}^{p} \alpha_i^h h_i(E(y)) - \sum_{i=1}^{p} \alpha_i^h h_i(E(x)) - \sum_{i=1}^{l} \alpha_i^G G_i(E(y)) \\
&+ \sum_{i=1}^{l} \alpha_i^G G_i(E(x)) - \sum_{i=1}^{l} \alpha_i^H H_i(E(y)) + \sum_{i=1}^{l} \alpha_i^H H_i(E(x)) \geq 0.
\end{aligned}$$

Using the feasibility of y and x for MPEC and MWDMPEC, respectively, we obtain

$$F(E(y)) \geq F(E(x)),$$

and the proof is complete. □

Theorem 4.4. *(Strong Duality) If $\tilde{y}$ is a global optimal solution of MPEC, such that the ENNAMCQ is satisfied at $\tilde{y}$ and index sets $I_g, \delta, \beta, \kappa$ defined accordingly. Let F, g_i $(i \in I_g)$, h_i $(i \in j^+)$, $-h_i$ $(i \in j^-)$, G_i $(i \in \delta^- \cup \beta_H^-)$, $-G_i$ $(i \in \delta^+ \cup \beta_H^+ \cup \beta^+)$, H_i $(i \in \kappa^- \cup \beta_G^-)$ and $-H_i$ $(i \in \kappa^+ \cup \beta_G^+ \cup \beta^+)$ fulfill the assumption of the Theorem 4.3. Then, there exists $\tilde{\alpha}$, such that $(\tilde{y}, \tilde{\alpha})$ is a global optimal solution of MWDMPEC and corresponding objective values of MPEC and MWDMPEC are equal.*

Proof. The proof is similar to the proof of Theorem 4.2. □

5. Conclusions

We have shown that E-M-stationary condition is sufficient for global optimality under some MPEC and E-convexity assumptions. We have also formulated the Wolfe type and Mond-Weir type dual models for the MPEC. Further, we established weak and strong duality theorems relating to the MPEC and two dual models.

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