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Γ-SEMIHYPERGROUPS AND THEIR PROPERTIES

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Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. The concept of Γ*-semihypergroups is a generalization of semigroups, a generalization of semihypergroups and a generalization of* Γ*-semigroups. In this paper, we define the notion of ideal, prime ideal, extension of an ideal in* Γ*-semihypergroups then we prove some results in respect and present many examples of* Γ*-semihypergroup. Also, we introduce the notions of quotient* Γ*-semihypergroup by using a congruence relation, and introduce the notion of right Noetherian* Γ*-semihypergroups. Finally, we study some properties of fundamental relations on a special kind of* Γ*-semihypergroups.*

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1. Introduction

In 1986, Sen and Saha [28] defined the notion of a Γ-semigroup as a generalization of a semigroup. One can see that Γ-semigroups are a generalizations of semigroups. Many classical notions of semigroups have been extended to Γ-semigroups and a lot of results on Γ-semigroups are published by a lot of mathematicians, for instance, Chattopadhyay [3, 4], Hila [16, 17], Saha [24], Sen and et. al. [25, 26, 27, 28, 32] and Seth [29].

Let S and Γ be non-empty sets. Then S is called a Γ*-semigroup* if there exists a map $S \times \Gamma \times S \longrightarrow S$, written (a, γ, b) by $a\gamma b$, such that satisfies the identities $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$. Let S be an arbitrary semigroup and Γ any non-empty set. Define a map $S \times \Gamma \times S \longrightarrow S$ by $a\alpha b = ab$ for all $a, b \in S$ and $\alpha \in \Gamma$. It is easy to see that S is a Γ-semigroup. Thus a semigroup can be considered as a Γ-semigroup. Many classical notions of semigroups have been extended to Γ-semigroups.

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Hyperstructures represent a natural extension of classical algebraic structures and they were introduced by the French mathematician F. Marty [20]. Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set.

Let H be a non-empty set. Then the map $\circ : H \times H \longrightarrow \wp^*(H)$ is called a *hyperoperation*, where $\wp^*(H)$ is the family of non-empty subsets of H. $(H, \circ)$ is called a *semihypergroup* if for every $x, y \in H$, we have $x \circ (y \circ z) = (x \circ y) \circ z$. If for every $x \in H$, $x \circ H = H = H \circ x$, then $(H, \circ)$ is called a *hypergroup*. In the above definition, if A and B are two non-empty subsets of H and $x \in H$, then we define

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b, \quad x \circ A = \{x\} \circ A \text{ and } A \circ x = A \circ \{x\}.$$

Since then, hundreds of papers and several books have been written on this topic, see [5, 7, 8, 30]. A recent book on hyperstructures [7] points out on their applications in cryptography, codes, automata, probability, geometry, lattices, binary relations, graphs and hypergraphs. Another book [8] is devoted especially to the study of hyperring theory. Several kinds of hyperrings are introduced and analyzed. The volume ends with an outline of applications in chemistry and physics, analyzing several special kinds of hyperstructures: e-hyperstructures and transposition hypergroups. The theory of suitable modified hyperstructures can serve as a mathematical background in the field of quantum communication systems. The concept of H_v-structures [30] constitute a generalization of the well-known algebraic hyperstructures (hypergroup, hyperring, hypermodule and so on). Actually some axioms concerning the above hyperstructures such as the associative law, the distributive law and so on are replaced by their corresponding weak axioms. Also, many authors studied different aspects of semihypergroups, for instance, P. Bonansinga and P. Corsini [2], Corsini [6], Davvaz [9], Davvaz and N.S. Poursalavati [10], Fasino and Freni [12], Gutan [14], Hasankhani [15], Leoreanu [19] and Onipchuk [23]. The concept of a Γ-semihypergroup is a generalization of a semigroup, a generalization of a semihypergroup and a generalization of a Γ-semigroup. Recently, Davvaz and et. al. [22] introduced the notion of Γ-semihypergroups.

2. Γ-semihypergroups and examples

In this section, we recall the concept of a Γ-semihypergroup and give some examples.

Definition 2.1. Let S and Γ be two non-empty sets. Then S is called a Γ-*semihypergroup* if each $\gamma \in \Gamma$ be a hyperoperation on S, i.e., $x\gamma y \subseteq S$ for every $x, y \in S$, and for every $\alpha, \beta \in \Gamma$ and $x, y, z \in S$ we have the associative property that is $x\alpha(y\beta z) = (x\alpha y)\beta z$. If for every $\gamma \in \Gamma$, (S, γ) is a hypergroup, then S is called a Γ-*hypergroup*. The Γ-semihypergroup S is called *commutative* if for every $x, y \in S$ and for every $\gamma \in \Gamma$, we have $x\gamma y = y\gamma x$. A non-empty subset A of S is called a Γ-*subsemihypergroup* of S if $A\Gamma A \subseteq A$.

Let A and B be two non-empty subsets of S. Then, we define

$$A\Gamma B = \bigcup_{\gamma\in\Gamma} A\gamma B = \cup\{a\gamma b \mid a \in A,\ b \in B \text{ and } \gamma \in \Gamma\}.$$

Let $(S, \circ)$ be a semihypergroup and let $\Gamma = \{\circ\}$. Then S is a Γ-semihypergroup. So every semihypergroup is a Γ-semihypergroup.

Now, we give some other examples of Γ-semihypergroups.

Example 2.1. Let S be a non-empty set and let Γ be a non-empty subset of S. If we define $x\gamma y = \{x, \gamma, y\}$, for every $x, y \in S$ and $\gamma \in \Gamma$, then S is a Γ-semihypergroup.

Example 2.2. Let $(S, \circ)$ be a hypergroup and $\Gamma = \{\alpha, \beta\}$. We define $x\alpha y = S$ and $x\beta y = x \circ y$, for every $x, y \in S$. Then S is a Γ-semihypergroup.

Example 2.3. Let S be a semigroup and $P_1, P_2, \cdots, P_k$ be non-empty subsets of S. Let $\Gamma = \{\alpha_1, \alpha_2, \cdots, \alpha_k\}$. We define $x\alpha_i y = xP_i y$ for every $x, y \in S$ and $\alpha_i \in \Gamma$, $1 \leq i \leq k$. Then S is a Γ-semihypergroup.

Example 2.4. Let $(S, \leq)$ be a totally ordered set and Γ be a non-empty subset of S. We define $x\gamma y = \{z \in S \mid z \geq \max\{x, \gamma, y\}\}$, for every $x, y \in S$ and $\gamma \in \Gamma$. Then S is a Γ-semihypergroup.

Let S be a Γ-semihypergroup. We define a relation ρ on $S \times \Gamma$ as follows:

$$(x, \alpha)\rho(y, \beta) \Longleftrightarrow x\alpha s = y\beta s, \forall s \in S.$$

Obviously ρ is an equivalence relation. Let $[x, \alpha]$ denote the equivalence class containing (x, α). Let $M = \{[x, \alpha] : x \in S, \alpha \in \Gamma\}$. We define hyperoperation $\circ$ on M as follows: $[x, \alpha] \circ [y, \beta] = \{[z, \beta] \mid z \in x\alpha y\}$, for all $x, y, z \in S$ and $\alpha, \beta, \gamma \in \Gamma$. Since $(x\alpha y)\beta z = x\alpha(y\beta z)$ in S, then

$$[x, \alpha] \circ ([y, \beta] \circ [z, \gamma]) = ([x, \alpha] \circ [y, \beta]) \circ [z, \gamma].$$

Thus, hyperoperation $\circ$ is associative, so $(M, \circ)$ is a semihypergroup. This semihypergroup is called the *left operator semihypergroup* of S.

Example 2.5. If we put $S = \{1, 2, 3, 4, 5\}$ and $\Gamma = \{4, 5\}$ in Example 2.4, then we have

$$[1, 4] = \{(1, 4), (2, 4), (3, 4), (4, 4)\},$$
$$[1, 5] = \{(1, 5), (2, 5), (3, 5), (4, 5), (5, 5), (5, 4)\}.$$

So $M = \{[1, 4], [1, 5]\}$ and the table of hyperoperation $\circ$ is as follows:

$\circ$	$[1,4]$	$[1,5]$
$[1,4]$	M	$[1,5]$
$[1,5]$	$[1,5]$	$[1,5]$

We see that $(M, \circ)$ is a semihypergroup.

3. Ideals of Γ-semihypergroup

In this section, we define the notion of a Γ-hyperideal of a Γ-semihypergroup and study some properties of it. Also, we consider the extension of a Γ-hyperideal in the commutative Γ-semihypergroups. Finally, prime Γ-hyperideals are defined.

Definition 3.1. A non-empty subset I of a Γ-semihypergroup S is called a *left (right)* Γ-*hyperideal*, "*ideal*, for short" of S, if $S\Gamma I \subseteq I$ $(I\Gamma S \subseteq I)$. S is called a *left (right) simple* Γ-*semihypergroup* if has no proper left (right) ideal. S is called a *simple* Γ-*semihypergroup* if S has no proper ideal both left and right.

Example 3.1. Consider Example 2.4. Put $S = \mathbb{N}$ with natural order. Then the subset $I_n = \{n, n+1, n+2, \cdots\}$ is an Ideal of S, for every $n \in \mathbb{N}$.

Let S be a Γ-semihypergroup and $\alpha \in \Gamma$, if we define $a \circ b = a\alpha b$ for every $a, b \in S$, then $(S, \circ)$ becomes a semihypergroup, we denote this semihypergroup by S_α.

Theorem 3.1. *Let S be a Γ-semihypergroup. Then S is a simple Γ-semihypergroup if and only if S_α is a hypergroup for every $\alpha \in \Gamma$.*

Proof. Let S be a simple Γ-semihypergroup and $\alpha \in \Gamma$, we show that S_α is a hypergroup. For this, we verify the reproduction axiom. Let $I = a\alpha S$ where $a \in S$. Then I is a right ideal of S, indeed $I\Gamma S = (a\alpha S)\Gamma S \subseteq a\alpha S = I$. Since S has no proper right ideal, then $I = a\alpha S = S$ so S_α is a hypergroup.

Conversely, let $\phi \neq I$ be a left ideal of S. Let $s \in S$ and $a \in I$. Since S_α is a hypergroup so there exists $t \in S$ such that $s \in t \circ a = t\alpha a \subseteq S\alpha I \subseteq I$, so $S = I$. Similarly, one can prove that S has no proper right ideal. Therefore S is simple. □

Corollary 3.1. *Let S be a Γ-semihypergroup. If for some $\alpha \in \Gamma$, S_α is a hypergroup, then for every $\beta \in \Gamma$, S_β is a hypergroup.*

Proof. Since S_α is a hypergroup, then by previous theorem, S is a simple Γ-semihypergroup. Thus, for every $\beta \in \Gamma$, S_β is a hypergroup. □

Corollary 3.2. *Let S be a Γ-semihypergroup. If for some $\alpha \in \Gamma$, S_α is a hypergroup, then S is a Γ-semihypergroup.*

Proof. By Corollary 3.1, it is trivial. □

Definition 3.2. Let A be a non-empty subset of a Γ-semihypergroup S. Then intersection of all ideals of S containing A is an ideal of S generated by A, and denoted by $< A >$.

Lemma 3.1. *Let S be a Γ-semihypergroup. If A is a non-empty subset of S, then $< A >= A \cup A\Gamma S \cup S\Gamma A \cup S\Gamma A\Gamma S$.*

Proof. Let $B = A \cup A\Gamma S \cup S\Gamma A \cup S\Gamma A\Gamma S$. Then B is an ideal of S, because

$$\begin{aligned} B\Gamma S &= (A \cup A\Gamma S \cup S\Gamma A \cup S\Gamma A\Gamma S)\Gamma S \\ &= A\Gamma S \cup A\Gamma(S\Gamma S) \cup S\Gamma A\Gamma S \cup S\Gamma A\Gamma(S\Gamma S) \\ &\subseteq A\Gamma S \cup A\Gamma S \cup S\Gamma A\Gamma S \cup S\Gamma A\Gamma S \\ &= A\Gamma S \cup S\Gamma A\Gamma S \subseteq B. \end{aligned}$$

Similarly, $S\Gamma B \subseteq B$, so B is an ideal of S.

Now, we show that if C is an ideal of S contains A, then $B \subseteq C$. Since $A \subseteq C$ and C is an ideal of S, then we have $A\Gamma S \subseteq C\Gamma S \subseteq C$ and $S\Gamma A \subseteq S\Gamma C \subseteq C$. Hence, $A\Gamma S\Gamma A \subseteq C\Gamma A \subseteq C$, therefore $B \subseteq C$ and the proof is completed. □

One can see that, if S is a commutative Γ-semihypergroup and $\phi \neq A \subseteq S$, then $< A >= A \cup A\Gamma S$.

Definition 3.3. Let I be an ideal of commutative Γ-semihypergroup S and $\phi \neq A \subseteq S$. Then the extension of I by A defined as follows:

$$(A : I) = \{x \in S \mid A\Gamma x \subseteq I\}.$$

If $A = \{a\}$, then we also write $< \{a\} : I >$ as $< a : I >$. If I is an ideal of S and $\phi \neq A \subseteq B \subseteq S$, then $(B : I) \subseteq (A : I)$.

Lemma 3.2. *Let S be a commutative Γ-semihypergroup. If I is an ideal of S, $\phi \neq A \subseteq S$ and $\gamma \in \Gamma$, then the following statement are true:*

(1) *$(A : I)$ is an ideal of S;*
(2) *$I \subseteq (A : I) \subseteq (A\Gamma A : I) \subseteq (A\gamma A : I)$;*
(3) *If $A \subseteq I$, then $(A : I) = S$.*

Proof. (1) Let $x \in (A : I)$, $y \in S$ and $\gamma \in \Gamma$. Then

$$A\Gamma(x\gamma y) = (A\Gamma x)\gamma y \subseteq I\gamma y \subseteq I\Gamma S \subseteq I$$

so $(A : I)$ is an ideal of S.

(2) If $x \in I$, then $I\Gamma x \subseteq S\Gamma I \subseteq I$. Thus $x \in (A : I)$. If $x \in (A : I)$, then $(A\Gamma A)\Gamma x = A\Gamma(A\Gamma x) \subseteq A\Gamma I \subseteq I$. Thus $x \in (A\Gamma A : I)$. Finally, if $x \in (A\Gamma A : I)$, then $(A\gamma A)\Gamma x \subseteq (A\Gamma A)\Gamma x \subseteq I$. So $x \in (A\gamma A : I)$.

(3) Let $A \subseteq I$ and $x \in S$. Then $A\Gamma x \subseteq I\Gamma S \subseteq I$, so $x \in (A : I)$. Hence $(A : I) = S$.

□

Lemma 3.3. *Let S be a commutative Γ-semihypergroup. Let I be an ideal of S and $\phi \neq A \subseteq S$. Then*

$$(A : I) = \bigcap_{a \in A} (a : I) = (A \setminus I : I).$$

Proof. By Lemma 3.2, we have $(A : I) \subseteq \bigcap_{a\in A}(a : I)$. Let $x \in \bigcap_{a\in A}(a : I)$. Then, $a\Gamma x \subseteq I$ for all $a \in A$. So $\bigcap_{a\in A}(a : I) \subseteq (A : I)$. Hence $(A : I) = \bigcap_{a\in A}(a : I)$. By Lemma 3.2(3), we have $(A : I) = \bigcap_{a\in A}(a : I) = (A\setminus I : I)$. □

Definition 3.4. A proper ideal P of a Γ-semihypergroup S is called a *prime ideal*, if for every ideal I, J of S, $I\Gamma J \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$. If Γ-semihypergroup S is commutative, then a proper ideal P is prime if and only if $a\Gamma b \subseteq P$ implies $a \in P$ or $b \in P$, for any $a, b \in S$.

Example 3.2. Consider Example 2.4. Put $S = \Gamma = \{1, 2, \cdots n\}$ for some $n \in \mathbb{N}$. Then all ideals of S are of the form $I_i = \{i, i+1, \cdots n\}$, for every $i \in S$ and I_2 is a prime ideal of S.

Theorem 3.2. *Let S be a Γ-semihypergroup and P be a left ideal of S. Then P is prime if and only if for all $x, y \in S$,*

$$x\Gamma S\Gamma y \subseteq P \text{ implies } x \in P \text{ or } y \in P.$$

Proof. If $x, y \in S$ and $x\Gamma S\Gamma y \subseteq P$, then $S\Gamma x\Gamma S\Gamma y \subseteq S\Gamma P \subseteq P$. Since $S\Gamma x$ and $S\Gamma y$ are left ideals of S, we have either $S\Gamma x \subseteq P$ or $S\Gamma y \subseteq P$. Suppose that $S\Gamma x \subseteq P$. Let $I = S\Gamma x \cup \{x\}$ be the left ideal of S generated by x. Then $I\Gamma I \subseteq S\Gamma x \subseteq P$ whence we obtain $I \subseteq P$. Hence $x \in P$. Similarly, we can show that if $S\Gamma y \subseteq P$ then $y \in P$.

Conversely, suppose that $I\Gamma J \subseteq P$ and $I \nsubseteq P$. Then, we show that $J \subseteq P$. Let $x \in I \setminus P$. Then for all $y \in J$ we have $x\Gamma S\Gamma y \subseteq I\Gamma J \subseteq P$. Since $x \notin P$, then $y \in P$, so $J \subseteq P$. □

Lemma 3.4. *Let S be a commutative Γ-semihypergroup and let I be an ideal of S. Then I is a prime ideal of S if and only if $(A : I) = I$ for all $A \nsubseteq I$.*

Proof. Assume that I is a prime ideal of S and $A \nsubseteq I$. Choose $a \in A$ such that $a \notin I$. Let $x \in (A : I)$. Then $a\Gamma x \subseteq I$. Since I is prime, thus $x \in I$. So $(A : I) \subseteq I$. Now, by 3.2(ii) we have $(A : I) = I$.

Conversely, let $(A : I) = I$, for every $A \nsubseteq I$. If A, B are ideals of S such that $A\Gamma B \subseteq I$ and $A \nsubseteq I$, then $B \subseteq (A : I) = I$, so I is a prime ideal of S. □

4. Right Noetherian Γ-semihypergroups

In this section, we introduce the notion of right Noetherian Γ-semihypergroups.

Definition 4.1. Let S be a Γ-semihypergroup. Then S is said right Noetherian, if S satisfies the ascending chain condition on right ideals. That is, for any sequence of right ideals $\{I_i\}_{i=1}^{\infty}$ of S such that $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_i \cdots$, there exists $n \in \mathbb{N}$ such that $I_m = I_n$ for each $m \in \mathbb{N}, m \geq n$.

Example 4.1. Let $(F, +, \cdot)$ be a field and G be a subgroup of $(F \setminus \{0\}, \cdot)$. Let $F/G = \{aG \mid a \in F\}$ and $\Gamma = \{\alpha, \beta\}$. Then consider the hyperoperations α and β as follows: $(aG)\alpha(bG) = \{cG \mid c \in aG + bG\}$ and $(aG)\beta(bG) = F/G$. Then F/G is a simple Γ-semihypergroup, so it is right Noetherian.

Example 4.2. Let $S = \{1, 2, 3. \cdots\}$ and $\Gamma = \{\alpha_1, \alpha_2, \cdots, \alpha_k\}$ for some $k \in \mathbb{N}$. Consider the hyperoperations α_i as follows: $x\alpha_i y = \{z \in S \mid z \leq \min\{x, i, y\}\}$.

Then S is a Γ-semihypergroup. Now, we find a chain of right ideals of S that is not satisfied ascending chain condition. For every $n \in \mathbb{N}$, let $I_n = \{1, 2, \cdots, n\}$. Then I_n is an ideal of S and we have $I_1 \subsetneq I_2 \subsetneq \cdots$. So S is not right Noetherian Γ-semihypergroup.

Example 4.3. Let $A_n = (n-1, n)$, $S = \bigcup_{n=1}^{\infty} A_n$ and $\Gamma = \{\alpha_i \mid i \in \mathbb{N}\}$. For every $x, y \in S$ and $\alpha_i \in \Gamma$, we define $x\alpha_i y = A_{n.i.m}$, where $x \in A_n$ and $y \in A_m$. Then S is a right Noetherian Γ-semihypergroup.

Let $(H, \circ)$ be a semihypergroup. A non-empty subset I of H is called a *left (right) hyperideal* of H, if $H \circ I \subseteq I$ $(I \circ H \subseteq I)$.

Let S be a Γ-semihypergroup and M be the left operator semihypergroup of it. Then for $A \subseteq M$ we define $A^+ = \{x \in S \mid [x, \alpha] \in M, \ \forall \ \alpha \in \Gamma\}$. Similarly, for $I \subseteq S$ we define $I^{+'} = \{[x, \alpha] \in M : x\alpha s \subseteq I, \forall s \in S\}$.

Theorem 4.1. *Let S be a Γ-semihypergroup and M be its left operator semihypergroup. Then the following statements are true:*

(i) *If A is a right hyperideal of M, then A^+ is a right ideal of S.*
(ii) *If I is a right ideal of S then, $I^{+'}$ is a right hyperideal of M.*

Proof. (i) Let $x \in A^+$, $y \in S$ and $\alpha \in \Gamma$. Then $[x, \alpha] \in A$ and since A is a hyperideal of M, thus $[x, \alpha] \circ [y, \alpha] \subseteq A$. So $\{[t, \alpha] : t \in x\alpha y\} \subseteq A$ then $x\alpha y \subseteq A^+$. Therefore A^+ is a right ideal of S.

(ii) Let $[x, \alpha] \in I^{+'}$ and $[y, \beta] \in M$. Then for all $s \in S$, $x\alpha s \subseteq I$. Now,

$$[x, \alpha] \circ [y, \beta] = \{[t, \beta] : t \in x\alpha y\} \subseteq I^{+'}$$

Therefore, $I^{+'}$ is a hyperideal of M. □

Let S be a Γ-semihypergroup and M be the left operator hypergroup of it. Let I be an ideal of S and A be a hyperideal of M. Then it is easy to see that $I \subseteq (I^+)^{+'}$ and $A \subseteq (A^{+'})^+$. In the following theorem we prove that if I and A are prime then the equality holds.

Theorem 4.2. *Let S be a Γ-semihypergroup and M be the left operator hypergroup of it. Let P be a right prime ideal of S. Then $P = (P^{+'})^+$.*

Proof. Suppose that P be prime and $x \in (P^{+'})^+$. Then $x\Gamma S \subseteq P$. So $x\Gamma S\Gamma x \subseteq x\Gamma S \subseteq P$. Since P is a prime right ideal of S, then by Theorem 3.2, $x \in P$. So $(P^{+'})^+ \subseteq P$. Therefore, $(P^{+'})^+ = P$. □

Let S be a Γ-semihypergroup. If their exist elements $e \in S$ and $\delta \in \Gamma$ such that $e\delta x = x$ for every $x \in S$, then S is said to have a *left unity.* It is easy to check that if S has a left unity, then $[e, \delta]$ is a left unity of the left operator semihypergroup M.

Theorem 4.3. *Let S be a Γ-semihypergroup and M be its left operator semihypergroup. If I is a right ideal of S, then $I = (I^{+'})^+$.*

Proof. Let $x \in (I^{+'})^{+}$. Then $[x, \alpha] \in I^{+'}$ for every $\alpha \in \Gamma$. So $x\alpha s \subseteq I$ for every $s \in S$. Since S has a left unity, thus $x \in I$. So $I = (I^{+'})^{+}$. □

Theorem 4.4. *Let S be a Γ-semihypergroup with a left unity. If the left operator semihypergroup M of S is right Noetherian, then S is right Noetherian.*

Proof. Let $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ be an ascending chain of right ideals of S. Then $I_1^{+'} \subseteq I_2^{+'} \subseteq I_3^{+'} \subseteq \cdots$ is an ascending chain of right ideals of M. Since M is right Noetherian, thus there exists a positive integer n such that $I_n^{+'} = I_{n+k}^{+'}$. Hence previous theorem $I_n = (I_n^{+'})^{+} = (I_{n+k}^{+'})^{+} = I_{n+k}$, for $k = 1, 2, \cdots$. So S is right Noetherian. □

Let S be a Γ-semihypergroup. If every non-empty set of right ideals of S, partially ordered by set inclusion, has a maximal element, we say that maximum condition holds for right ideals of S. That is, for each non-empty set $\mathcal{A}$ of ideals of S, there is an element $M \in \mathcal{A}$ such that there is no element $T \in \mathcal{A}$ such that $T \supset M$. Equivalently, if $T \in \mathcal{A}$ such that $T \supseteq M$, then $T = M$.

Theorem 4.5. *Let S be a Γ-semihypergroup. Then the following are equivalent:*

(i) *S is right Noetherian;*
(ii) *S satisfies the maximum condition for right ideals;*
(iii) *Every right ideal of S is finitely generated.*

Proof. (i)⇒(ii) Assume by contradiction that there is a non-empty set of ideals of S, say $\mathcal{A}$, which has no maximum element. If $I_1 \in \mathcal{A}$ then there exists an element $I_2 \in \mathcal{A}$ such that $I_1 \subsetneq I_2$; since $\mathcal{A}$ has no maximum element. Also, there exists an element $I_3 \in \mathcal{A}$, such that $I_2 \subsetneq I_3$. By continuing this process we have the acceding chain $I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots$, which is impossible.

(ii)⇒(iii) Let I be an ideal of S. Then we show that I is finitely generated. Let $\mathcal{A} = \{< A >$: A is a finite subset of I$\}$. By (ii), $\mathcal{A}$ has a maximal element, say $< A_0 >$ where A_0 is finite subset of I. Now, if $a \in I$ then $< A_0 \cup \{a\} > \in \mathcal{A}$. Then by maximality of $< A_0 >$ we have $a \in < A_0 >$. Therefore, I is a finitely generated ideal of S.

(iii)⇒(i) Let $\{I_i : i \in \mathbb{N}\}$ be a sequence of ideals of S, such that $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_i \subseteq \cdots$, and let $I = \bigcup_{i=1}^{\infty} I_i$. One can easily see that I is an ideal of S. Then, by (iii), there exist $a_1, a_2, \cdots, a_t \in S$ such that $I = < a_1, a_2, \cdots, a_t >$. Clearly $a_1, a_2, \cdots, a_t \in I$. Let $i_k \in \mathbb{N}$ such that $a_k \in I_{i_k}$, for $k = 1, 2, \cdots, t$. We put $n := \max\{i_1, i_2, \cdots, i_t\}$. Since $i_k \leq n$ for all $k = 1, 2, \cdots, t$, we have $I_{i_k} \subseteq I_n$, for all $k = 1, 2, \cdots, t$. Hence $a_k \in I_n$, for all $k = 1, 2, \cdots, t$. Since I_n is an ideal of S, thus $< a_1, a_2, \cdots, a_t > \subseteq I_n$. So $I = < a_1, a_2, \cdots, a_t > \subseteq I_n \subseteq I$. Then, $I_n = I$. So $I_m = I_n$, for all $m \in \mathbb{N}$ such that $m \geq n$. Indeed, let $m \geq n$. Then $I = I_n \subseteq I_m \subseteq I$, thus $I_m = I_n$. □

5. (S,Γ)-semihypergroups

In this section, we introduce the quotient Γ-semihypergroup by using a congruence relation. Also, we investigate the fundamental relations on Γ-semihypergroups. Then we present a special kind of Γ-semihypergroups which is called (S,Γ)-semihypergroup and (Γ,Δ)-semihypergroup.

An equivalence relation R on a Γ-semihypergroup S is called a *congruence relation*, if for any $a,b,c\in S$, $\gamma\in\Gamma$ and for every $t\in a\gamma c$, there exists $t'\in b\gamma c$ such that tRt'.

Let R be a congruence relation on a Γ-semihypergroup S. We set $S/R=\{R(x)\mid x\in S\}$ and $\dot{\Gamma}=\{\dot{\gamma}\mid \gamma\in\Gamma\}$ and $\dot{\gamma}\in\dot{\Gamma}$ and for every $x,y\in S$ we define $R(x)\dot{\gamma}R(y)=\{R(z)\mid z\in x\gamma y\}$.

Lemma 5.1. *If R is a congruence relation on a Γ-semihypergroup S, then S/R is a $\dot{\Gamma}$-semihypergroup.*

Proof. Let $x,x',y,y'\in S$ and $\gamma\in\Gamma$. We show that if xRx' and yRy' then $R(x)\dot{\gamma}R(y)=R(x')\dot{\gamma}R(y')$. Suppose that $R(z)\in R(x)\dot{\gamma}R(y)$. Then $z_1\in x\gamma y$, where z_1Rz', thus $R(z)=R(z_1)=R(z')$, so $R(x)\dot{\gamma}R(y)\subseteq R(x')\dot{\gamma}R(y')$. Similarly, one can see that $R(x')\dot{\gamma}R(y')\subseteq R(x)\dot{\gamma}R(y)$. Now, let $x,y,z\in S$ and $\alpha,\beta\in\Gamma$. Then

$$\begin{aligned}(R(x)\dot{\alpha}R(y))\dot{\beta}R(z) &= (\{R(t)\mid t\in x\alpha y\})\dot{\beta}R(z)\\ &= \{R(s)\mid s\in t\beta z, t\in x\alpha y\}\\ &= \{R(s)\mid s\in (x\alpha y)\beta z\}\\ &= \{R(s)\mid s\in x\alpha(y\beta z)\}\\ &= R(x)\dot{\alpha}(R(y)\dot{\beta}R(z)).\end{aligned}$$

Therefore, S/R is a Γ-semihypergroup. □

Let S be a Γ-semihypergroup and R be an equivalence relation on S. If A and B are non-empty subsets of S, then

$$A\bar{\bar{R}}B \text{ means that } \forall a\in A, b\in B, \text{ we have } aRb.$$

An equivalence relation R is called *strongly regular on the right (on the left)*, if for all $x\in S$, aRb implies $(a\alpha x)\bar{\bar{R}}(b\alpha x)$ $((x\alpha a)\bar{\bar{R}}(x\alpha b))$, for every $\alpha\in\Gamma$.

Definition 5.1. Let S be a Γ-semihypergroup and θ be an equivalence relation on S. We say that θ is a *fundamental relation* on S, if θ is the smallest strongly regular equivalence relation on S.

In fact, the fundamental relation θ is the smallest equivalence relation on Γ-semihypergroup S such that the quotient S/θ is a $\dot{\Gamma}$-semigroup. The fundamental relation was introduced on hypergroups by Koskas [18], and studied by many authors, for example, Corsini, Davvaz, Freni, Leoreanu, Vougiouklis and others [5, 7, 1, 11, 13, 21, 30, 31].

Let S be a Γ-semihypergroup and $a,b\in S$. For every $n\in\mathbb{N}$ we define the relation ρ on S as follows:

$$(a,b) \in \rho_n \Longleftrightarrow \begin{array}{l} \exists\ a_1, \dots a_{n+1} \in S, \gamma_1, \dots, \gamma_n \in \Gamma \\ \ni \{a,b\} \subseteq a_1\gamma_1 a_2\gamma_2 \dots a_n\gamma_n a_{n+1}, \end{array}$$

Now, set $\rho = \bigcup_{n=1}^{\infty} \rho_n$. It is easy to check that ρ is reflexive and symmetric. Let ρ^* be the transitive closure of the relation ρ.

If $a_1, \dots a_{n+1} \in S$ and $\gamma_1, \dots, \gamma_n \in \Gamma$, then we use following notation

$$a_1\gamma_1 a_2\gamma_2 \dots \gamma_{n-1} a_{n-1}\gamma_n a_{n+1} = \prod_{i=1}^{n} a_i\gamma_i a_{i+1}.$$

Theorem 5.1. [22]. *Let S be a Γ-semihypergroup. Then the relation ρ^* is a strongly regular equivalence relation on S.*

Corollary 5.1. [22]. *Let S be a Γ-semihypergroup. Then the quotient S/ρ^* is a $\dot{\Gamma}$-semigroup.*

Theorem 5.2. [22]. *Let S be a Γ-semihypergroup. Then the equivalence relation ρ^* is the smallest strongly regular equivalence relation on S and so $\rho^* = \theta$, where θ is the fundamental relation on S.*

Now, we present a way to obtain a Γ-semihypergroup with a semigroup.

Definition 5.2. Let $(\Gamma, \cdot)$ be a semigroup and $\{A_\gamma\}_{\gamma\in\Gamma}$ be a collection of nonempty disjoint sets and $S = \bigcup_{\gamma\in\Gamma} A_\gamma$. For every $x, y \in S$ and $\gamma \in \Gamma$ the we define $x\gamma y = A_{\gamma_x\gamma\gamma_y}$, where $x \in A_{\gamma_x}$ and $y \in A_{\gamma_y}$ for some $\gamma_x \in \Gamma$ and $\gamma_y \in \Gamma$. Then S is a Γ-semihypergroup and we call it a *(S,Γ)-semihypergroup*.

Theorem 5.3. *Let S be a (S,Γ)-semihypergroup. Then S is Γ-heypergroup if and only if Γ is a group.*

Proof. Let S be a Γ-heypergroup. We show that for every $\alpha, \beta \in \Gamma$, there exists $\gamma \in \Gamma$ such that $\alpha\gamma = \beta$. Choose $x \in A_\beta$ and $y \in A_\alpha$. Since S is a Γ-heypersemigroup, thus by Corollary 3.1, for every $\theta \in \Gamma$, S_θ is a hypergroup. So there exists $z \in S$. Then there exists $\delta \in \Gamma$ such that $z \in A_\delta$ and $x \in y\theta z = A_{\alpha\theta\delta}$, since $\{A_\alpha\}_{\alpha\in\Gamma}$ are disjoint, then $A_\beta = A_{\alpha\theta\delta}$ this means that $\alpha(\theta\delta) = \beta$.

Conversely, let Γ be a group. Then by Corollary 3.1, we should show that S_θ is a hypergroup for every $\theta \in \Gamma$. If $x \in S$, then there exists $\beta \in \Gamma$ such that $x \in A_\beta$. So

$$x\theta S = \bigcup_{y\in A_\alpha} A_{\beta\theta\alpha} = \bigcup_{\gamma\in\Gamma} A_\gamma = S.$$

Therefore, $(S.\theta)$ is a hypergroup. □

Now, we present a way to obtain a Γ-semihypergroup with a Δ-semigroup.

Definition 5.3. Let Γ be a Δ-semigroup, $\{A_\gamma\}_{\gamma\in\Gamma}$ be a collection of nonempty distinct sets and $S = \bigcup_{\gamma\in\Gamma} A_\gamma$. For every $x, y \in S$ and $\delta \in \Delta$, we define $x\delta y = A_{\alpha\delta\beta}$ where $x \in A_\alpha$ and $y \in A_\beta$ for some $\alpha, \beta \in \Gamma$. Then S is a Δ-semihypergroup and we call a *(Γ, Δ)-semihypergroup*.

We will prove some properties of (Γ, Δ)-semihypergroups.

Lemma 5.2. *Let S be a (Γ, Δ)-semihypergroups. Then S is a commutative Δ-semihypergroup if and only if Γ is a commutative Δ-semihypergroup.*

Proof. The proof is trivial. □

Lemma 5.3. *Let S be a (Γ, Δ)-semihypergroups. If I is an ideal of Γ, then $S_I = \bigcup_{\theta \in I} A_\theta$ is an ideal of S. Conversely, if $S_I = \bigcup_{\theta \in I} A_\theta$ is an ideal of S, then I is an ideal of Γ.*

Proof. Let $I \trianglelefteq \Gamma$, $x \in S$ and $\delta \in \Delta$. If $x \in A_\alpha$ and $\alpha \in \Gamma$, then

$$x\delta S_I = x\delta \left(\bigcup_{\theta \in I} A_\theta \right) = \left(\bigcup_{\theta \in I} x\delta A_\theta \right) = \bigcup_{\alpha \in I} A_{\alpha\delta\theta} \subseteq S_I.$$

Conversely, suppose that S_I is an ideal of S. If $\gamma \in \Gamma$, $\delta \in \Delta$ and $\theta \in I$, we should show that $\gamma\delta\theta \subseteq I$. Choose $x \in A_\gamma$ and $y \in A_\theta$. Then $x\delta y \subseteq S_I$, on the other hand $x\delta y = A_{\gamma\delta\theta} \subseteq S_I$. So there exists $\theta^{'} \in I$ such that $A_{\gamma\delta\theta} = A_{\theta'}$, so $\gamma\delta\theta = \theta' \in I$. □

Lemma 5.4. *Let S be a commutative (Γ, Δ)-semihypergroups and P be an ideal of Γ. Then P is a prime ideal of Γ if and only if $S_P = \bigcup_{\theta \in P} A_\theta$ is a prime ideal of S.*

Proof. Let P be a prime ideal of Γ and $a\Delta b \subseteq S_P$, where $a \in A_\alpha$, $b \in A_\beta$ and $\alpha, \beta \in \Gamma$. Then $\bigcup_{\delta \in \Delta} A_{\alpha\delta\beta} \subseteq S_P$. Hence $\alpha\Delta\beta \subseteq P$. Since P is a prime ideal of Γ, thus $\alpha \in P$ or $\beta \in P$. Therefore, $a \in A_\alpha \subseteq S_P$ or $b \in A_\beta \subseteq S_P$.

Conversely, let S_P be a prime ideal of S and $\alpha\Delta\beta \subseteq P$. If we choose $x \in A_\alpha$ and $y \in A_\beta$, then $x\Delta y = A_{\alpha\Delta\beta} \subseteq S_P$ so $x \in S_P$ or $y \in S_P$. Therefore, $\alpha \in P$ or $\beta \in P$. □

Let S be a Γ-semigroup and $\dot{S}$ be a $\dot{\Gamma}$-semigroup. If there exists a map $\Phi : S \longrightarrow \dot{S}$ and a bijection $f : \Gamma \longrightarrow \dot{\Gamma}$ such that

$$\Phi(x\gamma y) = \Phi(x) f(\gamma) \Phi(y),$$

for all $x, y \in S$ and $\gamma \in \Gamma$. Then we say (Φ, f) is a *homomorphism* between S and $\dot{S}$. Also, if Φ is a bijection then (Φ, f) is called an *isomorphism*, and S and $\dot{S}$ are *isomorphic*.

If Γ is a Δ-semigroup, then we say that Γ is idempotent whenever $\Gamma = \Gamma\Delta\Gamma$. If S is a (Γ, Δ)-semihypergroup where Γ is an idempotent Δ-semigroup and ρ^* is the fundamental relation on it, then by Corollary 5.1 the quotient of S on ρ^* is a $\dot{\Gamma}$-semigroup and we will prove that it is isomorphic to Γ.

Theorem 5.4. *Let S be a (Γ, Δ)-semihypergroup such that $S = \bigcup_{\gamma \in \Gamma} A_\gamma$ and Γ is a idempotent Δ-semigroup. If ρ^* is the fundamental relation on S then $S/\rho^* \cong \Gamma$.*

Proof. If $x \in S$ then there exists $\gamma_x \in \Gamma$ such that $x \in A_{\gamma_x}$. Now, we define the map Φ as follows:

$$\begin{array}{rl}\Phi : & S/\rho^* \longrightarrow \Gamma \\ & \rho^*(x) \longmapsto \gamma_x\end{array}$$

and $f : \dot{\Gamma} \longrightarrow \Gamma$ by $f(\dot{\gamma}) = \gamma$, for all $\dot{\gamma} \in \dot{\Gamma}$.

Let $\rho^*(x) = \rho^*(y)$. Then there exist $x = x_1, x_2, \cdots, x_n = y \in S$ such that $x = x_1\rho x_2 \rho x_3 \cdots, x_{n-1}\rho x_n = y$. Thus, there exist $k_1, k_2, \cdots k_n \in \mathbb{N}, \{u_{ij} \in A_{\gamma_{ij}} : 1 \leq i \leq n, 1 \leq j \leq k_i\}$ and $\{\delta_{ij} \in \Gamma : 1 \leq i \leq n-1, 1 \leq j \leq k_i - 1\}$ such that

$$\{x_i, x_{i+1}\} \subseteq \prod_{j=1}^{k_i-1} u_{ij}\delta_{ij}u_{i(j+1)} = A_{\prod_{j=1}^{k_i-1}\gamma_{ij}\delta_{ij}\gamma_{i(j+1)}},$$

for $1 \leq i \leq n-1$. Now, since the elements of $\{A_\gamma : \gamma \in \Gamma\}$ are disjoint and for every $1 \leq i \leq n$, we have

$$x_i \in A_{\prod_{j=1}^{k_i-1}\gamma_{(i-1)j}\delta_{(i-1)j}\gamma_{(i-1)(j+1)}} \cap A_{\prod_{j=1}^{k_i-1}\gamma_{ij}\delta_{ij}\gamma_{i(j+1)}},$$

then

$$A_{\prod_{j=1}^{k_i-1}\gamma_{(i-1)j}\delta_{(i-1)j}\gamma_{(i-1)(j+1)}} = A_{\prod_{j=1}^{k_i-1}\gamma_{ij}\delta_{ij}\gamma_{i(j+1)}}.$$

If we put $\gamma = \prod_{j=1}^{k_1-1}\gamma_{1j}\delta_{1j}\gamma_{1(j+1)}$ then $x, y \in A_\gamma$, so

$$\Phi(\rho^*(x)) = \gamma = \Phi(\rho^*(y)).$$

Thus Φ is well-defined.

Now, we show that (Φ, f) is a homomorphism. If $x, y \in S$ and $\delta \in \Delta$ then $x \in A_{\gamma_1}$ and $y \in A_{\gamma_2}$ for some $\gamma_1, \gamma_2 \in \Gamma$. Then

$$\begin{aligned}\Phi(\rho^*(x)\dot{\delta}\rho^*(y)) &= \Phi(\{\rho^*(z) : z \in x\delta y\}) \\ &= \Phi(\{\rho^*(z) : z \in A_{\gamma_1\delta\gamma_2}\}) \\ &= \Phi(\rho^*(x))\delta\Phi(\rho^*(y)) \\ &= \Phi(\rho^*(x))f(\dot{\delta})\Phi(\rho^*(y)).\end{aligned}$$

Let $\Phi(\rho^*(x)) = \Phi(\rho^*(y))$ and $\gamma \in \Gamma$ such that $x, y \in A_\gamma$. Then there exist $\alpha, \beta \in \Gamma$ and $\delta \in \Delta$ such that $\gamma = \alpha\delta\beta$. Now, if we choose an element $z_1 \in A_\alpha$ and $z_2 \in A_\beta$, then $\{x, y\} \subseteq A_\gamma = A_{\alpha\delta\beta} = z_1\delta z_2$. Thus $\rho^*(x) = \rho^*(y)$. Obviously, Φ is onto. Therefore, (Φ, f) is an isomorphism and the proof is completed. □

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