REDUCTION OF THE ELEMENTARY BODIES TO SYSTEMS OF MATERIAL POINTS WITH THE SAME INERTIA PROPERTIES

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In această lucrare se propun cateva modele matematice de reducere a corpurilor elementare la sisteme de puncte materiale cu aceleaşi proprietăţi de inerţie, astfel acestea vor avea: acelaşi centru de masă şi acelaşi tensor de inerţie. Plecând de la aceste modele, orice corp poate fi redus la un sistem de puncte materiale, iar determinarea proprietăţilor de inerţie devine mult mai simplă. De asemenea, corpurile reale au fost împărţite în patru clase, un aspect teoretic necesar pentru a explica modul în care se pot reduce corpurile. Această abordare de determinare a proprietăţilor de inerţie ale corpurilor este una nouă, neîntâlnită în nicio altă lucrare de specialitate.

In this paper the author proposes some mathematical models of reduction of the elementary bodies to systems of material points with the same inertia properties, which means: the same centre of mass and the same tensor of inertia. Starting from these models, each body can be reduced to a system of material points and the possibility to calculate all these properties becomes easier. Also the real bodies were divided in four classes to make a better explication of the reduction method. The reduction operation is an original one.

Keywords: centre of mass, reduction of a body, tensor of inertia, elementary body, order of body

1. Introduction

The centre of mass represents the point where the entirely weight (mass) of a body can be concentrated with the same mechanical effect like the natural body, where the distribution of the weight is a continuous function.

The moments of inertia measure how the mass of a body is distributed with respect to a point, an axis or a plane. The general expression for a system of „*n*" material points is, [1÷5]:

$$J_{\Theta} = \sum_{i=1}^{n} m_i \lambda_i^2 \qquad (1)$$

Where: Θ is the symbol of the element with respect to which is defined this moment of inertia, respectively: a point, an axis or a plane and λ_i - is the distance between the point A_i of mass m_i and the reference element: the point, the axis or the plane.

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The relation (1) shows that the moments of inertia are strictly positive. If we know the centre of mass and the moments of inertia about the main axes of some coordinates system, the inertia properties of the body are known.

The products of inertia are defined with respect to two axes of an orthogonal reference system; for example, if the reference system is Oxyz, the products of inertia with respect to the axes Ox, Oy have the expression, [1÷6]:

$$J_{xy} = \sum_{i=1}^{n} m_i x_i y_i \tag{2}$$

For a continuous body, the sums from relations (1) and (2) become integrals on the volume of the body, D, so:

$$J_{\Theta} = \int_{D} \lambda^2 \cdot dm;\ J_{xy} = \int_{D} xy \cdot dm \tag{3}$$

For a system of material points, the calculus of inertia properties means only algebraical sums, but to determine inertia properties for the continuous bodies supposes, generally, a very difficult integral calculus.

To determine the position of the centre of mass or the inertia properties, it means to divide the composite body in the elementary bodies for which relations of calculus are known.

Thus actually, an *elementary body* is considered a body for which the relations of calculus of the position of the centre of mass and the moments of inertia or products of inertia are known.

2. The order of a body

We consider necessary to use some concepts for classification of the bodies. So, depending on the dimensions, one can classify the bodies in four categories:

-*order 0 bodies* are the bodies which can be replaced by a point or a system of the material points (a point has „zero” dimensions);

- *order 1 bodies* are the bodies which can be replaced by segment of line, curve arc or a composed curve (a curve has a single dimension), which are sometimes called “bars”;

- *order 2 bodies* are the bodies which can be replaced by plane surface, a curved surface or a composed surface (a surface has two dimensions) – which are sometimes called “plates”;

- *order 3 bodies* are the real bodies which can have as exterior faces the plane surfaces or curve surfaces (the volumes have three dimensions). The bodies which have only the plane exterior surfaces - the polyhedrons - are called *simple bodies.*

The elementary simple bodies are the bodies with known inertia properties, which, through adding or extracting, can form any simple bodies from the same class with them.

3. The reduction of bodies to order 0 bodies

In introduction, we explained why it is easier to calculate the inertia properties for the system of material points. In the same mode, if we compare with the reduction of the force systems, we want to find an easy way to determine the inertia properties on a simple model. The reduction means to replace a body by an inferior order body. The elementary body, from this point of view, is a body which can be reduced to an order 0 body. All reduction operations must propose a body which has the same centre of mass and the same moments of inertia and products of inertia.

3.1. The reduction of a straight bar to an order 0 body

Theorem 1: **Any straight bar, an order 1 body, can be reduced to an order 0 body formed by a system of three material points, as follows: each end of bar has 1/6 of the mass of the body and the centre of mass has the 2/3 of the mass of the body.**

Proof: Let us consider a bar, AB, of mass *m*, and known coordinates of the ends $A(x_A, y_A)$ and $B(x_B, y_B)$ (Figure 1a), in the plane xOy. In conformity with [1], the moments of inertia and products of inertia different from zero are:

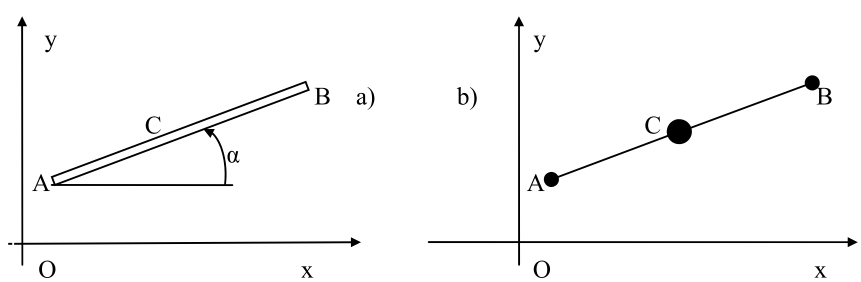

Figure 1

$$J_x = \frac{m}{3}(y_A^2 + y_B^2 + y_A y_B); J_y = \frac{m}{3}(x_A^2 + x_B^2 + x_A x_B); J_z = J_x + J_y$$
$$J_{xy} = m\left(\frac{x_A y_A + x_B y_B}{3} + \frac{x_A y_B + x_B y_A}{6}\right) \tag{4}$$

The proposed model (Figure 1b), in conformity with relations (1) and (2), has the following moments of inertia and products of inertia:

$$J_x = \frac{m}{6}y_A^2 + \frac{m}{6}y_B^2 + \frac{2m}{3}\left(\frac{y_A + y_B}{2}\right)^2 = \frac{m}{3}(y_A^2 + y_B^2 + y_A y_B);$$

$$J_y = \frac{m}{6}x_A^2 + \frac{m}{6}x_B^2 + \frac{2m}{3}\left(\frac{x_A + x_B}{2}\right)^2 = \frac{m}{3}(x_A^2 + x_B^2 + x_A x_B);\ J_z = J_x + J_y \qquad (5)$$

$$J_{xy} = \frac{m}{6}x_A y_A + \frac{m}{6}x_B y_B + \frac{2m}{3}\left(\frac{x_A + x_B}{2}\right)\left(\frac{y_A + y_B}{2}\right) = m\left(\frac{x_A y_A + x_B y_B}{3} + \frac{x_A y_B + x_B y_A}{6}\right)$$

Owing to the symmetry, the centre of the mass is obviously identical with the centre of mass of the model. The demonstration is obvious.

3.2. The reduction of the elementary order 2 bodies to an order 0 body

We found two elementary order 2 bodies: the *triangle* and the *parallelogram*. It is well known that the latter body, the parallelograms, include a large family of bodies: the rhombus, the rectangles and the squares. For any elementary body we considered *n*, the number of the vertices of the body, so that, *n*=3 for the triangles and *n*=4 for the parallelograms.

Theorem 2: **Any elementary order 2 bodies can be reduced to an order 0 body formed by a system of „n+1"material points, as follows: each vertex of the body has 1/12 of the mass of the body and the centre of mass has the rest of this mass: 3/4 - for the triangles and, respectively, 2/3 - for the parallelograms**.

Proof: The demonstration is made in xOy plane, see Figures 2 and 3.

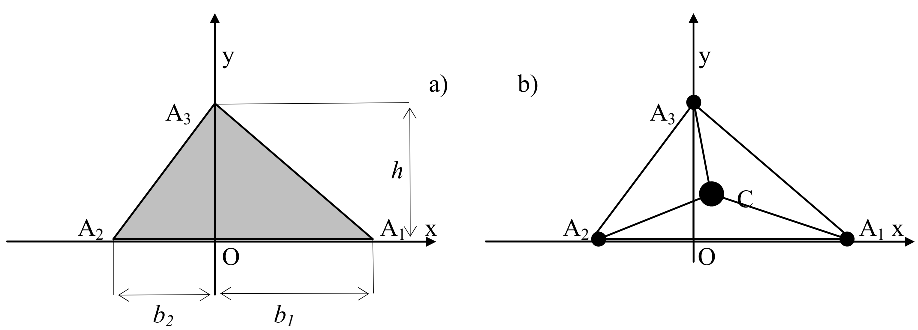

Figure 2

The triangle

Let us consider a triangle $A_1A_2A_3$, situated in xOy plane, with the vertex coordinates: $A_1(x_1 = b_1, 0, 0)$, $A_2(x_2 = -b_2, 0, 0)$, $A_3(0,\ y_3 = h,\ 0)$ and with the mass *m*. If the centre of mass is in the point C, the known coordinates of the C point are: $C\left(x_C = \frac{b_1 - b_2}{3};\ y_C = \frac{h}{3};\ z_C = 0\right)$. If the sum is noted with: $b_1+b_2=b$, in conformity with [1], the expressions of the moments of inertia and the products of inertia are:

$$J_x = \frac{mh^2}{6};\ J_y = \frac{m(b_1^3 + b_2^3)}{6b};\ J_{xy} = \frac{mh(b_1 - b_2)}{12} \tag{6}$$

The proposed model, presented in figure 2b, has the following moments of inertia and products of inertia:

$$x_C = \frac{\sum_{i=1}^{4} m_i x_i}{m} = \frac{b_1 - b_2}{3};\ y_C = \frac{\sum_{i=1}^{4} m_i y_i}{m} = \frac{h}{3}$$

$$J_x = \left(\frac{m}{12} + \frac{3m}{4\cdot 9}\right)h^2 = \frac{mh^2}{6};\ J_y = \frac{m}{12}\left(b_1^2 + b_2^2\right) + \frac{3m}{4}\frac{(b_1 - b_2)^2}{9} = \frac{m(b_1^2 + b_2^2 - b_1 b_2)}{6}; \tag{7}$$

$$J_{xy} = \frac{3m}{4}\frac{b_1 - b_2}{3}\frac{h}{3} = \frac{mh(b_1 - b_2)}{12}$$

Because the moment of inertia with respect to the axis Oz is equal to the sum $J_z = J_x + J_y$, it is not specified separately. The equalities between expressions (6) and (7) demonstrate the first part of definition 2.

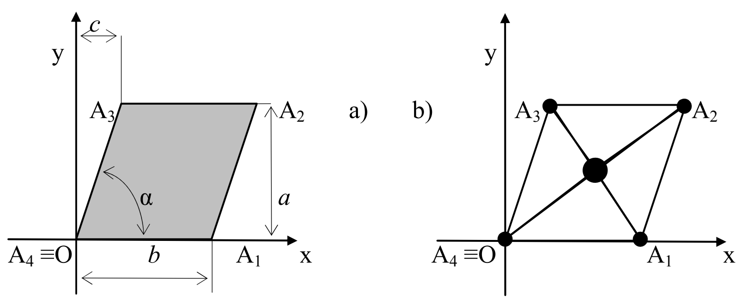

Figure 3

The parallelogram

Let us consider thee parallelogram, from Figure 3a, with the vertices $A_{i,}$ (i=1-4), situated in xOy plane.

The vertex coordinates are: $A_1(b, 0)$, $A_2(b+c, a)$, $A_3(c, a)$ and $A_4(0, 0)$.

In conformity with [1], the coordinates of the centre of mass are:

$$x_C = \frac{1}{4}\sum_{i=1}^{4} x_i = \frac{1}{2}(b + c);$$
$$y_C = \frac{1}{4}\sum_{i=1}^{4} y_i = \frac{a}{2} \tag{8}$$

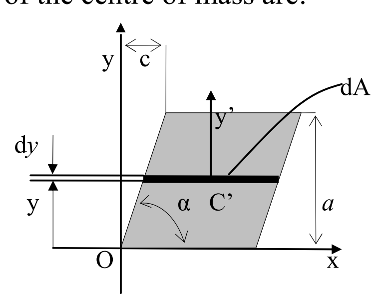

Figure 4

The moments of inertia and the products of inertia with respect to the reference system are determined using the integral calculus (Figure 4), so:

$$J_x = \int_0^a y^2 \rho dA = \rho b \int_0^a y^2 dy = \frac{\rho b a^3}{3} = \frac{ma^2}{3}$$

$$J_y = \int_0^a \frac{\rho b^2}{12} dA + \int_0^a x_{C'}^2 \rho dA = \frac{mb^2}{12} + \rho b \int_0^a (\frac{b}{2} + y\text{ctg}\alpha)^2 dy = \frac{m(b^2+c^2)}{3} + \frac{mbc}{2} \quad (9)$$

$$J_{xy} = \int_0^a y x_{C'} \rho dA = \rho b \int_0^a y(\frac{b}{2} + y\text{ctg}\alpha) dy = ma\left(\frac{b}{4} + \frac{c}{3}\right)$$

For the proposed model (see figure 3b), the moments of inertia and the products of inertia are:

$$x_C = \frac{\sum_{i=1}^{5} m_i x_i}{m} = \frac{\frac{m}{12}(2c+2b) + \frac{2m}{3}\frac{b+c}{2}}{m} = \frac{b+c}{2}; y_C = \frac{\sum_{i=1}^{5} m_i y_i}{m} = \frac{\frac{m}{12}2a + \frac{2m}{3}\frac{a}{2}}{m} = \frac{a}{2}$$

$$J_x = 2 \cdot \frac{m}{12}a^2 + \frac{2m}{3}\frac{a^2}{4} = \frac{ma^2}{3}; \quad (10)$$

$$J_y = \frac{m}{12}b^2 + \frac{m}{12}(b+c)^2 + \frac{m}{12}c^2 + \frac{2m}{3}\frac{(b+c)^2}{4} = \frac{m(b^2+c^2)}{3} + \frac{mbc}{2};$$

$$J_{xy} = \frac{m}{12}(b+c)a + \frac{m}{12}ca + \frac{2m}{3}\frac{1}{4}(b+c)a = ma\left(\frac{b}{4} + \frac{c}{3}\right)$$

The equality between expression (8), (9) and (10) demonstrate the second part of theorem 2.

3.3. The reduction of the elementary order 3 bodies to an order 0 body

In the polyhedron family, we will name the pyramids and the prisms as *simple polyhedra*. It is already demonstrated that a polyhedron can be divided in prism and pyramids, added or extracted. We found two elementary order 3 bodies: any tetrahedron and any prism with a parallelogram base. In fact any pyramid can be divided in a number of tetrahedra and any prism can be divided in a number of prisms with a parallelogram base and tetrahedra.

3.4. The reduction of a tetrahedron

Theorem 3.1: **Any tetrahedron, order 3 body, can be reduced to an order 0 body formed by a system of 5 material points as follows: each vertex of the tetrahedron has 1/20 of the mass of the body and the centre of mass has the 4/5 of the mass, the rest.**

Proof: First, we calculate the inertia properties of the tetrahedron in classic way.

3.4.1. The determination of the centre of mass and the moments of inertia

Let us consider a tetrahedron, $A_1A_2A_3A_4$, with the face $A_1A_2A_3$ situated in xOy plane and the vertex A_1 on the Ox axis and the vertex A_4 on the Oz axis.The vertex coordinates of the tetrahedron are: $A_1(x_1, 0, 0)$, $A_2(x_2, y_2, 0)$, $A_3(x_3, y_3, 0)$ and $A_4(0, 0, z_4)$ (Figure 5).

To obtain the values of the inertia properties we make the integral calculus for that tetrahedron. In Figure 5 is drawn the tetrahedron on which we make a section with two parallel plans, at height „*z*", we obtain the elementary volume d*V*, proportional with the elementary mass d*m*.

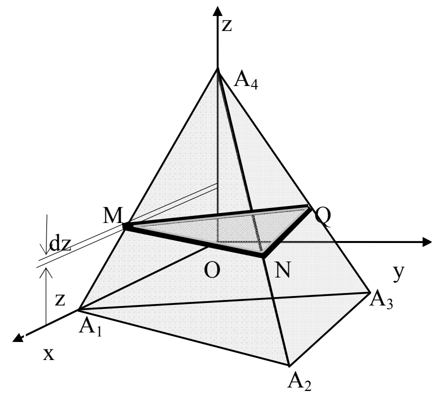

Figure 5

To determine the position of the centre of the mass we use the relations:

$$x_C = \frac{\int_D x dV}{\int_D dV}; y_C = \frac{\int_D y dV}{\int_D dV}; z_C = \frac{\int_D z dV}{\int_D dV}; \tag{11}$$

$$dV = \frac{1}{2}(x_M y_N + x_N y_Q - x_Q y_N - x_M y_Q)dz \tag{12}$$

Because the triangles ΔA_4MN and $\Delta A_4A_1A_2$ are similar, and also the triangles ΔA_4NQ and $\Delta A_4A_2A_3$ are similar too, it results:

$$\begin{gathered} x_M = \frac{x_1}{z_4}(z_4 - z); x_N = \frac{x_2}{z_4}(z_4 - z); y_N = \frac{y_2}{z_4}(z_4 - z); \\ x_Q = \frac{x_3}{z_4}(z_4 - z); y_Q = \frac{y_3}{z_4}(z_4 - z) \end{gathered} \tag{13}$$

In expressions (11) *x, y, z* are the coordinates of the centre of mass of the elementary volume, the triangle ΔQNM, thus:

$$\begin{gathered} x = \frac{x_M + x_N + x_Q}{3} = \frac{x_1 + x_2 + x_3}{3z_4}(z_4 - z); \\ y = \frac{y_N + y_Q}{3} = \frac{y_2 + y_3}{3z_4}(z_4 - z); \ z = z \end{gathered} \tag{14}$$

The volume of pyramid is:

$$V = \frac{z_4}{3} A_{\Delta A_1 A_2 A_3} = \frac{z_4}{6}(x_1 y_2 + x_2 y_3 - x_3 y_2 - x_1 y_3) \quad (15)$$

If we replace the term from (15) in (12), between the elementary volume and the volume we found the relation:

$$dV = \frac{1}{2}\frac{x_1 y_2 + x_2 y_3 - x_3 y_2 - x_1 y_3}{z_4^2}(z_4 - z)^2 dz = \frac{3V}{z_4^3}(z_4 - z)^2 dz \quad (16)$$

Hence, the coordinates of the centre of mass of the pyramid are:

$$x_C = \frac{\int_0^{z_4} 3V \frac{x_1 + x_2 + x_3}{3z_4^4}(z_4 - z)^3 dz}{V} = \frac{V \frac{x_1 + x_2 + x_3}{z_4^4}\frac{z_4^4}{4}}{V} = \frac{x_1 + x_2 + x_3}{4};$$

$$y_C = \frac{\int_0^{z_4} 3V \frac{y_2 + y_3}{3z_4^4}(z_4 - z)^3 dz}{V} = \frac{y_2 + y_3}{4};\ z_C = \frac{\int_0^{z_4} \frac{3V}{z_4^3}(z_4 - z)^2 z dz}{V} = \frac{z_4}{4} \quad (17)$$

The relations (17) show the formula of the coordinates of the centre of mass of a tetrahedron. In fact, each coordinate is a quarter of the sum of the vertex coordinates with respect to the same axis.

For the determination of the moments of inertia and products of inertia we use the reduction operation for the triangle MNQ. So, the moments of inertia and products of inertia of the elementary body of the volume d*V* and mass d*m*, are, in fact, the moment of inertia of the triangle NMQ. So, for a homogenous piramid of mass *m* (ρ=const.), we get:

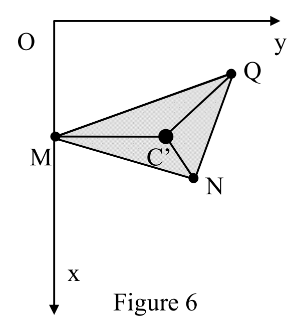

Figure 6

$$dm = \rho dV = \frac{3\rho V}{z_4^3}(z_4 - z)^2 dz = \frac{3m}{z_4^3}(z_4 - z)^2 dz \quad (18)$$

In conformity with reduction operation, the vertices M, N and Q of the triangle have: *dm*/12 and the centre of mass, C', has 3 *dm*/4, see Figure 6.

The elementary moment of inertia with respect to Ox axis is:

$$dJ_x = \sum m_i (y_i^2 + z_i^2) = \frac{dm}{12}(y_N^2 + y_Q^2 + 3z^2) + \frac{3dm}{4}\left[\frac{(y_N + y_Q)^2}{9} + z^2\right] \quad (19)$$

$$dJ_x = \frac{3m(z_4 - z)^4}{z_4^5}\frac{y_2^2 + y_3^2 + y_2 y_3}{6}dz + \frac{3m(z_4 - z)^2 z^2}{z_4^3}dz \quad (20)$$

By integrating this, we obtain:

$$J_x = \frac{m}{z_4^5}\frac{y_2^2 + y_3^2 + y_2 y_3}{2}\int_0^{z_4}(z_4 - z)^4 dz + \frac{3m}{z_4^3}\int_0^{z_4} z^2 (z_4 - z)^2 dz = m\frac{y_2^2 + y_3^2 + y_2 y_3 + z_4^2}{10} \quad (21)$$

Similarly, we can calculate the moment of inertia with respect to the Oy axis:

$$dJ_y = \sum m_i(x_i^2 + z_i^2) = \frac{dm}{12}(x_M^2 + y_N^2 + y_Q^2 + 3z^2) + \frac{3dm}{4}\left[\frac{(x_M + y_N + y_Q)^2}{9} + z^2\right] \quad (22)$$

After integrating it is obtained:

$$J_y = m\frac{x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_2x_3 + x_3x_1 + z_4^2}{10} \quad (23)$$

And with respect to the Oz axis:

$$dJ_z = \sum m_i(x_i^2 + y_i^2) = \frac{dm}{12}(x_M^2 + x_N^2 + x_Q^2 + y_N^2 + y_Q^2) + \frac{3dm}{4}\left[\frac{(x_M + x_N + x_Q)^2}{9} + \frac{(y_N + y_Q)^2}{9}\right] \quad (24)$$

In expression (24) all the terms are already calculated in relations (21) and (23), so the moment of inertia with respect to the Oz axis is:

$$J_z = m\frac{x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_2x_3 + x_3x_1 + y_2^2 + y_3^2 + y_2y_3}{10} \quad (25)$$

3.4.2. The determination of the products of inertia

The product of inertia of the elementary triangle is:

$$dJ_{xy} = \sum m_i x_i y_i = \frac{dm}{12}(x_N y_N + x_Q y_Q) + \frac{3dm}{4}\frac{(x_M + x_N + x_Q)(y_N + y_Q)}{9} \quad (26)$$

$$dJ_{xy} = \frac{3m(z_4 - z)^4}{12z_4^5}\left[2(x_2y_2 + x_3y_3) + (x_1y_2 + x_1y_3 + x_2y_3 + x_3y_2)\right]dz \quad (27)$$

After the integration in relation (27), we obtain:

$$J_{xy} = \frac{m}{20}\left[2(x_2y_2 + x_3y_3) + (x_1y_2 + x_1y_3 + x_2y_3 + x_3y_2)\right] \quad (28)$$

For the other products of inertia we make a similar calculus, so:

$$dJ_{yz} = \sum m_i y_i z_i = \frac{dm}{12}(y_N z + y_Q z) + \frac{3dm}{4}\frac{(y_N + y_Q)}{3}z \quad (29)$$

Replacing:

$$dJ_{yz} = \frac{dm}{3}(y_N + y_Q)z = \frac{m}{z_4^4}(y_2 + y_3)(z_4 - z)^3 z \quad (30)$$

After integration calculus:

$$J_{yz}=\frac{m(y_2+y_3)z_4}{20} \quad (31)$$

Similar:

$$J_{zx}=\frac{m(x_1+x_2+x_3)z_4}{20} \quad (32)$$

In conformity with theorem 3.1, the moments of inertia of the proposed model, are:

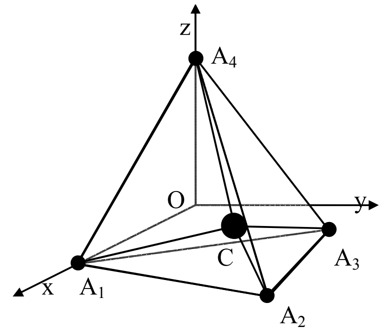

Figure 7

$$J_x=\frac{m}{20}y_2^2+\frac{m}{20}y_3^2+\frac{m}{20}z_4^2+\frac{4m}{5}\frac{(y_2+y_3)^2+z_4^2}{16}=\frac{m}{10}(y_2^2+y_3^2+y_2y_3+z_4^2) \quad (33)$$

$$J_y=\frac{m}{20}x_1^2+\frac{m}{20}x_2^2+\frac{m}{20}x_1^2+\frac{m}{20}z_4^2+\frac{4m}{5}\frac{(x_1+x_2+x_3)^2+z_4^2}{16}= \\ =\frac{m}{10}(x_1^2+x_2^2+x_3^2+x_1x_2+x_1x_3+x_2x_3+z_4^2) \quad (34)$$

$$J_z=\frac{m}{20}x_1^2+\frac{m}{20}(x_2^2+y_2^2)+\frac{m}{20}(x_3^2+y_3^2)+\frac{4m}{5}\frac{(x_1+x_2+x_3)^2+(y_2+y_3)^2}{16}= \\ =\frac{m}{10}(x_1^2+x_2^2+x_3^2+x_1x_2+x_1x_3+x_2x_3+y_2^2+y_3^2+y_2y_3) \quad (35)$$

The products of inertia of the proposed model are:

$$J_{xy}=\frac{m}{20}(x_1y_2+x_1y_3+2x_2y_2+x_2y_3+x_3y_2+2x_3y_3) \\ J_{yz}=\frac{m}{20}(y_2z_4+y_3z_4);\quad J_{zx}=\frac{m}{20}(x_1+x_2+x_3)z_4 \quad (36)$$

Relations (33)-(36) prove the validity of the proposed model.

3.5. The reduction of a prism with parallelogram faces

Theorem 3.2: **Any prism with parallelogram bases, an order 3 body, can be reduced to an order 0 body formed by a system of 9 material points as follows: each vertex of the prism has 1/24 of the mass of the prism and the centre of mass has 2/3 of the mass (the rest).**

Proof:

A prism with parallelogram bases $A_1A_2A_3A_4A_5A_6A_7A_8$, as represented in Figure

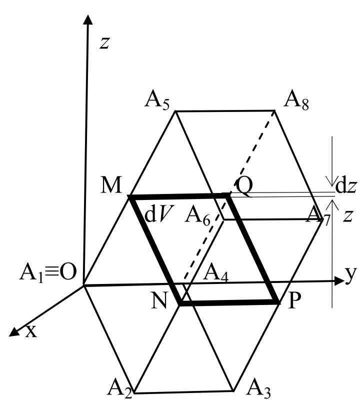

Figure 8

11, has, with respect to the Cartesian coordinate system Oxyz, the points O≡A_1 and the base $A_1A_2A_3A_4$ situated in xOy plane, with the edge A_1A_4 situated on Oy axis.

The coordinates of the prism vertices are: $A_1(0, 0, 0)$, $A_2(x_2, y_2, 0)$, $A_3(x_3, y_3, 0)$, $A_4(0, y_4, 0,)$, $A_5(x_5, y_5, h)$, $A_6(x_6, y_6, h)$, $A_7(x_7, y_7, h)$, $A_8(x_8, y_8, h)$.

The unit vector of generator direction is: $\vec{e} = \cos\alpha\vec{i} + \cos\beta\vec{j} + \cos\gamma\vec{k}$

The vectors of the edges have the form: $\overrightarrow{OA_5} = \overrightarrow{A_2A_6} = \overrightarrow{A_3A_7} = \overrightarrow{A_4A_8} = \lambda\vec{e}$

To simplify the notations, it is easier to write the coordinates as a function of scalars. We note the scalar: $h = \lambda\cos\lambda$, and the vertex coordinates as:

$$\begin{aligned} &x_2 = a; y_2 = c; x_3 = a; y_3 = b + c; x_4 = 0; y_4 = b; \\ &x_5 = \lambda\cos\alpha;\ y_5 = \lambda\cos\beta; x_6 = a + \lambda\cos\alpha;\ y_6 = c + \lambda\cos\beta; \\ &x_7 = a + \lambda\cos\alpha;\ y_7 = b + c + \lambda\cos\beta;\ x_8 = a + \lambda\cos\alpha;\ y_8 = b + \lambda\cos\beta; \end{aligned} \tag{37}$$

3.5.1. The integral calculus of the moments and the products of inertia

The prism was sectioned with two parallel plans at the height „z" and dz distance between them. The elementary body has the mass and the volume:

$$dm = \rho dV;\ dV = x_2 y_4 dz = abdz \tag{38}$$

If we make the notation: OM=u, we can write the coordinates expression:

$$z = u\cos\gamma = z_M = z_N = z_Q = z_P \tag{39}$$

$$\begin{aligned} &x_M = u\cos\alpha; y_M = u\cos\beta;\ x_N = a + u\cos\alpha;\ y_N = c + u\cos\beta; \\ &x_Q = a + u\cos\alpha; y_Q = b + c + u\cos\beta; x_P = u\cos\alpha; y_Q = b + u\cos\beta; \end{aligned} \tag{40}$$

The moment of inertia of the elementary body with respect to Ox axis is:

$$dJ_x = dm\frac{b^2 + c^2}{12} + dm\frac{(y_M + y_N + y_P + y_Q)^2 + 16z^2}{16} \tag{41}$$

Integrating:

$$J_x = \frac{m(b^2 + c^2 + h^2 + y_5^2)}{3} + \frac{m(b + c)y_5}{2} \tag{42}$$

The moment of inertia of the elementary body with respect to Oy axis is:

$$dJ_y = dm\frac{a^2}{12} + dm\frac{(x_M + x_N + x_P + x_Q)^2 + 16z^2}{16} \tag{43}$$

Integrating:

$$J_y = \frac{ma^2}{12} + \rho\frac{ab}{16}\int_0^h [(2a + 4\frac{\cos\alpha}{\cos\lambda}z)^2 + 16z^2]dz = \frac{m(a^2 + h^2 + x_5^2)}{3} + \frac{max_5}{2} \tag{44}$$

The moment of inertia of the elementary body with respect to Oz axis is:

$$dJ_z = dm\frac{a^2+b^2+c^2}{12} + dm\frac{(x_M+x_N+x_P+x_Q)^2+(y_M+y_N+y_P+y_Q)^2}{16} \quad (45)$$

Integrating:

$$J_z = \frac{m(a^2+b^2+c^2+x_5^2+y_5^2)}{3} + \frac{m(ax_5+by_5+cy_5)}{2} \quad (46)$$

The product of inertia of the elementary body with respect to Ox and Oy axes is:

$$dJ_{xy} = dm\frac{(x_M+x_N+x_P+x_Q)(y_M+y_N+y_P+y_Q)}{16} \quad (47)$$

Integrating:

$$J_{xy} = \frac{mx_5y_5}{3} + \frac{m[(b+c)x_5+ay_5]}{4} \quad (63)$$

$$J_{yz} = \frac{m(b+c)h}{16} + \frac{my_5h}{12};$$

$$J_{zx} = \frac{mah}{16} + \frac{mx_5h}{12} \quad (48)$$

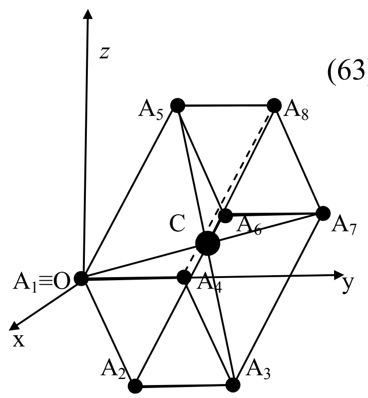

Figura 9

The proposed model is presented in Figure 9. The coordinates of the centre of mass can be determinate with the formula:

$$x_C = \frac{1}{8}\sum_{i=1}^{8} x_i;\, y_C = \frac{1}{8}\sum_{i=1}^{8} y_i;\, z_C = \frac{1}{8}\sum_{i=1}^{8} z_i; \quad (49)$$

The moments of inertia of this model can be calculated with the following relations. The moment of inertia with respect to Ox axis is:

$$J_x = \frac{m}{24}y_2^2 + \frac{m}{24}y_3^2 + \frac{m}{24}y_4^2 + \frac{m}{24}(y_5^2+h^2) + \frac{m}{24}(y_6^2+h^2) + \frac{m}{24}(y_7^2+h^2) + \frac{m}{24}(y_8^2+h^2) + \frac{2m}{3}(y_C^2+\frac{1}{4}h^2) \quad (50)$$

Replacing the values from relations (37) and (49) in (50), we obtain:

$$J_x = \frac{m(b^2+c^2+h^2+y_5^2)}{3} + \frac{m(b+c)y_5}{2} \quad (51)$$

The moment with respect to the Oy axis:

$$J_y = \frac{m}{24}x_2^2 + \frac{m}{24}x_3^2 + \frac{m}{24}x_4^2 + \frac{m}{24}(x_5^2+h^2) + \frac{m}{24}(x_6^2+h^2) + \frac{m}{24}(x_7^2+h^2) + \frac{m}{24}(x_8^2+h^2) + \frac{2m}{3}(x_C^2+\frac{1}{4}h^2) \quad (52)$$

Replacing the values from relations (37) and (49) in (52), we obtain:

$$J_y = \frac{m(a^2 + h^2 + x_5^2)}{3} + \frac{max_5}{2} \quad (53)$$

The moment with respect to the Oz axis:

$$J_z = \frac{m}{24}(x_2^2 + y_2^2) + \frac{m}{24}(x_3^2 + y_3^2) + \frac{m}{24}y_4^2 + \frac{m}{24}(x_5^2 + y_5^2) + \frac{m}{24}(x_6^2 + y_6^2) + + \frac{m}{24}(x_7^2 + y_7^2) + \frac{m}{24}(x_8^2 + y_8^2) + \frac{2m}{3}(x_C^2 + y_C^2) \quad (54)$$

$$J_z = \frac{m(a^2 + b^2 + c^2 + x_5^2 + y_5^2)}{3} + \frac{m(ax_5 + by_5 + cy_5)}{2} \quad (55)$$

The product of inertia with respect to Ox and Oy axes:

$$J_{xy} = \frac{m}{24}x_2y_2 + \frac{m}{24}x_3y_3 + \frac{m}{24}x_5y_5 + \frac{m}{24}x_6y_6 + \frac{m}{24}x_7y_7 + \frac{m}{24}x_8y_8 + \frac{2m}{3}x_Cy_C \quad (56)$$

$$J_{xy} = \frac{mx_5y_5}{3} + \frac{m[(b+c)x_5 + ay_5]}{4} \quad (57)$$

The product of inertia with respect to Oy and Oz axes:

$$J_{yz} = \frac{m}{24}y_5h + \frac{m}{24}y_6h + \frac{m}{24}y_7h + \frac{m}{24}y_8h + \frac{2m}{3}y_C\frac{h}{2} \quad (58)$$

$$J_{yz} = \frac{m(b+c)h}{16} + \frac{my_5h}{12} \quad (59)$$

The product of inertia with respect to Oz and Ox axes:

$$J_{zx} = \frac{m}{24}x_5h + \frac{m}{24}x_6h + \frac{m}{24}x_7h + \frac{m}{24}x_8h + \frac{2m}{3}x_C\frac{h}{2} \quad (60)$$

Finally:

$$J_{zx} = \frac{mah}{16} + \frac{mx_5h}{12} \quad (61)$$

The centre of mass of the proposed model, C', can be easy verified using similar relations with (7).

If the relations (51), (53), (55), (57), (59) and (61) are compared with the relations from chapter 3.5.1, they are similar and the proposed model has the same inertia properties thus the theorem is proved.

4. Conclusions

In fact, all the applied sciences use models to study the natural phenomenon and the reduction of the bodies help us to work easier and to calculate the proprieties with a minimum mathematical effort. This reduction operation is an original approach which can open a way in this direction.

First, we find the elementary bodies which can be reduced. For the polyhedron family order 3 bodies and, also, for the degenerated polyhedron solids, order 2 or order 1 bodies, we find the following elementary bodies: any tetrahedron, any prism with parallelogram bases, any triangle, any parallelogram and any straight bar. Because any polyhedron can be formed with these elementary bodies, which are presented above, in fact any polyhedron can be reduced to a system of material points.

Also, we know how to extend this reduction operation for other solid family and we suppose that it will be extended for the particular solid families.

Let us remark that the reduction operation presents many advantages if it is compared with the theoretical study. Also, it is easy to apply this operation in computer programs. We know that, at this time, there are many computer programs which can calculate all the inertia properties for any solid body that can be designed in CAD technologies, and our approach can improve the future programs. Let us consider this study a step in this direction.

We can not evaluate now all the applications of this method but we are sure that many applications would be possible, and not only in theoretical mechanics. As a consequence of the ideas presented in this paper, there is the possibility to reduce other bodies to inferior order bodies. We will present all these in other papers.

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