ADVANCES ON HESSIAN STRUCTURES

Gabriel Bercu[1], Claudiu Corcodel[2], Mihai Postolache[3]

Demonstrăm că geodezicele unei varietăţi Riemanniene plate sunt transversale pe varietatea de nivel constant. Considerăm apoi varietatea $M = \mathbb{R}^2$, înzestrată cu metrica Riemanniană $\bar{g}(x, y) = \operatorname{diag}(g(x)\ ,\ 1)$, unde g este o funcţie reală pozitivă, de clasă C^∞. Dată o funcţie f de clasă C^∞, determinăm metrica produsă de Hessiana $h = \nabla_g^2 f$, indicăm condiţii suficiente ca metricile h şi $\bar{g}$ să genereze aceeaşi conexiune şi iniţiem un studiu al structurilor Hessiene iterative 2D [5]*. Apoi, folosind tehnicile din* [7]*, introducem noi exemple de funcţii autoconcordante. Dacă* §1 *conţine fundamente teoretice introductive,* §2 *conţine rezultate originale, iar* §3 *unifică aceste rezultate cu cele din lucrările noastre* [5], [6].

We show that the geodesics of a flat Riemannian space are transversal to the constant level manifold. Next, consider the manifold $M = \mathbb{R}^2$, endowed with the Riemannian metric $\bar{g}(x, y) = \operatorname{diag}(g(x)\ ,\ 1)$, where g is a positive real function of C^∞-class. Given a function f of C^∞-class, we determine the metric produced by the Hessian $h = \nabla_g^2 f$ and provide sufficient conditions for h and $\bar{g}$ to give rise to the same connection. Then we initiate a study on iterative 2D *Hessian structures* [5]*. Using essentially the techniques from* [7]*, we introduce new examples of self-concordant functions. While* §1 *introduces the general setting,* §2 *contains new results, and* §3 *unifies these results with our works* [5], [6].

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1. Introduction and general setting

The Hessian Riemannian structures are very useful tools for practical problems. To give some examples, we underline their applications in economic theory, in system modeling and optimization as well as in statistical theory. That is why they are intensively studied by famous scientists in the world. In [8], a version of Hessian metrics is used by S. Y. Cheng and S. T. Yau to define a canonical Riemannian metric on any convex domain, using the solution of a Monge-Ampère equation. In [19], T. Sasaki studied hyperbolic affine hyperspheres via Hessian metrics, while in [24] B. Totaro studied the curvature of a Hessian metric. In [17], Y. Nesterov and M. J. Todd studied the Riemannian geometry defined on a convex subset of $\mathbb{R}^n$ by the Hessian of a self-concordant barrier function. In [31], E. Vinberg used this

[1]Lecturer, Dunărea de Jos University of Galaţi, Department of Mathematics, 800008 Domnească Street, No. 47, Galaţi, Romania, e-mail: gbercu@ugal.ro

[2]Teacher, Technical College of Nehoiu, Buzău County, Romania

[3]Professor, Faculty of Applied Sciences, University "Politehnica" of Bucharest, Romania

class of metrics to define a canonical Riemannian metric on an arbitrary convex cone. Closely related to this topic is the research of Shima on Hessian manifolds (see [20]÷[22]) and the recent surveys on Hessian metrics by Duistermaat [12] and Shima-Yagi [23]. For other relevant issues on this field, we refer the reader to the following works: [15], [16], [30].

Inspired and motivated by the ongoing work on this topic reported above, this paper aims to establish some new results on Hessian structures. It is organized as follows. Next, in this section, the notations and assumptions are given. In §2 we study the geodesics of a flat space, while in §3, we join our main results with those in [5]. Finally, we conclude the paper and suggest possible further development.

A pseudo-Riemannian metric of signature (p, q) on a smooth manifold M of dimension $n = p + q$ is a smooth symmetric differentiable 2-form g on M such that, at each point x of M, g_x is non-degenerate on T_xM with the signature (p, q). We call (M, g) *a pseudo-Riemannian manifold*, [21], [22].

Given a pseudo-Riemannian manifold (M, g), the fundamental theorem of pseudo-Riemannian geometry states that there exists a unique linear connection ∇_g on M, called the Levi-Civita connection (of g), such that the following two assertions hold good:

a) ∇_g is metric (i.e. $\nabla_g g = 0$); b) ∇_g is torsion-free (i.e. $T = 0$).

If $(U, x^1, \dots, x^n)$ is a coordinate chart on M, then the Christoffel symbols Γ^k_{ij} of the Levi-Civita connection are related to the functions g_{ij} by the formulas

$$\Gamma^k_{ij} = \frac{1}{2} g^{k\ell} \left(\frac{\partial g_{\ell i}}{\partial x^j} + \frac{\partial g_{j\ell}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^\ell} \right),$$

while the curvature R has the components

$$R^\ell_{ijk} = \frac{\partial \Gamma^\ell_{ki}}{\partial x^j} - \frac{\partial \Gamma^\ell_{ji}}{\partial x^k} + \Gamma^r_{ki}\Gamma^\ell_{jr} - \Gamma^r_{ji}\Gamma^\ell_{kr}. \tag{1}$$

Note that in local coordinates a geodesic $\gamma(t) = (x^i(t))_{i=1,\dots,n}$ satisfies a system of n second order differential equations

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0, \qquad i = 1, \dots, n.$$

If $f \colon M \to \mathbb{R}$ is a smooth function, then the second covariant derivative

$$\nabla^2_g f = \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial f}{\partial x^k} \right) dx^i \otimes dx^j$$

is called the Hessian of f, [23], [27].

Let us suppose that the Hessian $h = \nabla^2_g f$ is non-degenerate. Then h is a pseudo-Riemannian metric which produces the Levi-Civita connection ∇_h and the Christoffel symbols $\bar{\Gamma}^k_{ij}$.

Throughout this paper, we shall use the following notations:

$$f_{,i} = \frac{\partial f}{\partial x^i}; \quad f_{,ij} = \frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^m_{ij} f_{,m}; \quad f_{,ijk} = \frac{\partial f_{,ij}}{\partial x^k} - \Gamma^\ell_{ki} f_{,\ell j} - \Gamma^\ell_{kj} f_{,\ell i}. \tag{2}$$

We have, [3]

Theorem 1.1. *Let $f,^{pk}$ be the contravariant components of the pseudo-Riemannian metric $h_{pk} = f_{,pk}$ and R^m_{ijk} be the components of the curvature tensor field produced by the pseudo-Riemannian metric g_{ij}. Then the components of Levi-Civita connection ∇_h are given by the following formula*

$$\bar{\Gamma}^p_{ij} = \Gamma^p_{ij} + \frac{1}{2} f,^{kp} [f_{,ijk} + (R^m_{ikj} + R^m_{jki}) f_{,m}].$$

As a direct consequence of Theorem 1.1, we can find the geodesics of this manifold as

$$\ddot{x}^p + \left[\Gamma^p_{ij} + f,^{pk} \left(\frac{1}{2} f_{,ijk} + R^\ell_{ikj} f_{,\ell}\right)\right] \dot{x}^i \dot{x}^j = 0, \quad p = 1, \ldots, n.$$

This result is generalization of Theorem 2.1 from [17] in the pseudo-Riemannian case, and it is used in [4] for finding the explicit form of geodesics of a Hessian manifold. Moreover, in [4] a solution study is provided by means of asimptotic approach and computer experiments.

2. **Geodesics of a flat space**

Consider (M, g) be a flat space. Then we have

$$\Gamma^p_{ij} = 0, \quad R^l_{ijk} = 0, \quad f_{,ij} = \frac{\partial^2 f}{\partial x^i \partial x^j}, \quad f_{,ijk} = \frac{\partial^3 f}{\partial x^i \partial x^j \partial x^k}.$$

According to Theorem 1.1, the components of the Levi-Civita connection ∇_h are given by the formula

$$\bar{\Gamma}^p_{ij} = \frac{1}{2} f,^{pk} \frac{\partial^3 f}{\partial x^i \partial x^j \partial x^k}, \tag{3}$$

and the system of geodesics becomes

$$\ddot{x}^p + \frac{1}{2} f,^{pk} f_{,ijk} \dot{x}^i \dot{x}^j = 0, \quad p = 1, \ldots, n. \tag{4}$$

Looking at the equations (4), we remark that we have

$$\frac{\partial}{\partial x^k}\Big(2 \frac{\partial f}{\partial x^i} \ddot{x}^i + \frac{\partial^2 f}{\partial x^i \partial x^j} \dot{x}^i \dot{x}^j\Big) = 0,$$

so there exists a real constant a such that

$$2 \frac{\partial f}{\partial x^i} \ddot{x}^i + \frac{\partial^2 f}{\partial x^i \partial x^j} \dot{x}^i \dot{x}^j = 2a. \tag{5}$$

But (5) can be written as $\dfrac{\partial f}{\partial x^i} \ddot{x}^i + \sum\limits_{i \leq j} \dfrac{\partial^2 f}{\partial x^i \partial x^j} \dot{x}^i \dot{x}^j = a$, or

$$\frac{d}{dt}\Big(\frac{\partial f}{\partial x^i} \dot{x}^i\Big) = a. \tag{6}$$

From (6), there exists b such that $\dfrac{\partial f}{\partial x^i} \dot{x}^i = at + b$, or $\dfrac{d}{dt}(f(x(t)) = at + b$. It follows

$$f(x(t)) = \frac{at^2}{2} + bt + c.$$

Theorem 2.1. *Let (M, g) be a flat Riemannian space and $f\colon M \to \mathbb{R}$ a C^2 regular function. The geodesics of the Riemannian manifold $(M, f_{,ij})$ are transversal to the constant level manifold $f(x) = c$.*

Using (3) we deduce that when the source manifold (M, g) is a flat one, then the curvature tensor field $\bar{R}$ of the pseudo-Riemannian manifold (M, h), with $h = \nabla_g^2 f$ has the components

$$\bar{R}_{aijk} = -\frac{1}{4} f,^{pr} \left(f_{,kip} f_{,jra} - f_{,jip} f_{,kra}\right).$$

3. Advances on Hessian structures

The goal of this section is to join the present research with our recent results published in [5]. That is why, we have to remember some background introduced in detail in [5] and then introduce the results. Theorem 3.1 and Theorem 3.2, whose background is introduced bellow, offer a different alternative to the result in [5].

Let us consider the manifold $M = \mathbb{R}^2$, endowed with the Riemannian metric $\bar{g}$ of diagonal type, $\bar{g}(x, y) = \operatorname{diag}(g(x)\ ,\ 1)$. Here g is a positive real function of C^∞-class. It results that $\bar{g}$ produces a Riemannian connection whose components are given by

$$\Gamma_{11}^1 = \frac{d}{dx} \ln \sqrt{g}, \quad \Gamma_{12}^1 = \Gamma_{21}^1 = \Gamma_{11}^2 = \Gamma_{21}^2 = \Gamma_{22}^2 = \Gamma_{12}^2 = \Gamma_{22}^1 = 0. \tag{7}$$

Using (1), we find the components of the curvature tensor as follows:

$$R_{ijk}^h = 0, \quad \text{for all } i, j, k, h \text{ in } \{1, 2\}.$$

We underline the role of metric $\bar{g}$ in the study of our problems stated bellow.

On M, let be given the real function $f(x, y) = p(x) + r(y)$, where p and r are real functions, defined on $\mathbb{R}$, of C^∞-class. By (2), we get

$$f_{,11} = p'' - \Gamma_{11}^1 p', \quad f_{,12} = f_{,21} = 0, \quad f_{,22} = r'', \tag{8}$$

and respectively

$$\begin{aligned} f_{,111} &= \frac{d}{dx}\left(p'' - \Gamma_{11}^1 p'\right) - 2\Gamma_{11}^1 \left(p'' - \Gamma_{11}^1 p'\right), \\ f_{,112} &= f_{,121} = f_{,211} = f_{,122} = f_{,212} = f_{,221} = 0, \\ f_{,222} &= r'''. \end{aligned}$$

We impose $h = \nabla_{\bar{g}}^2 f$ be positive definite. Since

$$h(x, y) = \operatorname{diag}\left(p''(x) - \Gamma_{11}^1(x) p'(x)\ ,\ r''(y)\right) \tag{9}$$

this means

$$p''(x) - \Gamma_{11}^1(x) p'(x) > 0, \qquad r''(y) > 0. \tag{10}$$

If relations (10) are satisfied, then h is a new Riemannian metric, which produces the Levi-Civita connection ∇_h and the Christoffel symbols $\bar{\Gamma}_{ij}^p$ given by Theorem

1.1. In our case, we get

$$\bar{\Gamma}^1_{11} = \frac{1}{2}\frac{\frac{d}{dx}\left(p'' - \Gamma^1_{11}p'\right)}{p'' - \Gamma^1_{11}p'},$$
$$\bar{\Gamma}^2_{11} = \bar{\Gamma}^1_{12} = \bar{\Gamma}^1_{21} = \bar{\Gamma}^2_{12} = \bar{\Gamma}^2_{21} = \bar{\Gamma}^1_{22} = 0,$$
$$\bar{\Gamma}^2_{22} = \frac{1}{2}\frac{r'''}{r''}.$$

$\star$ PROBLEM 1. Find the functions f whose Hessian $h = \nabla^2_{\bar{g}} f$ produces the same connection as the initial metric $\bar{g}$.

To solve this problem, we have to write the explicit form of the following two conditions:

$$p''(x) - \Gamma^1_{11}(x)p'(x) = g(x), \quad r''(y) = 1, \tag{11}$$

given that $\Gamma^1_{11}(x) = \frac{1}{2}\frac{g'(x)}{g(x)}$.

If we denote $p' = s$, the first equation in (11) leads to $2gs' - g's = 2g^2$. Having in mind this equality, we get $\left(\frac{s^2}{g}\right)' = 2s$, therefore we have

$$\frac{p'^2}{2p} = g. \tag{12}$$

• If we think relation (12) as an equation with the unknown g, then we obtain

Theorem 3.1. *On the Riemannian manifold*

$$(\mathbb{R}^2, \bar{g}), \quad \bar{g}(x,y) = \mathrm{diag}\Big(\frac{p'^2(x)}{2p(x)}\ ,\ 1\Big)$$

the Hessian of the function $f(x,y) = p(x) + \frac{1}{2}y^2 + ay + b$*, where* a *and* b *are real constants, produces the same connection as the initial metric* $\bar{g}$*.*

• If we think relation (12) as an equation with the unknown p, then we obtain

Theorem 3.2. *Let be given the Riemannian manifold*

$$(\mathbb{R}^2, \bar{g}), \quad \bar{g}(x,y) = \mathrm{diag}\left(g(x)\ ,\ 1\right).$$

If we suppose that the function p *satisfies equation* (12)*, then the Hessian of the function* $f(x,y) = p(x) + \frac{1}{2}y^2 + ay + b$*, where* a *and* b *are real constants, produces the same connection as the initial metric* $\bar{g}$*.*

$\star$ PROBLEM 2. Initiate a study of iterative structures Hessian, mathematical concept strongly required by practical problems in Optimization.

To start our iterative process, we need to write the metric h in (9) in an equivalent form. In this respect, we use the explicit form of Γ^1_{11} in (7) and the form of $f_{,11}$ in (8). After calculation, we obtain

$$h(x,y) = \nabla^2_{\bar{g}} f(x,y) = \operatorname{diag}\left(\frac{p'(x)}{2}\left(\ln\frac{p'^2}{g(x)}\right)', \ r''(y)\right) \tag{13}$$

We introduce the notations

$$u_1(x) = \frac{p'(x)}{2}\left(\ln\frac{p'^2}{g(x)}\right)', \quad v_1(y) = r''(y), \tag{14}$$

and using (14) we set

$$h_1(x,y) := h(x,y) = \nabla^2_{\bar{g}} f(x,y) = \operatorname{diag}\left(u_1(x)\ , \ v_1(y)\right),$$

and remark that

$$\begin{aligned}
&\bar{\Gamma}^1_{11} = \frac{1}{2}\frac{u_1'}{u_1},\\
&\bar{\Gamma}^2_{11} = \bar{\Gamma}^1_{12} = \bar{\Gamma}^1_{21} = \bar{\Gamma}^2_{12} = \bar{\Gamma}^2_{21} = \bar{\Gamma}^1_{22} = 0,\\
&\bar{\Gamma}^2_{22} = \frac{1}{2}\frac{v_1'}{v_1}.
\end{aligned}$$

To continue our iterative process, we suppose that $k := h_2 = \nabla^2_{h_1} f$ has the components k_{ij}, where $k_{ij} = \dfrac{\partial^2 f}{\partial x^i \partial x^j} - \bar{\Gamma}^p_{ij} f_{,p}$. After calculations, we find

$$k_{11}(x) = \frac{p'(x)}{2}\left(\ln\frac{p'^2(x)}{u_1(x)}\right)', \quad k_{12} = k_{21} = 0, \quad k_{22}(y) = \frac{r'(y)}{2}\left(\ln\frac{r'^2(y)}{v_1(y)}\right)',$$

If consider $u_2 := k_{11}$, and $v_2 = k_{22}$, we have

$$h_2(x,y) := k(x,y) = \nabla^2_{h_1} f(x,y) = \operatorname{diag}\left(u_2(x)\ , \ v_2(y)\right).$$

But $h_2 := k$ produces the connection $\tilde{\Gamma}$, having the components

$$\begin{aligned}
&\tilde{\Gamma}^1_{11} = \frac{1}{2}\frac{u_2'}{u_2},\\
&\tilde{\Gamma}^2_{11} = \tilde{\Gamma}^1_{12} = \tilde{\Gamma}^1_{21} = \tilde{\Gamma}^2_{12} = \tilde{\Gamma}^2_{21} = \tilde{\Gamma}^1_{22} = 0,\\
&\tilde{\Gamma}^2_{22} = \frac{1}{2}\frac{v_2'}{v_2}.
\end{aligned}$$

As a result of our iterative process we state

Theorem 3.3. *For each positive integer, n, we have*

$$h_{n+1}(x,y) = \nabla^2_{h_n} f = \operatorname{diag}\left(u_{n+1}(x)\ , \ v_{n+1}(y)\right), \quad n \geq 1,$$

where

$$u_{n+1}(x) = \frac{p'(x)}{2}\left(\ln\frac{p'^2(x)}{u_n(x)}\right)', \quad v_{n+1}(y) = \frac{r'(y)}{2}\left(\ln\frac{r'^2(y)}{v_n(y)}\right)',$$

and u_1, v_1 are given by (14).

$\star$ PROBLEM 3. Given $f\colon \mathbb{R}^2 \to \mathbb{R}$ an arbitrary function, introduce new examples of k-self-concordant functions [7], defined on the Riemannian manifold $(\mathbb{R}^2, \bar{g})$.

Using the background in [7], we find:

1) If $g(x) = e^{2x}$, then $f(x, y) = e^x$ is k-self-concordant;

2) If $g(x) = \dfrac{1}{x^2}$, $x > 0$, then $f(x, y) = \ln x$ is k-self-concordant.

The above two examples suggest to consider the function f depending on the variable x only.

We introduce without proof the result in the following, a proof of this result will be given in a forthcoming work.

Proposition 3.1. *Let us consider $f(x, y) := f(x)$ and denote $p = \dfrac{1}{f'}\left(\dfrac{f'^2}{g}\right)'$. The function f is k-self-concordant if and only if $p > 0$ and ${p'}^2 \le 2k^2 g p^3$.*

In our work [6], the reader can find applications of this criterion in finding new classes of self-concordant functions.

4. **Conclusion and further development**

In this work, we showed that the geodesics of a flat Riemannian space are transversal to the constant level manifold. Next, we introduced some advances on Hessian structures. We solved the problem of finding 2D Riemannian manifolds [3], [5], [28] with the property that the original metric and the associated Hessian metric give rise to the same connection, but the problem is still open for other classes of manifolds. For this case study, we indicate two wide classes of Hessian metrics having this property. We also initiated a research on iterative 2D Hessian structures, strongly required in Optimization, [25], and found a new relevant class of self-concordant functions. The results of our work give an up to date link between differential geometry and applied (experimental) sciences, see [1] by S. Amari for geometrical methods in Statistics, [2] by P. L. Antonelli for mathematical modeling in Ecology, [11] by J. Donato for geometrical methods in Information Theory, [25] by Constantin Udrişte for Optimization Methods on Manifolds. Regarding different but related viewpoints, the authors address the reader to these treatises and to the research works [9], [10], [13], [14], [18], [26], [29] as well.

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