EXISTENCE RESULTS FOR AN IMPULSIVE PANTOGRAPH DIFFERENTIAL EQUATIONS WITHIN EXPONENTIAL KERNEL

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This manuscript deals with the existence results for an impulsive pantograph integro-differential equations (IPIDE) through Caputo-Fabrizio (CF) operator. Certain novel existence findings are shown using fixed point approaches. Finally, two numerical examples are provided in the work to demonstrate the application of our theoretical findings.

Keywords: Fractional-order, Caputo-Fabrizio operator (CFO), Pantograph equation, Existence and uniqueness, fixed point.

MSC2020: 26A33, 34A37.

1. Introduction

Fractional differential equations (FDEs) have gotten a lot of attention in recent decades from academics in physics, biology, chemistry, and other disciplines of science and engineering [6, 8, 16, 29, 43, 48]. Because of the wide range of applications, this subject has been researched using Riemman-Liouvile, Caputo, Hilfer, Atangana-Baleanu-Caputo and Hadamard type fractional derivatives [7, 18, 19, 23, 25–27, 37–40]. To extend the application of fractional calculus, a number of authors have proposed a new class of fractional derivatives with distinct kernels. The most popular definitions offered by Riemann-Liouville and the Caputo version have the flaw of having a singular kernel [13]. A recent unique concept of fractional derivative was proposed by Caputo and Fabrizio [21]. They excluded the singular kernel from their definition. The authors of [35] investigated the characteristics of the new notion using the information provided by [21]. In the recent two years, several results on the new CFO have been established. Numerous mathematicians have contributed to the development of CFO, as evidenced by the publications [2–5, 9–12, 20, 22, 30, 35, 44] and references thereto.

In several areas of science and technology, including physics, chemical technology, population dynamics, and the natural sciences, evolution processes can abruptly change state or be susceptible to short-term perturbations. In order to characterize these sorts of perturbations, a new class of differential equations termed impulsive differential equations (IDEs) was developed. The books [14, 17, 31] provide an insight to the notion of IDEs.

In the 1960s, British Railways sought to increase the speed of the electric locomotive. A significant invention was the pantograph, which captures electricity from an above line. J. R. Ockendon and A. B. Tayler examined the pantograph head motion on an electric locomotive in [41]. In the

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process of solving this issue, they encountered a peculiar delay differential equation of the form

$$\begin{aligned} p'(\varsigma) &= cp(\varsigma) + dp(\mu\varsigma), \quad \varsigma > 0, \\ p(0) &= p_0, \end{aligned}$$

where $0 < \mu < 1$. This specific type of delay differential equation was referred to as a pantograph equation at the time that the article was published in 1971. The pantograph equation is useful in a variety of domains, including electrodynamics and biology (for example, see [36]). Many researchers [32–34] have investigated both the properties of the analytic solution to this problem and numerical approaches. It is of interest to examine the fractional model of the pantograph equations due to its significance in several applicable disciplines. Recently, very few authors have concentrated on fractional pantograph equation [1, 15, 24, 28, 42, 45–47]. In particular, in [15], authors examined the existence of solutions of nonlinear FPE. Later, Vivek et al. [45, 46] analyzed the qualitative properties of PE via Hilfer fractional derivatives. Later, in [42], authors investigated the positive solution of FPDE with mixed axioms under suitable fixed point theorem. According to a survey of numerous recent studies, the topic of impulsive pantograph differential equations via CFO of the form (**1**.1)-(**1**.3) has not yet been addressed by anyone. This is the primary motive for this work.

In light of the foregoing, in the first part of this manuscript, we examine the existence and uniqueness findings for a class of IPDE through CFO of the form

$$({}^{CF}D^{\vartheta}_{\varsigma}p)(\varsigma) = f(\varsigma, p(\varsigma), p(\mu\varsigma)), \quad \varsigma \in [0,\xi], \quad \varsigma \neq \varsigma_q, \quad q = 1,2\ldots,\ell \qquad (\mathbf{1}.1)$$

$$\Delta p(\varsigma_q) = p(\varsigma_q^+) - p(\varsigma_q^-) = \Lambda_q(p(\varsigma_q)), \quad q = 1,2\ldots,\ell \qquad (\mathbf{1}.2)$$

$$p(0) = p_0, \qquad (\mathbf{1}.3)$$

where ${}^{CF}D^{\vartheta}_{\varsigma_q}$ is the CFFD of order $\vartheta \in (0,1)$, $0 < \mu < 1$, $\xi > 0$, $f : [0,\xi] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and ς_q satisfy $0 = \varsigma_0 < \varsigma_1 < \cdots < \varsigma_\ell < \varsigma_{\ell+1} = \xi$, $p(\varsigma_q^+) = \lim_{\varepsilon\to 0} p(\varsigma_q + \varepsilon)$ and $p(\varsigma_q^-) = \lim_{\varepsilon\to 0} p(\varsigma_q - \varepsilon)$ represents the right and left limits of $p(\varsigma)$ at $\varsigma = \varsigma_q$ respectively, $p_0 \in \mathbb{R}$. $\Lambda_q \in C(\mathbb{R},\mathbb{R})$ is a given function.

In the second part, we examine the existence and uniqueness findings for a class of IPIDE via CFO of the form

$$({}^{CF}D^{\vartheta}_{\varsigma_q}p)(\varsigma) = f\left(\varsigma, p(\varsigma), \int_0^{\varsigma} h(\varsigma,s)p(\mu s)ds\right), \quad \varsigma \in [0,\xi], \quad \varsigma \neq \varsigma_q, q = 1,2\ldots,\ell \qquad (\mathbf{1}.4)$$

with the conditions (**1**.2)-(**1**.3), where $f : [0,\xi] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and $h(\varsigma,s)$ is continuous for all $(\varsigma,s) \in [0,\xi] \times [0,\xi]$ and we can find a positive constant H in a way that $\max_{\varsigma,s\in[0,\xi]} \|h(\varsigma,s)\| = H$.

In general, we analyze the existence results of the model (**1**.1)-(**1**.4), when $p_0 \in \mathbb{X}$, $f : [0,\xi] \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$, $\Lambda_q \in C(\mathbb{X},\mathbb{X}), q = 1,2,\ldots,\ell$ are given functions and $\mathbb{X}$ is real or complex Banach space with a norm $\|\cdot\|$.

The significant result of the study can be summed up as follows:

1. This work investigates the existence of a $\mathscr{PC}$ solution to IPIDE through CFO for the system (**1**.1)-(**1**.4).
2. The fixed point theorems of Banach and Krasnoselskii are used to derive the primary insights. Finally, we will present a few examples to demonstrate how our primary conclusions might be used.
3. This is the first attempt, as far as we know, to handle the IPIDE with CFO for the system (**1**.1)-(**1**.4).

We will now proceed to a description of the work. Section 2 presents the concept of a piecewise continuous solution for our issue, as well as several notations and a review of some concepts and previous findings. The first result is based on the Banach contraction principle, while the second result is based on Krasnoselskii's fixed point theorem, which we give in Section 3. In the following section (Section 4), we will show how our key conclusions can be applied by presenting a few examples.

2. Preliminaries

This section discusses the CFO's basic definitions and findings, which will help us prove our main points.

The functions designated by the notation $L^1([0,\xi],\mathbb{X})$ that are integrable in the Bochner concept with reference to the Lebesgue measure and come furnished with the notation

$$\|p\|_{L^1} = \int_0^{\xi} \|p(x)\|dx$$

are referred to as $p:[0,\xi]\to\mathbb{X}$.

Definition 2.1. *[21]* The CFO integral order $0<\vartheta<1$ for a function $g\in L^1([0,\xi])$ described by

$$({}^{CF}I_0^{\vartheta}g)(\zeta) = \frac{2(1-\vartheta)}{N(\vartheta)(2-\vartheta)}g(\zeta) + \frac{2\vartheta}{N(\vartheta)(2-\vartheta)}\int_0^{\zeta} g(x)dx, \quad \zeta\geq 0, \tag{2.1}$$

where $N(\vartheta)$ is normalization constant depending on ϑ that satisfies the condition $N(0)=1$ and $N(1)=2$.

Remark 2.1. If we take $N(\vartheta)=\dfrac{2}{2-\vartheta}$, then (**2**.1) becomes

$$({}^{CF}I_0^{\vartheta}g)(\zeta) = (1-\vartheta)g(\zeta) + \vartheta\int_0^{\zeta} g(x)dx, \quad \zeta\geq 0.$$

Definition 2.2. *[21]* The CFO of the function $g\in AC([0,\xi])$ is described by

$$({}^{CF}D_0^{\vartheta}g)(\zeta) = \frac{(2-\vartheta)N(\vartheta)}{2(1-\vartheta)}\int_0^{\zeta} e^{\left(-\frac{\vartheta}{1-\vartheta}(\zeta-x)\right)} g'(x)dx, \vartheta\in(0,1), \zeta\in[0,\xi], \tag{2.2}$$

where the notation $AC([0,\xi]):[0,\xi]\to\mathbb{X}$ signifies the space of all functions that are absolutely continuous.

At this juncture, it should be pointed out that ${}^{CF}D_0^{\vartheta}g=0$ iff g is a constant function.

Remark 2.2. If we take $N(\vartheta)=\dfrac{2}{2-\vartheta}$, then (**2**.2) becomes

$$({}^{CF}D_0^{\vartheta}g)(\zeta) = \frac{1}{(1-\vartheta)}\int_0^{\zeta} e^{\left(-\frac{\vartheta}{1-\vartheta}(\zeta-x)\right)} g'(x)dx, \quad \zeta\in[0,\xi].$$

Lemma 2.1. *[35]* Let $g\in L^1([0,\xi])$. Then a function $p\in\mathscr{C}([0,\xi])$ is a solution of the following system

$$\begin{aligned} ({}^{CF}D_0^{\vartheta}p)(\varsigma) &= g(\varsigma), \quad \varsigma\in[0,\xi],\\ p(0) &= p_0, \end{aligned} \tag{2.3}$$

iff p fulfills the subsequent integral equation

$$p(\varsigma) = p_0 - \frac{2(1-\vartheta)}{(2-\vartheta)N(\vartheta)}g(0) + \frac{2(1-\vartheta)}{(2-\vartheta)N(\vartheta)}g(\varsigma) + \frac{2\vartheta}{(2-\vartheta)N(\vartheta)}\int_0^{\varsigma} g(s)ds. \tag{2.4}$$

Remark 2.3. *(i) The above Lemma* **2**.*1 is true only when* $g(0)=0$. *Then* (**2**.4) *becomes*

$$p(\varsigma) = p_0 + \frac{2(1-\vartheta)}{(2-\vartheta)N(\vartheta)}g(\varsigma) + \frac{2\vartheta}{(2-\vartheta)N(\vartheta)}\int_0^{\varsigma} g(s)ds. \tag{2.5}$$

For our convenience, we denote

$$A_\vartheta = \frac{2(1-\vartheta)}{(2-\vartheta)N(\vartheta)} \quad \textit{and} \quad B_\vartheta = \frac{2\vartheta}{(2-\vartheta)N(\vartheta)}.$$

Then (**2**.5) *can be written as*

$$p(\varsigma) = p_0 + A_\vartheta g(\varsigma) + B_\vartheta \int_0^\varsigma g(s)ds. \tag{2.6}$$

(ii) Suppose $g(0) \neq 0$*, then* (**2**.4) *is a solution of the subsequent system*

$$\begin{aligned} ({}^{CF}D_0^\vartheta p)(\varsigma) &= g(\varsigma) - g(0)e^{-\frac{\vartheta}{1-\vartheta}\varsigma}, \quad \varsigma \in [0,\xi], \\ p(0) &= p_0. \end{aligned} \tag{2.7}$$

From the above Remark **2**.3, we conclude that (**2**.6) is the solution of the system (**2**.3).

Constructing the piece-wise continuous functions is the first step that has to be taken before defining the solution to the given system (**1**.1)-(**1**.4). We address it in depth here.

For a given $\xi > 0$, let $\mathbb{I}_q = (\varsigma_q, \varsigma_{q+1}]$, $q = 1,2,\ldots,\ell$. Consider the subsequent space of $\mathscr{PC}$ functions:

$$\begin{aligned} \mathscr{PC}([0,\xi],\mathbb{X}) = \{ & p : [0,\xi] \to \mathbb{X}, p_q \in \mathscr{C}(\mathbb{I}_q,\mathbb{X}) \quad \text{for} \quad q = 0,1,2,\ldots,\ell, \\ & \text{and there exist } p(\varsigma_q^-) \text{ and } p(\varsigma_q^+) \text{ with } p(\varsigma_q) = p(\varsigma_q^-), q = 1,2,\ldots,\ell \}. \end{aligned}$$

Checking that $\mathscr{PC}([0,\xi],\mathbb{X})$ is a Banach space with norm $\|\cdot\|$ is not difficult.

Definition 2.3. *A function* $p \in \mathscr{PC}$ *is said to be a solution of* (**1**.1)-(**1**.3) *if it fulfills* $p(0) = p_0, ({}^{CF}D_\varsigma^\vartheta p)(\varsigma) = f(\varsigma, p(\varsigma), p(\mu\varsigma))$ *with* $f(0, p(0), p(0)) = 0$*; for* $\varsigma \in \mathbb{I}_q$, $q = 0,1,2,\ldots,\ell$*, and* $\Delta p(\varsigma_q) = \Lambda_q(p(\varsigma_q^-))$, $q = 1,2\ldots,\ell$.

We first investigate the linear system and find its solution before moving on to the nonlinear system (**1**.1)-(**1**.3).

Lemma 2.2. *Let* $g : [0,\xi] \to \mathbb{X}$ *be a continuous function with* $g(0) = 0$*. A function* $p \in \mathscr{PC}$ *is a solution of the fractional integral equation*

$$p(\varsigma) = \begin{cases} p_0 + A_\vartheta g(\varsigma) + B_\vartheta \int_0^\varsigma g(s)ds, & \varsigma \in [0,\varsigma_1], \\ p_0 + A_\vartheta g(\varsigma) + B_\vartheta \int_0^\varsigma g(s)ds + \Lambda_1(p(\varsigma_1)), & \varsigma \in (\varsigma_1,\varsigma_2], \\ \vdots & \\ p_0 + A_\vartheta g(\varsigma) + B_\vartheta \int_0^\varsigma g(s)ds + \sum_{q=1}^{\ell} \Lambda_q(p(\varsigma_q)), & \varsigma \in (\varsigma_q,\varsigma_{q+1}]. \end{cases} \tag{2.8}$$

iff p is a solution of the subsequent problem

$$\begin{cases} ({}^{CF}D_\varsigma^\vartheta p)(\varsigma) = g(\varsigma), \quad t \in [0,\xi], \quad \varsigma \neq \varsigma_q, \quad q = 1,2\ldots,\ell, \\ \Delta p(\varsigma_q) = \Lambda_q(p(\varsigma_q)), \quad q = 1,2\ldots,\ell, \\ p(0) = p_0. \end{cases} \tag{2.9}$$

Based on the above Lemma **2**.2, we can define the solution of the given system (**1**.1)-(**1**.3).

Lemma 2.3. *A function $p \in \mathscr{PC}$ is a solution of the system* $(\mathbf{1}.1)$-$(\mathbf{1}.3)$ *iff p fulfills the subsequent integral equation*

$$p(\varsigma) = \begin{cases} p_0 + A_\vartheta f(\varsigma, p(\varsigma), p(\mu\varsigma)) \\ +B_\vartheta \int_0^\varsigma f(s, p(s), p(\mu s))ds, & \varsigma \in [0, \varsigma_1], \\ p_0 + A_\vartheta f(\varsigma, p(\varsigma), p(\mu\varsigma)) \\ +B_\vartheta \int_0^\varsigma f(s, p(s), p(\mu s))ds + \Lambda_1(p(\varsigma_1)), & \varsigma \in (\varsigma_1, \varsigma_2], \\ \vdots \\ p_0 + A_\vartheta f(\varsigma, p(\varsigma), p(\mu\varsigma)) \\ +B_\vartheta \int_0^\varsigma f(s, p(s), p(\mu s))ds + \sum_{q=1}^{\ell} \Lambda_q(p(\varsigma_q)), & \varsigma \in (\varsigma_q, \varsigma_{q+1}]. \end{cases} \qquad (\mathbf{2}.10)$$

3. Existence Results

Under suitable fixed point theory [22], the existence findings for $(\mathbf{1}.1)$-$(\mathbf{1}.3)$ are examined in this section.

The following requirements must be met in order to use the fixed point theorems discussed above:

(A1) (i) The function $f : [0, \xi] \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ is continuous and we can find a positive constant $\mathscr{L}_f$ in a way that

$$\|f(\varsigma, p(\varsigma), p(\mu\varsigma)) - f(\varsigma, \bar{p}(\varsigma), \bar{p}(\mu\varsigma))\| \leq 2\mathscr{L}_f \|p - \bar{p}\|,$$

for each $\varsigma \in [0, \xi], \mu \in (0, 1), p, \bar{p} \in \mathbb{X}$.

(ii) There exist positive constants $\mathscr{L}_f, \widetilde{\mathscr{L}_f} > 0$ in ways that

$$\|f(\varsigma, p(\varsigma), p(\mu\varsigma))\| \leq 2\mathscr{L}_f \|p\| + \widetilde{\mathscr{L}_f}, \quad \varsigma \in [0, \xi], \mu \in (0, 1), p \in \mathbb{X}.$$

(A2) (i) The functions $\Lambda_q : \mathbb{X} \to \mathbb{X}, q = 1, 2, \ldots, \ell$ are continuous and we can find a positive constant $\mathcal{M}_{\Lambda_q}$ in ways that

$$\|\Lambda_q(u) - \Lambda_q(\bar{u})\| \leq \mathcal{M}_{\Lambda_q} \|u - \bar{u}\|, \quad \text{for all} \quad u, \bar{u} \in \mathbb{X}$$

and $\mathcal{M}_\Lambda = \max\{\mathcal{M}_{\Lambda_1}, \mathcal{M}_{\Lambda_2}, \ldots, \mathcal{M}_{\Lambda_\ell}\}$.

(ii) There exists a positive constant $\widetilde{\mathcal{M}}_\Lambda = \max\{\widetilde{\mathcal{M}}_{\Lambda_1}, \widetilde{\mathcal{M}}_{\Lambda_2}, \ldots, \widetilde{\mathcal{M}}_{\Lambda_\ell}\}$ in a way that

$$\|\Lambda_q(p)\| \leq \widetilde{\mathcal{M}}_{\Lambda_q} \|p\|, q = 1, 2, \ldots, \ell, p \in \mathbb{X}.$$

Theorem 3.1. *Suppose f and $\Lambda_q, q = 1, 2, \ldots, \ell$ are satisfy the conditions (A1)(i) and (A2)(i). If*

$$\widetilde{\mu} = [\mu^* \mathscr{L}_f + l\mathcal{M}_\Lambda] < 1, \qquad (\mathbf{3}.1)$$

where $\mu^* = 2(A_\vartheta + B_\vartheta \xi)$, and $1 - \widetilde{\mu} > 0$, then the system $(\mathbf{1}.1)$-$(\mathbf{1}.3)$ *has a unique solution on* $[0, \xi]$.

Proof. We transform the system (**1**.1)-(**1**.3) into a fixed point problem. Consider the operator $\Upsilon : \mathscr{PC}([0,\xi],\mathbb{X}) \to \mathscr{PC}([0,\xi],\mathbb{X})$ by

$$(\Upsilon p)(\varsigma) = \begin{cases} p_0 + A_\vartheta f(\varsigma, p(\varsigma), p(\mu\varsigma)) \\ +B_\vartheta \int_0^\varsigma f(s, p(s), p(\mu s))ds, \quad \varsigma \in [0,\varsigma_1], \\ p_0 + A_\vartheta f(\varsigma, p(\varsigma), p(\mu\varsigma)) + B_\vartheta \int_0^\varsigma f(s, p(s), p(\mu s))ds \\ +\Lambda_1(p(\varsigma_1)), \quad \varsigma \in (\varsigma_1, \varsigma_2], \\ \vdots \\ p_0 + A_\vartheta f(\varsigma, p(\varsigma), p(\mu\varsigma)) + B_\vartheta \int_0^\varsigma f(s, p(s), p(\mu s))ds \\ + \sum_{q=1}^{\ell} \Lambda_q(p(\varsigma_q)), \quad \varsigma \in (\varsigma_q, \varsigma_{q+1}]. \end{cases} \tag{3.2}$$

Now, we show that $\Upsilon B_Q \subset B_Q$. To do this, let $f(\cdot,0,0) = 0, \max\{\Lambda_q(0), q = 1,2,\dots,\ell\} = 0$ and let $B_Q = B(0,Q) = \{p \in \mathscr{PC}([0,\xi],\mathbb{X}) : \|p\|_{\mathscr{PC}} \leq Q\}$ with radius $Q \geq \frac{\|p_0\|}{1-\widetilde{\mu}}$, where $\widetilde{\mu} = \mu^* \mathscr{L}_f + l\mathcal{M}_\Lambda$ and $\mu^* = 2(A_\vartheta + B_\vartheta \xi)$.

For each $\varsigma \in [0,\varsigma_1]$ and $p \in B_Q$, we sustain

$$\begin{aligned} \|(\Upsilon p)(\varsigma)\| &= \left\| p_0 + A_\vartheta f(\varsigma, p(\varsigma), p(\mu\varsigma)) + B_\vartheta \int_0^\varsigma f(s, p(s), p(\mu s))ds \right\| \\ &\leq \|p_0\| + (A_\vartheta + B_\vartheta \varsigma_1)\, 2\mathscr{L}_f Q \\ &\leq Q. \end{aligned}$$

Further, for each $\varsigma \in \mathbb{I}_q$, $q = 1,2,\dots,\ell$, and $p \in B_Q$, we obtain

$$\begin{aligned} \|(\Upsilon p)(\varsigma)\| &\leq \|p_0\| + \left[(A_\vartheta + B_\vartheta \varsigma_{q+1})\, 2\mathscr{L}_f + \ell \mathcal{M}_{\Lambda_q}\right] Q \\ &\leq Q. \end{aligned}$$

Thus, for $\varsigma \in [0,\xi]$, and $p \in \mathscr{PC}$, we have

$$\|\Upsilon(p)\|_{\mathscr{PC}} \leq \|p_0\| + (\mu^* \mathscr{L}_f + l\mathcal{M}_\Lambda)Q \leq Q.$$

This demonstrates that the operator Υ causes the ball B_Q to be transformed into itself. Next, for $p, \overline{p} \in \mathscr{PC}$ and $\varsigma \in [0,\varsigma_1]$, we sustain

$$\|(\Upsilon p)(\varsigma) - (\Upsilon \overline{p})(\varsigma)\| \leq (A_\vartheta + B_\vartheta \varsigma_1)\, 2\mathscr{L}_f \|p - \overline{p}\|_{\mathscr{PC}}.$$

For $\varsigma \in (\varsigma_1, \varsigma_2]$, we get

$$\|(\Upsilon p)(\varsigma) - (\Upsilon \overline{p})(\varsigma)\| \leq [(A_\vartheta + B_\vartheta \varsigma_2)\, 2\mathscr{L}_f + \mathcal{M}_{\Lambda_1}]\, \|p - \overline{p}\|_{\mathscr{PC}}.$$

For $\varsigma \in (\varsigma_q, \varsigma_{q+1}]$, $q = 1,2,\dots,\ell$, we have

$$\|(\Upsilon p)(\varsigma) - (\Upsilon \overline{p})(\varsigma)\| \leq \left[(A_\vartheta + B_\vartheta \varsigma_{q+1})\, 2\mathscr{L}_f + \mathcal{M}_{\Lambda_q} \ell\right] \|p - \overline{p}\|_{\mathscr{PC}}.$$

Thus, for all $\varsigma \in [0,\xi]$, we obtain

$$\|(\Upsilon p) - (\Upsilon \overline{p})\|_{\mathscr{PC}} \leq [\mu^* \mathscr{L}_f + l\mathcal{M}_\Lambda]\, \|p - \overline{p}\|_{\mathscr{PC}}.$$

Based on (**3**.1) and the Banach fixed point theorem [22], we conclude that Υ contains a unique fixed point $p \in \mathscr{PC}$ that is a solution of the model (**1**.1)-(**1**.3) on $[0,\xi]$. □

Now, we prove the existence of solutions of (**1**.1)-(**1**.3) by utilizing Krasnoselskii's fixed point theorem (KFPT) [22].

Theorem 3.2. *Suppose that the conditions* $(A1)$ *and* $(A2)$ *hold with* $[2A_\vartheta \mathscr{L}_f + \ell \mathcal{M}_\Lambda] < 1$ *and* $1 - \widehat{\mu} > 0$*, where* $\widehat{\mu} = \mu^* \mathscr{L}_f + l\widetilde{\mathcal{M}}_\Lambda$*. Then the system* (**1**.1)-(**1**.3) *has at least one solution on* $[0,\xi]$.

Proof. Allow us to define two operators from (**2**.10) as follows:

$$(\Upsilon_1 p)(\varsigma)=\begin{cases} p_0+A_\vartheta f(\varsigma,p(\varsigma),p(\mu\varsigma)), \quad \varsigma\in[0,\varsigma_1],\\ p_0+A_\vartheta f(\varsigma,p(\varsigma),p(\mu\varsigma))+\Lambda_1(p(\varsigma_1)), \varsigma\in(\varsigma_1,\varsigma_2],\\ \vdots\\ p_0+A_\vartheta f(\varsigma,p(\varsigma),p(\mu\varsigma))+\sum_{q=1}^{\ell}\Lambda_q(p(\varsigma_q)), \varsigma\in(\varsigma_q,\varsigma_{q+1}].\end{cases} \tag{3.3}$$

and

$$(\Upsilon_2 p)(\varsigma)=B_\vartheta\int_0^\varsigma f(s,p(s),p(\mu s))ds,\ \varsigma\in[0,\xi]. \tag{3.4}$$

Let $B_Q=\{p\in\mathscr{PC}([0,\xi],\mathbb{X}):\|p\|_{\mathscr{PC}}\le Q\}$ with radius $Q\ge\dfrac{\|p_0\|+\mu^*\widetilde{\mathscr{L}_f}}{1-\widehat{\mu}}$, where $\widehat{\mu}=\mu^*\mathscr{L}_f+l\widetilde{\mathcal{M}}_\Lambda$ and $\mu^*=2(A_\vartheta+B_\vartheta\xi)$.

For each $\varsigma\in[0,\varsigma_1]$ and $p,p_1\in B_Q$, we find that

$$\begin{aligned}&\|\Upsilon_1 p(\varsigma)+\Upsilon_2 p_1(\varsigma)\|\\ &\le\|p_0\|+(A_\vartheta+B_\vartheta\varsigma_1)2\mathscr{L}_f Q+(A_\vartheta+B_\vartheta\varsigma_1)\widetilde{\mathscr{L}_f}\\ &\le Q.\end{aligned}$$

Further, for each $\varsigma\in\mathbb{I}_q$, $q=1,2,\dots,\ell$, and $p,p_1\in B_Q$, we obtain

$$\begin{aligned}&\|\Upsilon_1 p(\varsigma)+\Upsilon_2 p_1(\varsigma)\|\\ &\le\left\|p_0+A_\vartheta f(\varsigma,p(\varsigma),p(\mu\varsigma))+B_\vartheta\int_0^\varsigma f(s,p_1(s),p_1(\mu s))ds+\sum_{q=1}^{\ell}\Lambda_q(p(\varsigma_q))\right\|\\ &\le\|p_0\|+(A_\vartheta+B_\vartheta\varsigma_{q+1})\widetilde{\mathscr{L}_f}+\left[(A_\vartheta+B_\vartheta\varsigma_{q+1})2\mathscr{L}_f+\ell\widetilde{\mathcal{M}}_\Lambda\right]Q\\ &\le Q.\end{aligned}$$

Thus, for $\varsigma\in[0,\xi]$, and $p\in B_Q$, we have

$$\|\Upsilon_1(p)+\Upsilon_2(p_1)\|_{\mathscr{PC}}\le\|p_0\|+\frac{\mu^*}{2}\widetilde{\mathscr{L}_f}+[\mu^*\mathscr{L}_f+\ell\widetilde{\mathcal{M}}_\Lambda]Q\le Q.$$

Thus $\Upsilon_1(p)+\Upsilon_2(p_1)\in B_Q$. Next, we prove that Υ_1 is contraction. Since f is continuous, so is Υ_1, and letting $p,\overline{p}\in B_Q$, from (**3**.3) and (A1)(i), for each $\varsigma\in[0,\varsigma_1]$, we have

$$\begin{aligned}\|(\Upsilon_1 p)(\varsigma)-(\Upsilon_1\overline{p})(\varsigma)\|&\le A_\vartheta[\|f(\varsigma,p(\varsigma),p(\mu\varsigma))-f(\varsigma,\overline{p}(\varsigma),\overline{p}(\mu\varsigma))\|]\\ &\le 2A_\vartheta\mathscr{L}_f\|p-\overline{p}\|_{\mathscr{PC}}.\end{aligned}$$

From (A2)(i) and for $\varsigma\in(\varsigma_1,\varsigma_2]$, we get

$$\|(\Upsilon_1 p)(\varsigma)-(\Upsilon_1\overline{p})(\varsigma)\|\le[2A_\vartheta\mathscr{L}_f+\mathcal{M}_{\Lambda_1}]\,\|p-\overline{p}\|_{\mathscr{PC}}.$$

For $\varsigma\in(\varsigma_q,\varsigma_{q+1}]$, $q=1,2,\dots,\ell$, we have

$$\|(\Upsilon_1 p)(\varsigma)-(\Upsilon_1\overline{p})(\varsigma)\|\le\left[2A_\vartheta\mathscr{L}_f+\mathcal{M}_{\Lambda_q}\ell\right]\|p-\overline{p}\|_{\mathscr{PC}}.$$

Thus, for all $\varsigma\in[0,\xi]$, we obtain

$$\|(\Upsilon_1 p)-(\Upsilon_1\overline{p})\|_{\mathscr{PC}}\le[2A_\vartheta\mathscr{L}_f+\ell\mathcal{M}_\Lambda]\,\|p-\overline{p}\|_{\mathscr{PC}}.$$

Hence Υ_1 is a contraction. Continuity of f implies that the operator Υ_2 is continuous. Also Υ_2 is uniformly bounded on B_Q as

$$\begin{aligned}\|(\Upsilon_2 p)(\varsigma)\| &\leq \left\| B_\vartheta \int_0^\varsigma f(s,p(s),p(\mu s))ds \right\| \\ &\leq B_\vartheta \xi [2\mathscr{L}_f Q + \widetilde{\mathscr{L}_f}] = A,\end{aligned}$$

which implies that $\|\Upsilon_2 p\| \leq A$. Thus Υ_2 is uniformly bounded. To prove that the operator Υ_2 is compact, it remains to show that Υ_2 is equi-continuous. Now, for any $\tau_1, \tau_2 \in [0,\xi]$ with $\tau_1 < \tau_2$ and $p \in B_Q$, we find that

$$\begin{aligned}&\|(\Upsilon_2 p)(\tau_2) - (\Upsilon_2 p)(\tau_1)\| \\ &\leq \left\| B_\vartheta \int_0^{\tau_2} f(s,p(s),p(\mu s))ds - B_\vartheta \int_0^{\tau_1} f(s,p(s),p(\mu s))ds \right\| \\ &\leq \left\| B_\vartheta \int_0^{\tau_2} f(s,p(s),p(\mu s))ds + B_\vartheta \int_{\tau_1}^{0} f(s,p(s),p(\mu s))ds \right\| \\ &\leq B_\vartheta \int_{\tau_1}^{\tau_2} \|f(s,p(s),p(\mu s))\| ds \leq B_\vartheta (2\mathscr{L}_f Q + \widetilde{\mathscr{L}_f})(\tau_2 - \tau_1).\end{aligned} \tag{3.5}$$

From (**3**.5), we see that if $\tau_2 \to \tau_1$, then the right-hand side of (**3**.5) goes to zero, so $\|(\Upsilon_2 p)(\tau_2) - (\Upsilon_2 p)(\tau_1)\| \to 0$ as $\tau_1 \to \tau_2$. Thus, Υ_2 is equi-continuous. Also $\Upsilon_2(\mathbb{X}) \subset \mathbb{X}$, therefore Υ_2 is compact and, due to Arzela-Ascoli theorem, Υ has at least one fixed point. Hence the corresponding system has at least one solution. □

Next, we examine the existence and uniqueness results for the system (**1**.4) with the conditions (**1**.2)-(**1**.3).

Initially, we define the solution for the system (**1**.4) with the conditions (**1**.2)-(**1**.3).

Definition 3.1. *A function $p \in \mathscr{PC}$ is said to be a solution of* (**1**.4) *with the conditions* (**1**.2)-(**1**.3) *if it fulfills $p(0) = p_0$, $({}^{CF}D_\varsigma^\vartheta p)(\varsigma) = f\left(\varsigma, p(\varsigma), \int_0^\varsigma h(\varsigma,s)p(\mu s)ds\right)$ with $f(0,p(0),0) = 0$; for $\varsigma \in \mathbb{I}_q$, $q = 0,1,2,\ldots,\ell$, and $\Delta p(\varsigma_q) = \Lambda_q(p(\varsigma_q))$, $q = 1,2\ldots,\ell$.*

Lemma 3.1. *A function $p \in \mathscr{PC}$ is a solution of the system* (**1**.4) *with the conditions* (**1**.2)-(**1**.3) *iff p fulfills the subsequent integral equation*

$$p(\varsigma) = \begin{cases} p_0 + A_\vartheta f\left(\varsigma, p(\varsigma), \int_0^\varsigma h(\varsigma,s)p(\mu s)ds\right) \\ +B_\vartheta \int_0^\varsigma f\left(s, p(s), \int_0^s h(s,\tau)p(\mu\tau)d\tau\right) ds, \ \varsigma \in [0,\varsigma_1], \\ p_0 + A_\vartheta f\left(\varsigma, p(\varsigma), \int_0^\varsigma h(\varsigma,s)p(\mu s)ds\right) \\ +B_\vartheta \int_0^\varsigma f\left(s, p(s), \int_0^s h(s,\tau)p(\mu\tau)d\tau\right) ds + \Lambda_1(p(\varsigma_1)), \ \varsigma \in (\varsigma_1,\varsigma_2], \\ \vdots \\ p_0 + A_\vartheta f\left(\varsigma, p(\varsigma), \int_0^\varsigma h(\varsigma,s)p(\mu s)ds\right) \\ +B_\vartheta \int_0^\varsigma f\left(s, p(s), \int_0^s h(s,\tau)p(\mu\tau)d\tau\right) ds + \sum_{q=1}^{\ell} \Lambda_q(p(\varsigma_q)), \ \varsigma \in (\varsigma_q,\varsigma_{q+1}]. \end{cases}$$

Define the mapping $\overline{\Upsilon} : \mathscr{PC}([0,\xi],\mathbb{X}) \to \mathscr{PC}([0,\xi],\mathbb{X})$ by

$$(\overline{\Upsilon}p)(\varsigma) = \begin{cases} p_0 + A_\vartheta f\left(\varsigma, p(\varsigma), \int_0^\varsigma h(\varsigma,s)p(\mu s)ds\right) \\ +B_\vartheta \int_0^\varsigma f\left(s, p(s), \int_0^s h(s,\tau)p(\mu\tau)d\tau\right) ds, \varsigma \in [0,\varsigma_1], \\ p_0 + A_\vartheta f\left(\varsigma, p(\varsigma), \int_0^\varsigma h(\varsigma,s)p(\mu s)ds\right) \\ +B_\vartheta \int_0^\varsigma f\left(s, p(s), \int_0^s h(s,\tau)p(\mu\tau)d\tau\right) ds \\ +\Lambda_1(p(\varsigma_1)), \varsigma \in (\varsigma_1,\varsigma_2], \\ \vdots \\ p_0 + A_\vartheta f\left(\varsigma, p(\varsigma), \int_0^\varsigma h(\varsigma,s)p(\mu s)ds\right) \\ +B_\vartheta \int_0^\varsigma f\left(s, p(s), \int_0^s h(s,\tau)p(\mu\tau)d\tau\right) ds \\ +\sum_{q=1}^{\ell} \Lambda_q(p(\varsigma_q)), \varsigma \in (\varsigma_q,\varsigma_{q+1}]. \end{cases} \quad (\mathbf{3}.6)$$

In order to investigate (**1**.4) with (**1**.2)-(**1**.3), we must also mention the following conditions:

$(A1^*)$ (i) The function $f : [0,\xi]\times\mathbb{X}\times\mathbb{X} \to \mathbb{X}$ is continuous and we can find a positive constant $\mathscr{L}_f$ in a way that

$$\|f(\varsigma,u,v) - f(\varsigma,\bar{u},\bar{v})\| \leq \mathscr{L}_f[\|u-\bar{u}\| + \|v-\bar{v}\|],$$

for each $\varsigma \in [0,\xi], \mu \in (0,1), u,\bar{u},v,\bar{v} \in \mathbb{X}$.

(ii) There exist positive constants $\mathscr{L}_f, \widetilde{\mathscr{L}_f} > 0$ in ways that

$$\|f(\varsigma,u,v)\| \leq \mathscr{L}_f[\|u\| + \|v\|] + \widetilde{\mathscr{L}_f}, \quad \varsigma \in [0,\xi], \mu \in (0,1), p \in \mathbb{X}.$$

(A3) $h(\varsigma,s)$ is continuous for all $(\varsigma,s) \in [0,\xi]\times[0,\xi]$ and we can find a positive constant H in a way that $\max_{\varsigma,s\in[0,\xi]} \|h(\varsigma,s)\| = H$.

Theorem 3.3. *Suppose* $f, \Lambda_q, q = 1,2,\ldots,\ell$ *and* h *are satisfy the conditions* $(A1^*)$*, (A2)(i) and (A3). If*

$$\widetilde{\mu}_1 = [\mu_1^* \mathscr{L}_f(1+H\xi) + l\mathcal{M}_\Lambda] < 1, \quad (\mathbf{3}.7)$$

where $\mu_1^* = (A_\vartheta + B_\vartheta\xi)$ *and* $1-\widetilde{\mu}_1 > 0$*, then the system* (**1**.4) *with the conditions* (**1**.2)-(**1**.3) *has a unique solution on* $[0,\xi]$.

Finally, we prove the existence of solutions of (**1**.4) with the conditions (**1**.2)-(**1**.3) by utilizing Krasnoselskii's fixed point theorem (KFPT) [22].

Theorem 3.4. *Suppose that the conditions* $(A1^*)$, $(A2)$ *and* $(A3)$ *hold with* $[A_\vartheta\mathscr{L}_f(1+H\xi) + \ell\mathcal{M}_\Lambda] < 1$ *and* $1-\widetilde{\mu}_2 > 0$*, where* $\widetilde{\mu}_2 = \mu_1^*\mathscr{L}_f[1+H\xi] + \ell\widetilde{\mathcal{M}}_\Lambda$*. Then the system* (**1**.4) *with the conditions* (**1**.2)-(**1**.3) *has at least one solution on* $[0,\xi]$.

Proof. The proof of the above Theorems are very similar to Theorems **3**.1 and **3**.2, respectively, so we omit it here. □

4. Applications

Example 4.1.

Consider the subsequent impulsive pantograph system with CFO of the form

$$ {}^{CF}D^{\frac{1}{4}}p(\varsigma) = \frac{e^{-\varsigma}(\|p(\varsigma)\| + \|p\left(\frac{1}{2}\varsigma\right)\|)}{(9+e^{\varsigma})(1+\|p(\varsigma)\| + \|p\left(\frac{1}{2}\varsigma\right)\|)}, \ \varsigma \in [0,1], \varsigma_1 \neq \frac{1}{3}, \qquad (\mathbf{4}.1) $$

$$ p\left(\frac{1}{3}^{+}\right) - p\left(\frac{1}{3}^{-}\right) = \frac{\|p(\frac{1}{3})\|}{20(1+\|p(\frac{1}{3})\|)}, \qquad (\mathbf{4}.2) $$

$$ p(0) = 0. \qquad (\mathbf{4}.3) $$

Set $\vartheta = \frac{1}{4}, \ell = \xi = 1, \mu = \frac{1}{2}, A_\vartheta = \frac{3}{4}, B_\vartheta = \frac{1}{4}, N(\vartheta) = \frac{8}{7}$ and

$$ f(\varsigma,u,v) = \frac{e^{-\varsigma}(\|u\|+\|v\|)}{(9+e^{\varsigma})(1+\|u\|+\|v\|)}, (\varsigma,u,v) \in [0,1]\times[0,\infty)\times[0,\infty); $$

$$ \Lambda_1(p) = \frac{p}{20(1+p)}. $$

Let $u,v,\overline{u},\overline{v} \in [0,\infty)$ and $\varsigma \in [0,1]$. Then, we have

$$ \|f(\varsigma,u,v) - f(\varsigma,\overline{u},\overline{v})\| \leq \frac{e^{-\varsigma}}{(9+e^{\varsigma})}\left\|\frac{u+v}{1+u+v} - \frac{\overline{u}+\overline{v}}{1+\overline{u}+\overline{v}}\right\| = \frac{1}{10}[2\|u-\overline{u}\|]. $$

since

$$ \|f(\varsigma,p(\varsigma),p(\mu\varsigma)) - f(\varsigma,\overline{p}(\varsigma),\overline{p}(\mu\varsigma))\| \leq \frac{1}{10}[\|p-\overline{p}\| + \|p-\overline{p}\|] = \frac{1}{10}[2\|p-\overline{p}\|] $$

and

$$ \|f(\varsigma,p(\varsigma),p(\mu\varsigma))\| \leq \frac{1}{10}[2\|p\|]. $$

Thus, assumptions $(A1)(i)(ii)$ hold with $\mathscr{L}_f = \frac{1}{10}$ and $\widetilde{\mathscr{L}_f} = 0$. And for all $u,\overline{u} \in \mathscr{PC}([0,1],\mathbb{X})$, we have

$$ \|\Lambda_1(u) - \Lambda_1(\overline{u})\| \leq \frac{1}{20}\|u-\overline{u}\| \quad \text{and} \quad \|\Lambda_1(u)\| \leq \frac{1}{20}\|u\|. $$

Hence, assumptions $(A2)(i)(ii)$ hold with $\mathcal{M}_\Lambda = \frac{1}{20}$ and $\widetilde{\mathcal{M}}_\Lambda = \frac{1}{20}$. Furthermore

$$ \widetilde{\mu} = \mu_1^*\mathscr{L}_f + l\mathcal{M}_\Lambda = 2\left(A_\vartheta + B_\vartheta\xi\right)\mathscr{L}_f + \ell\mathcal{M}_\Lambda = 0.25. $$

Therefore, the condition (**3**.1) holds where $\widetilde{\mu} = 0.25 < 1$. Hence, in view of Theorem **3**.1, the given system (**4**.1)-(**4**.3) has a unique solution in $[0,1]$. Moreover

$$ 2A_\vartheta\mathscr{L}_f + \ell\mathcal{M}_\Lambda = 2\left(\frac{3}{4}\right)\left(\frac{1}{10}\right) + 1\left(\frac{1}{20}\right) = 0.2. $$

and $1-\widehat{\mu} = 0.25 > 0$. Then all the conditions of Theorem **3**.2 are also satisfied. Hence, the given system (**4**.1)-(**4**.3) has at least one solution in $[0,1]$.

Example 4.2.

Consider the subsequent impulsive pantograph integro-differential system with CFO of the form

$$^{CF}D^{\frac{1}{4}}p(\varsigma) = \frac{2+\|p(\varsigma)\|+\left\|\int_0^1 e^{\varsigma-s}p\left(\frac{1}{2}s\right)ds\right\|}{2e^{\varsigma+1}\left(1+\|p(\varsigma)\|+\left\|\int_0^1 e^{\varsigma-s}p\left(\frac{1}{2}s\right)ds\right\|\right)},$$

$$\varsigma\in[0,1],\ \varsigma_1\neq\frac{1}{3}, \tag{4.4}$$

$$p\left(\frac{1}{3}^+\right)-p\left(\frac{1}{3}^-\right) = \frac{\|p(\frac{1}{3})\|}{20(1+\|p(\frac{1}{3})\|)}, \tag{4.5}$$

$$p(0)=0. \tag{4.6}$$

Set $\vartheta=\frac{1}{4}, \ell=\xi=1, \mu=\frac{1}{2}, A_\vartheta=\frac{3}{4}, B_\vartheta=\frac{1}{4}, N(\vartheta)=\frac{8}{7}$ and

$$f(\varsigma,u,v) = \frac{2+\|u\|+\|v\|}{2e^{\varsigma+1}(1+\|u\|+\|v\|)},\ (\varsigma,u,v)\in[0,1]\times[0,\infty)\times[0,\infty)$$

$$\Lambda_1(p) = \frac{p}{20(1+p)}.$$

Let $u,v,\overline{u},\overline{v}\in[0,\infty)$ and $\varsigma\in[0,1]$. Then, we have

$$\|f(\varsigma,u,v)-f(\varsigma,\overline{u},\overline{v})\|\leq\frac{1}{2e^2}[\|u-\overline{u}\|+\|v-\overline{v}\|]$$

and

$$\|f(\varsigma,u,v)\|\leq\frac{1}{2e^2}[2+\|u\|+\|v\|].$$

Thus, assumptions $(A1^*)(i)(ii)$ and $(A3)$ hold with $\mathscr{L}_f=\frac{1}{2e^2}$, $\widetilde{\mathscr{L}_f}=0$ and $H=e$ respectively. In view of Example **4**.1, assumptions $(A2)(i)(ii)$ hold with $\mathcal{M}_\Lambda=\frac{1}{20}$ and $\widetilde{\mathcal{M}}_\Lambda=\frac{1}{20}$. Furthermore

$$\widetilde{\mu}_1=\mu_1^*\mathscr{L}_f(1+H\xi)+l\mathcal{M}_\Lambda = 1\left(\frac{1}{2e^2}\right)(1+e)+1\left(\frac{1}{20}\right)=0.3016$$

and

$$A_\vartheta\mathscr{L}_f(1+H\xi)+\ell\mathcal{M}_\Lambda=\frac{3}{4}\cdot\frac{1}{20}(1+e)+1\left(\frac{1}{20}\right)=0.1894.$$

Then all the conditions of Theorem **3**.3 and **3**.4 are satisfied. Hence, the given system (**4**.4)-(**4**.6) has at least one solution in $[0,1]$.

5. Conclusions

We have defined the existence theory of solutions to exponential kernel-type fractional order differential equations. We used the well-known Banach and Krasnoselskii fixed point theorems to develop the aforementioned theory. We use a contractive map to analyse the existence and uniqueness of the addressing model (**1**.1)-(**1**.3) in Theorem **3**.1. Theorem **3**.2 is constructed to explore the existence outcomes of the considered system (**1**.1)-(**1**.3) under condensing map circumstances. In Theorem **3**.3, we use a contractive map to analyse the existence and uniqueness of the addressing model (**1**.4) with the conditions (**1**.2)-(**1**.3). Under condensing map conditions, Theorem **3**.4 is employed to explore the existence outcomes of the considered system (**1**.4) with the conditions (**1**.2)-(**1**.3). There are intriguing examples provided to justify the observed findings. The findings are novel for impulsive differential equations incorporating CFO. With a suitable fixed point theorem to approximate controllability with instantaneous and non-instantaneous impulses for a variety of models, it may be possible to enhance the efficacy of such present research in the future.

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