FIXED POINT RESULTS VIA β-φ-CONTRACTIONS IN *JS*-METRIC SPACES

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In this paper, we introduce a new notion of nonlinear type contraction and utilize the same to prove some fixed point results for such type of contractions in JS-metric spaces [Fixed point theory and Applications 2015 (2015), 1-14]. Our newly established results generalize a host of the existing results including the classical results due to Banach, Ran and Reurings, Nieto and Rodríguez-López, Jleli and Samet and several others.

Keywords: Fixed point, JS-metric space, β-admissibility, φ-contraction.
MSC: 47H10, 54H25.

1. Introduction

The Banach contraction principle proved by S. Banach in his Ph.D. thesis [1] in 1922 continues to be the most inspiring result of metric fixed point theory. It asserts that every contraction mapping on a complete metric space admits a unique fixed point. In the last several decades there have been numerous generalizations of this fundamental result (see [2, 3, 8, 9, 19, 25] and references cited therein).

One very natural way of improving this seminal result is to enlarge the class of underlying spaces which has led to the introduction of several metrical spaces, namely rectangular metric space [13], generalized metric space [24], partial metric space [14, 15, 29], b-metric space [12, 20, 21], partial b-metric space [31], symmetric space [18], quasi metric space [27], quasi-partial metric space [28] and many more.

Another way of generalizing the Banach fixed point theorem is to enlarge the class of contraction mappings which has led to the introduction of a multitude of contractive conditions. An exhaustive demonstration of the same is available in [26]. For further works of this kind, one can see [2, 3, 5, 7–10, 17, 19, 22, 30].

In 2015, Jleli and Samet [6] introduced a new class of spaces, namely generalized metric space (often referred as JS-metric space) and extended the Banach contraction principle to such spaces. In this paper, we introduce the notion of β-φ-contraction on JS-metric spaces and obtain fixed point results for this kind of contractions. In doing so, we were essentially motivated by Matkowski, who obtained a similar result in his noted paper [2] for nonlinear type contraction mappings in metric spaces. These types of contractions are often patterned after the work of Boyd and Wong [3] and Browder [4].

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2. **Preliminaries**

In this section, we collect some notions that serve as background materials for our main results. To make our presentation possibly self-contained we recall the definition of JS-metric spaces and the analogous notions of convergence, Cauchy sequence and completeness in such spaces.

Throughout the manuscript $\mathbb{N}, \mathbb{R}$, and $\mathbb{R}^+$ denote the set of natural numbers, real numbers and non-negative real numbers respectively.

Let X be a nonempty set and $\mathcal{D} : X \times X \to [0, \infty]$ be a given map. For each $x \in X$, the set $C(\mathcal{D}, X, x)$ is defined as

$$C(\mathcal{D}, X, x) = \{(x_n) \mid \lim_{n\to\infty} \mathcal{D}(x_n, x) = 0\}.$$

Definition 2.1. *[6] Let X be a nonempty set and $\mathcal{D} : X \times X \to [0, \infty]$ be a given map. $\mathcal{D}$ is said to be a JS-metric on X if it satisfies the following conditions:*
(i) for every $x, y \in X$, $\mathcal{D}(x, y) = 0$ implies $x = y$;
(ii) for every $x, y \in X$, $\mathcal{D}(x, y) = \mathcal{D}(y, x)$;
(iii) there exists $k > 0$ such that

$$\mathcal{D}(x, y) \le k \limsup_{n\to\infty} \mathcal{D}(x_n, y),$$

for all $x, y \in X$ and $(x_n) \in C(\mathcal{D}, X, x)$.
The pair $(X, \mathcal{D})$ is said to be a JS-metric space if the above conditions hold.

The classes of metric spaces, b-metric spaces, dislocated metric spaces are well known examples of JS-metric spaces.

The notions of convergence, Cauchy sequence and completeness in JS-metric spaces are as follows.

Definition 2.2. *[6] A sequence (x_n) in $(X, \mathcal{D})$ is said to be $\mathcal{D}$-convergent to $x \in X$ if*

$$(x_n) \in C(\mathcal{D}, X, x).$$

Proposition 2.1. *[6] Limit of a $\mathcal{D}$-convergent sequence in a JS-metric space is unique.*

Definition 2.3. *[6] A sequence (x_n) in a JS-metric space $(X, \mathcal{D})$ is said to be $\mathcal{D}$-Cauchy if*

$$\lim_{m,n\to\infty} \mathcal{D}(x_n, x_m) = 0.$$

Definition 2.4. *[6] A JS-metric space is said to be complete if every $\mathcal{D}$-Cauchy sequence in X is $\mathcal{D}$-convergent.*

Now, we present the notion of comparison functions fulfilling all the assumptions of Matkowski [2].

Definition 2.5. *A monotone increasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a comparison function if*

$$\lim_{n\to\infty} \varphi^n(t) = 0,$$

for all $t \ge 0$. We denote the collection of all comparison functions by Φ.

The following lemmas will be used in the sequel.

Lemma 2.1. *For any $\varphi \in \Phi$, $\varphi(t) < t \ \forall t > 0$.*

Proof. Assume on contrary to the statement that there exist some $s > 0$ such that

$$s \le \varphi(s).$$

Using monotonicity of φ, we get

$$\varphi(s) \le \varphi^2(s).$$

Similarly, we get

$$\begin{aligned}\varphi^2(s) &\leq \varphi^3(s) \\ &\leq \varphi^4(s) \\ &\vdots \\ &\leq \varphi^n(s) \\ &\leq \varphi^{n+1}(s) \\ &\vdots\end{aligned}$$

Thus we have the following:

$$\begin{aligned}s \leq \varphi(s) \leq \varphi^2(s) &\leq \varphi^3(s) \\ &\vdots \\ &\leq \varphi^n(s) \\ &\to 0 \text{ as } n \to \infty.\end{aligned}$$

which contradicts the fact that $s > 0$. Therefore, our assumption that $s \leq \varphi(s)$ is wrong. Hence $\varphi(t) < t \ \forall t > 0$. □

Lemma 2.2. *For any comparison function φ, $\varphi(0) = 0$.*

Proof. Suppose on contrary to the statement that

$$\varphi(0) = \epsilon > 0.$$

Consider $\delta = \frac{\epsilon}{2}$. Using the above Lemma 2.1 for $\delta(> 0)$, we get

$$\varphi(\delta) < \delta.$$

Now, using monotonicity of φ, we obtain

$$\begin{aligned}2\delta = \epsilon &= \varphi(0) \\ &\leq \varphi(\delta)(\text{as } 0 < \delta) \\ &< \delta,\end{aligned}$$

which is contradictory as $\delta > 0$. Therefore our assumption that $\varphi(0) > 0$ is wrong. This completes the proof. □

Lemma 2.3. *For any $\varphi \in \Phi$, $\lim_{n\to\infty} \varphi(t_n) = 0$ if $\lim_{n\to\infty} t_n = 0$.*

Proof. Let $\epsilon > 0$ be any arbitrary positive number. As $\{t_n\} \to 0$, there exists $K \in \mathbb{N}$ such that

$$|t_n| < \frac{\epsilon}{2} \ \forall n \geq K.$$

Using the above Lemma 2.1, we obtain

$$\begin{aligned}|\varphi(t_n)| &= \varphi(t_n) \\ &< t_n \text{ (by Lemma 2.1)} \\ &< \frac{\epsilon}{2} < \epsilon,\end{aligned}$$

for all $n \geq K$. Thus we see that for any arbitrary positive number $\epsilon > 0$, there exists $K \in \mathbb{N}$ such that $|\varphi(t_n)| < \epsilon \ \forall n \geq K$. Therefore $\lim_{n\to\infty} \varphi(t_n) = 0$. This completes the proof. □

Definition 2.6. *[11] Let ψ denotes a monotonic increasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following condition:*

$$\sum_{n=0}^{\infty} \psi^n(t) < \infty,$$

for all $t > 0$. The collection of all ψ's satisfying the above property is denoted as Ψ.

Remark 2.1. *Observe that the class Ψ of the above described functions is a sub-class of the class of comparison functions Φ.*

The following Lemma can be found in [16].

Lemma 2.4. *Let $\psi \in \Psi$. Then*
(*a*) *$\psi(t) < t$ for all $t > 0$,*
(*b*) *$\psi(0) = 0$,*
(*c*) *ψ is continuous at 0.*

3. Main Results

Definition 3.1. *[23] Let f be a self mapping on X and $\alpha : X \times X \to [0, \infty)$. The mapping f is said to be triangular α-admissible if the following conditions hold:*
(i) $\alpha(x, y) \geq 1 \implies \alpha(fx, fy) \geq 1$;
(ii) $\alpha(x, y) \geq 1, \alpha(y, z) \geq 1 \implies \alpha(x, z) \geq 1$;
for all $x, y, z \in X$.

From here onwards, we use the term β-admissibility to represent triangular α-admissibility. Before we proceed to the main results, we introduce the notion of β-φ-contraction as follows:

Definition 3.2. *Let f be a self mapping on a JS-metric space $(X, \mathcal{D})$ such that*

$$\mathcal{D}(fx, fy) = \infty \implies \mathcal{D}(x, y) = \infty.$$

Then, the function f is said to be a β-φ contraction on X if there exist some $\beta : X \times X \to [0, \infty)$ and $\varphi \in \Phi$ such that

$$\mathcal{D}(x, y) < \infty \implies \beta(x, y)\mathcal{D}(fx, fy) \leq \varphi\mathcal{D}(x, y), \tag{1}$$

for all $x, y \in X$.

Remark 3.1. *By choosing $\beta(x, y) = 1$ for all $x, y \in X$ and $\varphi(t) = kt$ $\forall t \geq 0$, the Definition 3.2 deduces the Definition of k-contraction introduced in [6].*

For every $x \in X$, we define $\delta(\mathcal{D}, f, x)$ as the following

$$\delta(\mathcal{D}, f, x) = \sup\{D(f^i x, f^j x) | i, j \in \mathbb{N}\}.$$

Theorem 3.1. *Let $(X, \mathcal{D})$ be a complete JS-metric space and f be a β-φ-contraction mapping satisfying the following conditions:*
(i) f is β-admissible;
(ii) there exists $x_0 \in X$ such that $\beta(x_0, fx_0) \geq 1$ and $\delta(\mathcal{D}, f, x_0) < \infty$;
(iii) f is continuous.
Then f possesses a fixed point in X.

Proof. Given that there exists $x_0 \in X$ such that

$$\beta(x_0, fx_0) \geq 1 \text{ and } \delta(\mathcal{D}, f, x_0)(= k \geq 0) < \infty.$$

Define the Picard sequence (x_n) based at x_0, *i.e.*,

$$x_n = f^n x_0 \ \forall n \in \mathbb{N}.$$

Case I: If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then, x_n is a fixed point of f.
Case II: $x_n \neq x_{n+1}$ $\forall n \in \mathbb{N}$. As f is β-admissible and $\beta(x_0, x_1) = \beta(x_0, fx_0) \geq 1$, we have

$$\beta(x_1, x_2) = \beta(fx_0, fx_1) \geq 1.$$

By induction, we conclude

$$\beta(x_n, x_{n+1}) \geq 1 \ \forall n \in \mathbb{N}.$$

Now, using the β-φ-contraction assumption on f, we get

$$\begin{aligned} \mathcal{D}(x_n, x_{n+1}) &= \mathcal{D}(fx_{n-1}, fx_n) \\ &\leq \beta(x_{n-1}, x_n)\mathcal{D}(fx_{n-1}, fx_n) \\ &\leq \varphi\mathcal{D}(x_{n-1}, x_n). \end{aligned}$$

Using the second condition of β-admissibility, we have

$$\beta(x_m, x_n) \geq 1 \ \forall m, n \in \mathbb{N} : m < n.$$

Hence for $i < j$, we have

$$\begin{aligned} \mathcal{D}(f^{n+i}x_0, f^{n+j}x_0) &\leq \beta(f^{n+i-1}x_0, f^{n+j-1}x_0)\mathcal{D}(ff^{n+i-1}x_0, ff^{n+j-1}x_0) \\ &\leq \varphi\mathcal{D}(f^{n+i-1}x_0, f^{n+j-1}x_0) \\ &\vdots \\ &\leq \varphi^n\mathcal{D}(f^i x_0, f^j x_0). \end{aligned}$$

Now, as $\delta(\mathcal{D}, f, x_0) = k(\geq 0) < \infty$ and $\mathcal{D}(f^i x_0, f^j x_0) \leq \delta(\mathcal{D}, f, x_0)$, we have

$$\begin{aligned} \varphi^n\mathcal{D}(f^i x_0, f^j x_0) &\leq \varphi^n k \ (\varphi, \text{ and hence } \varphi^n \text{ being monotonic increasing}) \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Hence

$$\begin{aligned} \mathcal{D}(f^{n+i}x_0, f^{n+j}x_0) &\leq \varphi^n\mathcal{D}(f^i x_0, f^j x_0) \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Hence, the sequence $\{f^n x_0\}_{n\in\mathbb{N}}$ is a $\mathcal{D}$-Cauchy sequence. As $(X, \mathcal{D})$ is JS-complete, there exists some $x \in X$ such that $f^n x_0$ $\mathcal{D}$-converges to x.

The function f being continuous, we have

$$f^{n+1}x_0 = f(f^n x_0) \xrightarrow{\mathcal{D}} fx \text{ as } n \to \infty.$$

Owing to uniqueness of limit in JS-metric spaces, we conclude $fx = x$, *i.e.*, x is a fixed point of f. □

In the next theorem, we replace the continuity assmption (iii) in Theorem 3.1 with another assumption.

Theorem 3.2. *Let $(X, \mathcal{D})$ be a complete JS-metric space and f be a β-φ-contraction mapping satisfying the following conditions:*
(i) f is β-admissible;
(ii) there exists $x_0 \in X$ such that $\beta(x_0, fx_0) \geq 1$ and $\delta(\mathcal{D}, f, x_0) < \infty$;
(iii)′ if (x_n) is a sequence in X converging to $x \in X$ and $\beta(x_n, x_{n+1}) \geq 1$ $\forall n \in \mathbb{N}$, then $\beta(x_n, x) \geq 1$ $\forall n \in \mathbb{N}$.
Then f possesses a unique fixed point in X.

Proof. Proceeding on the lines of the earlier Theorem 3.1, we obtain that $\{f^n x_0\}$ is a $\mathcal{D}$-Cauchy sequence $\mathcal{D}$-converging to some $x \in X$. From hypothesis $(iii)'$, and the β-φ-contraction condition of f, we have

$$\begin{aligned} \mathcal{D}(x_{n+1}, fx) &\leq \beta(x_n, x)\mathcal{D}(x_{n+1}, fx) \\ &= \beta(x_n, x)\mathcal{D}(fx_n, fx) \\ &\leq \varphi\mathcal{D}(x_n, x). \end{aligned}$$

Now, since $\mathcal{D}(x_n, x) \to 0$ as $n \to \infty$, using Lemma 2.3 we obtain

$$\varphi\mathcal{D}(x_n, x) \to 0 \text{ as } n \to \infty.$$

Thus we see that $x_{n+1} = f^{n+1}x_0 \to fx$. Hence from uniqueness of limit in JS-metric space, we conclude that $fx = x$, *i.e.*, x is a fixed point of f in X. □

For uniqueness of the fixed point of the mapping f, we assume the following hypothesis.
(H): For all $x, y \in X$, there exists $z \in X$ such that $\beta(x, z) \geq 1$ and $\beta(y, z) \geq 1$.

Theorem 3.3. *If in addition to hypotheses of the earlier Theorem 3.1 (or Theorem 3.2), the above assumption (H) holds, then f possesses a unique fixed point in X.*

Proof. Existence of fixed point is already guaranteed by the earlier Theorem 3.1 (or Theorem 3.2). For uniqueness, let us assume that x and y be two fixed points of f. From hypothesis (H), there is $z \in X$ such that

$$\beta(x, z) \geq 1 \text{ and } \beta(y, z) \geq 1.$$

As f is β-admissible and x, y are fixed points of f, we get

$$\beta(x, f^n z) = \beta(f^n x, f^n z) \geq 1$$

and

$$\beta(y, f^n z) = \beta(f^n y, f^n z) \geq 1, \text{ for all } n \in \mathbb{N},$$

implying thereby

$$\begin{aligned} \mathcal{D}(x, f^n z) &= \mathcal{D}(fx, f(f^{n-1}z)) \\ &\leq \beta(x, f^{n-1}z)\mathcal{D}(fx, f(f^{n-1}z)) \\ &\leq \varphi\mathcal{D}(x, f^{n-1}z) \\ &\leq \varphi^2\mathcal{D}(x, f^{n-2}z) \\ &\vdots \\ &\leq \varphi^n\mathcal{D}(x, z) \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Thus we have $f^n z \to x$.

Similarly, we can show that $f^n z \to y$. From uniqueness of limit in JS-metric spaces we conclude $x = y$, *i.e.*, the fixed point of f in X is unique. □

Remark 3.2. *The conclusions of Theorems 3.1, 3.2 and 3.3 remain true if the class (of functions) Φ in the condition* (1) *is replaced by the class (of functions) Ψ.*

Example 3.1. *Consider the set $X = \mathbb{R}^+$ of nonnegative real numbers endowed with the JS-metric given as follows:*

$$\mathcal{D}(x, y) = \begin{cases} x + y, & \text{if } xy = 0, \\ \frac{x+y}{2}, & \text{otherwise.} \end{cases}$$

Define the mapping $f : X \to X$ *defined as*

$$f(x) = \begin{cases} \frac{x}{4}, & \text{if } 0 \leq x \leq 1, \\ 2x - \frac{7}{4}, & \text{otherwise.} \end{cases}$$

Also, consider the mapping $\beta : X \times X \to [0, \infty)$ *defined as*

$$\beta(x, y) = \begin{cases} 1, & \text{if } x, y \in [0, 1), \\ 0, & \text{otherwise.} \end{cases}$$

We can see that there exists $\frac{1}{3} \in X$ *such that* $\beta(\frac{1}{3}, f\frac{1}{3}) = \beta(\frac{1}{3}, \frac{1}{12}) = 1$ *and*

$$\begin{aligned} \delta(\mathcal{D}, f, \frac{1}{3}) &= \sup\{\mathcal{D}\big(f^i(\frac{1}{3}), f^j(\frac{1}{3})\big)|i, j \in \mathbb{N}\} \\ &= \sup\big\{\mathcal{D}(\frac{1}{3.4^i}, \frac{1}{3.4^j})|i, j \in \mathbb{N}\} \\ &= \frac{\frac{1}{12} + \frac{1}{12}}{2} = \frac{1}{12} < \infty. \end{aligned}$$

Also, f *is* β*-admissible as*

$$\beta(x, y) \geq 1 \implies x, y \in [0, 1) \implies fx, fy \in [0, 1) \implies \beta(fx, fy) \geq 1,$$

and

$$\beta(x, y) \geq 1, \beta(y, z) \geq 1 \implies x, y, z \in [0, 1) \implies fx, fz \in [0, 1) \implies \beta(fx, fz) \geq 1.$$

We will show that f *is a* β*-*φ*-contraction for* $\varphi(t) = \frac{2t}{3}$ *and* β *as defined above. We consider the following cases.*

Case I: For $x = y = 0$,

$$\mathcal{D}(fx, fy) = \mathcal{D}(0, 0) = 0 + 0 = 0 \text{ and } \beta(0, 0) = 1$$

So,

$$1.0 = 0 \leq \varphi(0) = 0,$$

and hence the contraction condition holds.

Case II: For $x = 0, y \in (0, 1)$ *we have*

$$\mathcal{D}(0, y) = 0 + y = y; \mathcal{D}(f0, fy) = \mathcal{D}(0, \frac{y}{4}) = 0 + \frac{y}{4} = \frac{y}{4} \text{ and } \beta(0, y) = 1,$$

and thus

$$1.\frac{y}{4} = \frac{y}{4} \leq \frac{y}{3} = \varphi(\frac{y}{2}),$$

i.e.,

$$\beta(0, y)\mathcal{D}(f0, fy) \leq \varphi\mathcal{D}(0, y).$$

Similarly, for $x \in (0, 1), y = 0$ *the contraction condition holds.*

Case III: When $x, y \in (0, 1)$

$$\beta(x, y) = 1, \ \mathcal{D}(x, y) = \frac{x+y}{2}$$

$$\mathcal{D}(fx, fy) = \mathcal{D}(\frac{x}{4}, \frac{y}{4}) = \frac{\frac{x}{4} + \frac{y}{4}}{2} = \frac{x+y}{8}.$$

and thus

$$\begin{aligned} 1.\frac{x+y}{8} &= \frac{x+y}{8} \\ &\leq \frac{x+y}{3} \\ &= \varphi(\frac{x+y}{2}), \end{aligned}$$

implying thereby

$$\beta(x, y)\mathcal{D}(fx, fy) \leq \varphi\mathcal{D}(x, y).$$

Finally, in case of $x \geq 1$ or $y \geq 1$, $\beta(x,y) = 0$ and hence the contraction condition holds trivially. Also, we can see that the function f is continuous on $\mathbb{R}^+$.

Thus all the hypotheses of our Theorem 3.1 hold. Therefore, f possesses a fixed point in $\mathbb{R}^+$ (We notice that $f(0) = 0$).

4. Consequences

With a view to highlight the realized improvements, we utilize our newly proved results to deduce some well known results of the existing literature.

Firstly, we derive the Banach contraction principle [1] following the lines of one of our main results (Theorem 3.1).

Theorem 4.1. *Let (X,d) be a complete metric space and $f : X \to X$ be a mapping satisfying*

$$d(fx, fy) \leq \rho d(x,y)\ \forall x, y \in X,$$

where $0 \leq \rho < 1$. Then f possesses a unique fixed point in X.

Proof. Define $\beta : X \times X \to [0,\infty)$ as $\beta(x,y) = 1$ for all $x, y \in X$. Take $\varphi(t) = \rho t$. Also for any $x_0 \in X$, we have

$$\begin{aligned} d(f^i x_0, f^j x_0) &\leq d(f^i x_0, x_0) + d(x_0, f^j x_0) \\ &= d(x_0, f^i x_0) + d(x_0, f^j x_0). \end{aligned} \tag{2}$$

Now,

$$\begin{aligned} d(f^n x_0, f^{n+1} x_0) &= d\big(f(f^{n-1}x_0), f(f^n x_0)\big) \\ &\leq \rho d(f^{n-1}x_0, f^n x_0) \\ &\leq \rho^2 d(f^{n-2}x_0, f^{n-1}x_0) \\ &\leq \rho^3 d(f^{n-3}x_0, f^{n-2}x_0) \\ &\ \ \vdots \\ &\leq \rho^n d(x_0, fx_0). \end{aligned}$$

Therefore,

$$\begin{aligned} d(x_0, f^k x_0) &\leq d(x_0, fx_0) + d(fx_0, f^2x_0) + d(f^2x_0, f^3x_0) + \ldots + d(f^{k-1}x_0, f^k x_0) \\ &\leq d(x_0, fx_0)(1 + \rho + \rho^2 + \rho^3 + \rho^4 + \rho^5 + \ldots + \rho^{k-1}) \\ &< d(x_0, fx_0)\frac{1}{1-\rho} \text{ for any } k \in \mathbb{N}, \end{aligned}$$

and hence using the above inequality 2, we obtain

$$d(f^i x_0, f^j x_0) < \frac{2}{1-\rho} d(x_0, fx_0) < \infty,$$

i.e., for any $x_0 \in X$, we have $\beta(x_0, fx_0) \geq 1$ and $\delta(d, f, x_0) < \infty$. Also, we know that any contraction function is (uniformly) continuous. Thus we see that all the hypotheses of our Theorem 3.1 are satisfied. Hence f has a fixed point in X.

For uniqueness of fixed point, we see that for any two fixed points x, y; there is $z = x$(or y) such that $\beta(x,z) \geq 1$ and $\beta(y,z) \geq 1$. Thus Theorem 3.3 guarantees uniqueness of fixed point. □

The following results, namely: Theorem 4.2 and Theorem 4.3 are consequences (in metric spaces) of Theorem 3.1 and Theorem 3.2 respectively which can be viewed as variants of relevant fixed point theorems contained in Samet et al. [11].

Theorem 4.2. *Consider a complete metric space (X,d) and a self mapping f on X satisfying the conditions below:*

(i) f *is* β-φ *contractive;*
(ii) f *is* β*-admissible;*
(iii) there exists $x_0 \in X$ *such that* $\beta(x_0, fx_0) \geq 1$*;*
(iv) f *is continuous;*
(v) for all $x, y \in X$*, there exists* $z \in X$ *such that* $\beta(x, z) \geq 1$ *and* $\beta(y, z) \geq 1$*.*
Then, f *has a unique fixed point in* X*.*

Proof. Every metric space being a JS-metric space, the conditions (i) and (iii) of Theorem 3.1 hold trivially. The key to prove this theorem is to show that $\delta(d, f, x_0) < \infty$. For that, we can use the exactly same technique as the one used in the earlier Theorem 4.1. Thus, all the hypotheses of our Theorems 3.1 and 3.3 are satisfied. Hence f possesses a unique fixed point in X. □

Theorem 4.3. *Consider a complete metric space* (X, d) *and a self mapping* f *on* X *which satisfies the conditions below:*
(i) f *is* β-φ *contractive;*
(ii) f *is* β*-admissible;*
(iii) there exists $x_0 \in X$ *such that* $\beta(x_0, fx_0) \geq 1$*;*
(iv)′ for each sequence (x_n) *in* X *converging to* x *and* $\beta(x_n, x_{n+1}) \geq 1$ $\forall n \in \mathbb{N}$*;* $\beta(x_n, x) \geq 1$ $\forall n \in \mathbb{N}$*);*
(v) for all $x, y \in X$*, there exists* $z \in X$ *such that* $\beta(x, z) \geq 1$ *and* $\beta(y, z) \geq 1$*.*
Then, f *has a unique fixed point in* X*.*

Proof. To prove this result, we use the condition $(iv)'$ instead of condition (iv) of the above Theorem 4.2. Observe that the condition $(iv)'$ here implies the condition $(iii)'$ of Theorem 3.2. Thus, all the hypotheses of Theorems 3.2 and 3.3 hold. Therefore, f admits a unique fixed point in X. □

The result of Ran and Reurings [9] can also be derived following the lines of the proof to Theorem 3.1.

Theorem 4.4. *Let* $(X, \preceq)$ *be a partially ordered complete metric space under the metric* d *and* T *a self mapping on* X *such that the following holds:*
(i) $d(Tx, Ty) \leq \rho d(x, y)$ $\forall x, y \in X$ *such that* $x \preceq y$*, for some* $\rho \in [0, 1)$*;*
(ii) there is some $x_0 \in X$ *such that* $x_0 \preceq Tx_0$*;*
(iii) T *is continuous and nondecreasing (i.e.,* $x \preceq y \implies Tx \preceq Ty$*).*
Then T *possesses a fixed point in* X*.*

Proof. Being a metric space, X is also a JS-metric space. The existence of some $x_0 \in X$ such that $\delta(d, f, x_0) < \infty$ can be shown in a similar fashion to the methods used in Theorem 4.1.

Now, we define $\beta : X \times X \to [0, \infty)$ and $\varphi : [0, \infty) \to [0, \infty)$ as follows:

$$\beta(x, y) = \begin{cases} 1, & \text{if } x \preceq y \\ 0, & \text{otherwise,} \end{cases} \quad \text{and } \varphi(t) = \rho(t) \ \forall t \geq 0.$$

From condition (i) and the definition of β, we have

$$\beta(x, y)d(Tx, Ty) \leq \rho d(x, y) \ \forall x, y \in X,$$

i.e., $T : X \to X$ is a β-φ-contraction on X for $\varphi(t) = \rho(t)$. The only thing that remains to be shown is β-admissibility of T. For that we consider $x, y \in X$ such that $\beta(x, y) \geq 1$.

From the definition of β, $\beta(x, y) \geq 1$ implies $x \preceq y$. The self mapping T being nondecreasing, we have $Tx \preceq Ty$ and hence $\beta(Tx, Ty) \geq 1$. Thus the first condition for β-admissibility holds.

Now, to show that the second condition holds we consider $x, y, z \in X$ such that $\beta(x,y) \geq 1$, $\beta(y,z) \geq 1$. From the definition of β, $\beta(x,y) \geq 1$ implies $x \preceq y$ and $\beta(y,z) \geq 1$ implies $y \preceq z$. Again, using the nondecreasing property of T, we get $Tx \preceq Ty$ and $Ty \preceq Tz$.

Now, we use the fact that '$\preceq$' is a partial order on X. Using the transitivity property of '$\preceq$', we have $Tx \preceq Tz$; as $Tx \preceq Ty$ and $Ty \preceq Tz$. Thus we get $\beta(Tx, Tz) \geq 1$ and hence the second condition for β-admissibility holds. Therefore, the condition (i) of our main result 3.1 holds.

Thus we have shown that X is a JS-metric space and T is a β-φ contraction on X satisfying all the conditions of Theorem 3.1. Hence T admits a fixed point. □

Now we show that the fixed point result due to Nieto and Rodrígez-López [10] can also be deduced from Theorem 3.2.

Theorem 4.5. *Let $(X, \preceq)$ be a partially ordered complete metric space under the metric d and T a self mapping on X such that the following holds:*

(i) There exists $\rho \in [0,1)$ such that $d(Tx, Ty) \leq \rho d(x,y)$ $\forall x, y \in X$ with $x \preceq y$;

(ii) there is some $x_0 \in X$ such that $x_0 \preceq Tx_0$;

(iii) T is nondecreasing w.r.t. '$\preceq$', i.e., $x \preceq y \implies Tx \preceq Ty$;

(iv) if (x_n) is a nondecreasing sequence (i.e. $x_n \preceq x_{n+1}$ $\forall n \in \mathbb{N}$) in X converging to x then $x_n \preceq x$ $\forall n \in \mathbb{N}$.

Then T admits a fixed point in X.

Proof. The proof of this theorem is identical to the proof of the Theorem 4.4. Unlike Theorem 4.4, we use the condition (iv) instead of the continuity of the involved map. Observe that this result is a consequence of Theorem 3.2 but not of Theorem 3.1.

It is worth mentioning here that the given condition (iv) implies condition $(iii)'$ of Theorem 3.2 provided β is identical to the one considered in Theorem 4.4. □

Now, we show that the main result contained in Jleli and Samet [6] can also be deduced from our results.

Theorem 4.6. *Let $(X, \mathcal{D})$ be a complete JS-metric space and f a self mapping on X satisfying the following conditions:*

(i) $\mathcal{D}\big(f(x), f(y)\big) \leq \rho \mathcal{D}(x,y)$ for every $(x,y) \in X \times X$ for some $\rho \in (0,1)$;

(ii) there exists $x_0 \in X$ such that $\delta(\mathcal{D}, f, x_0) < \infty$;

Then $f^n x_0$ converges to a fixed point of f, say w in X. In addition, if there is another fixed point w' of f such that $\mathcal{D}(w, w') < \infty$, then $w = w'$.

Proof. This result is a direct consequence of our result 3.1 (or 3.2) if we consider β and φ as follows:

$\beta : X \times X \to [0, \infty)$ such that $\beta(x,y) = 1$ $\forall x, y \in X$ and $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(t) = \rho t$ $\forall t > 0$.

Proceeding on the lines of our main results (3.1 or 3.2) we have: there exists some $w \in X$ such that $f^n x_0 \to w$. Now we use the given condition (i) to obtain the following

$$\begin{aligned} \mathcal{D}(f^{n+1}x_0, fw) &= \beta(f^n x_0, w)\mathcal{D}(ff^n x_0, fw) \\ &\leq \rho \mathcal{D}(f^n x_0, w) \\ &\to 0 \text{ as } n \to \infty, \end{aligned}$$

i.e., $f^{n+1}x_0$ $\mathcal{D}$-converges to fw. Hence from uniqueness of limit in JS-metric space, we conclude that $fw = w$, *i.e.*, w is a fixed point of f in X.

For uniqueness part, let w' be another fixed point such that $\mathcal{D}(w, w') < \infty$. using condition (i), we have

$$\mathcal{D}(w, w') = \mathcal{D}(fw, fw') \leq \rho \mathcal{D}(w, w').$$

As $\mathcal{D}(w, w') < \infty$ and $\rho \in (0, 1)$ the above inequality holds only when $\mathcal{D}(w, w') = 0$, *i.e.*, $w = w'$. Hence, fixed point of f is unique in X. This completes the proof. □

5. Conclusion

The existing literature of metric fixed point theory already contains numerous results on fixed points in JS-metric spaces. In this paper, we have considered a new type of nonlinear contraction in JS-metric spaces under the name 'β-ϕ contraction' and utilized the same to obtain fixed point results in such spaces. In the process, several classical results, e.g., the results due to Banach [1], Ran and Reurings [9], Nieto and Rodriguez Lopez [10] etc. have been generalized. Some other variants of various well known fixed point results, with similar assumptions as in Theorems 4.1-4.6, in spaces like b-metric spaces, dislocated metric spaces, modular spaces etc. can be derived as corollaries to our Theorems 3.1 and 3.2. For the sake of brievity we avoid the details.

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