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CONSIDERATIONS ON THE GLOBAL AND LOCAL FRACTAL DIMENSION OF BINARY AND GREY-LEVEL IMAGES

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Determinarea dimensiunii fractale constituie un mijloc de caracterizare a gradului de complexitate al unei forme. În această lucrare sunt prezentate câteva considerații asupra determinării dimensiunii fractale în cazul a trei categorii de imagini: imagini binare, imagini în tonuri de gri şi imagini binare cu caracter eterogen. Pentru prima categorie este propusă metoda box-counting extinsă, ce utilizează o tehnică originală de determinare a celei mai drepte porțiuni a curbei log-log. Apoi este prezentat spectrul dimensiunilor fractale menit să caracterizeze imaginile în tonuri de gri. Caracterul eterogen al structurilor este evidențiat prin histograma dimensiunilor fractale locale.

The determination of fractal dimension is a way to characterize the complexity of a form. In this paper, we've presented some considerations on determination of the fractal dimension for three categories of images: binary images, grey-level images and binary heterogeneous images. For the first category we propose the extended box-counting algorithm using an original technique for determination of the liniest portion of the log-log curve. Then we present a fractal spectrum meant to characterize the grey-level images. For the third category we propose the local-fractal dimension.

Keywords: fractal dimension, box-counting, fractal spectrum, local fractal dimension

Introduction

Fractal features describes closely the properties of natural forms. For this reason, the interest to this new mathematical field, fractal geometry, grows quickly. New techniques of fractal analysis are developed, these techniques proves their utility in real systems in various fields.

An important place is reserved by the fractal dimension, as a way to characterize the complexity of a form. In this paper, we present some considerations on the determination of global fractal dimension using the box-counting method, then we present the fractal spectrum for grey-level images and finely we describe an algorithm for determination of local-fractal dimension as a way to characterize the multifractal structures.

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Regarding the box-counting algorithm we've tested several methods for determination of the liniest portion of the log-log curve. Finally we've imagined a combination of these methods and we've obtained an original method based on the frequencies slopes histogram.

In order to characterize the complexity of a multifractal structure we propose the local dimensions histogram which associates to every pixel in the image a dimension using a technique similar with the box-counting method. This approach was imagined by G. Landini [5] for medical purposes.

All methods were tested on two categories of images: a first category contains Euclidian forms, whose Euclidian and topological dimension is known, the second category contains self-similar fractals whose fractal dimension is also known. For multifractal analysis we've used different combinations of these two categories.

1. The fractal dimension

An essential property of fractal object is that its size depends on the size unit used to measure it. This observation was depicted by Richardson: in order to find out the length of the coast line between Spain and Portugal, he consulted the encyclopedias of the two countries. He discovered that in the Spanish encyclopedia the line coast was evaluated at 987 km length, meanwhile the Portugal encyclopedia evaluated the same shape at 1214 km length. The strange phenomenon is explained by the usage of two different measure units in the two cases: the smaller unit has to describe more details of the line coast and, thus, the measure obtained was higher:

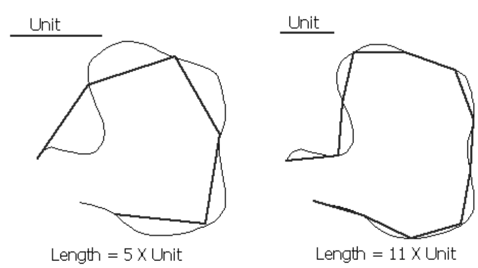


Fig. 1. The length of the line coast depends on the measure unit. The unit has to describe more details of the line coast and, thus, the measure obtained was higher

A famous fractal, the Koch Snowflake simulates this phenomenon:

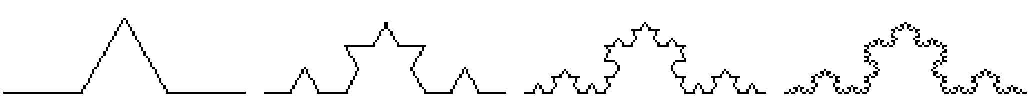


Fig. 2. The famous fractal Koch Snowflake

Repeating the rule of construction to infinite we obtain the Koch Snowflake which is a shape with infinite length.

A fractal space is a non-integral dimension space, dimension known as fractal dimension, into which the measure of the snowflake will not be null or infinite.

1.1. Self-similar dimension

A self-similar fractal means a fractal composed of several transformed copies of himself. In this case, the fractal dimension is evaluated as the degree of fragmentation of the object, using the formula:

$$D_f = \frac{\log(\text{Number\_copies})}{\log(\text{Magnification})}$$

The Koch Snowflake is an example of self-similar fractal composed by 4 copies of himself, at the magnification 3. So, the dimension of this object will be:

$$D_f = \frac{\log(4)}{\log(3)} = 1.2618 .$$

Self-similar fractals have interesting properties, but in real world not all fractals are self-similar, so it's necessary to define ways to measure them too.

1.2. The Hausdorff dimension

The Hausdorff dimension, known as Hausdorff-Besicovich dimension, is defined as the most efficient covering: consider d,s $\in$R and a set of test functions where N(s) is the number of spheres (cubes) of s-diameter needed to cover the given F set. Then, an unique non-integer value d=D_H exists, called the Hausdorff dimension of F, so that:

$$d < D_H \Rightarrow N(s) \xrightarrow[s \to 0]{} \infty$$
$$d > D_H \Rightarrow N(s) \xrightarrow[s \to 0]{} 0$$

Thus, the Hausdorff dimension is proportional with the minimum number of spheres N(s), of a given diameter s, needed to cover the measured object:

$$D \approx \frac{\log(N(s))}{\log(1/s)}$$

The measure of an object having D_H Hausdorff dimension is:

$$N(s) * s^{D_H}$$

2. Algorithms used for determination of fractal dimension

2.1. The box-counting algorithm

Although elegant the Hausdorff dimension is not easily to compute, alternative methods are used, such as box-counting technique. The box-counting algorithm evaluates the fractal dimension, function of the evolution of the object size in relation with the scale factor used.

If the limit exists, the box-counting dimension is given by the formula below:

$$D = \lim_{s \longrightarrow 0} \log \frac{N(s)}{\log(1/s)}$$

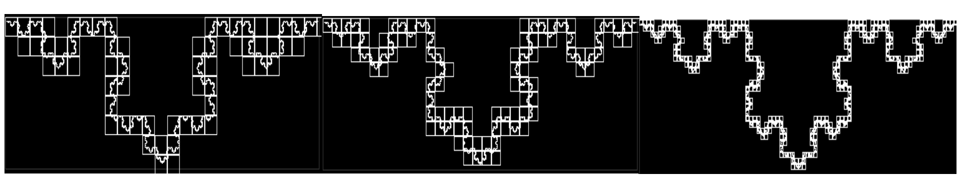
In practice, this limit converts slowly, thus is used an alternative way:

$$\log(N(s)) = D * \log\left(\frac{1}{s}\right)$$

is the equation of a line with the slope D, so we trace the log-log curve, containing (log(N(s), log(1/s)) points, then by linear regression (the least square method) we find the slope of the line, this is in fact the fractal dimension we are looking for.

2.1.1. The extended box-counting algorithm

In this study we used the extended box-counting algorithm [1][2]: we trace the log-log curve for consecutive values of covering square, starting with a minimal value corresponding to the finest detail of the image to a maximum value which is not bigger that the size of the object. The slope of the liniest portion of the slope is the box-counting dimension. For example, we consider a famous fractal - the Koch Snowflake:



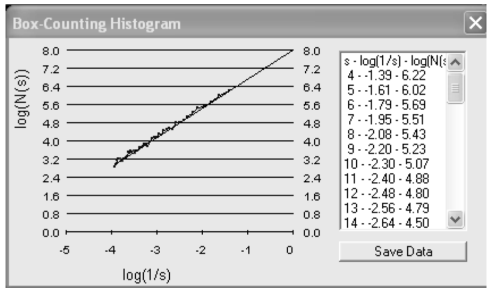


Fig. 3. The Koch Snowflake (binary image) covered with squares of decreasing size (first row)) The log-log curve (second row)

2.1.2. Determining the liniest portion of the log-log slope

Determining the liniest portion of the log-log slope is another essential problem. In our tests we've considered several methods which lead us to an original approach based on the frequencies slopes histogram. In the next paragraphs we'll shortly describe these methods:

Linear regression – the dimension we obtain using the least square method for all points in the log-log curve may be negatively influenced by the systematic errors caused by the unfavorable placement of the covering square. As the size of the covering square is decreasing, the distance between the points on the log-log curve and the regression line is growing, as we can observe in the image below - we consider a simple black and white image containing a straight line (remember that the shape we analyze is white on black background):

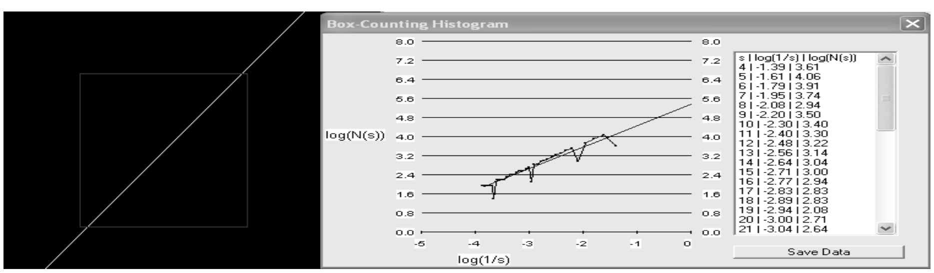


Fig. 4 The regression line deviates from the points on the log-log curve as the covering square size is decreasing; the phenomenon is caused by the systematic errors

The regression line for all points in the log-log curve is Y=0.90*X+5.34, its slope being 0.90, far from the 1-value Euclidian dimension of a line.

The second method we tested consists in elimination of the systematic errors, meaning the points on the log-log curve distanced (above a threshold) from most of the others. This approach was also imprecise as it implies the use of a parameter: how far must be a point to be considered a systematic error? A great number of points distanced from the others are yielding to negative results. For the same selection used above, we obtain a regression line with 0.92 slope value:

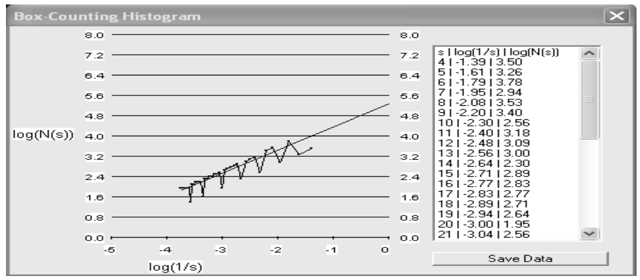


Fig. 5. Eliminating the systematic errors may improve the result, but not enough

Another approach we tested consists in determination of one line which binds the most points of the log-log curve. Every two points define a line, for every such line we count the points placed on it (under a threshold). Finally we choose the line with the greater number of points. The method implies a supplementary parameter: how far can be a point to be considered on the line? A smaller distance implies that the line binds not many points, so we'll obtain many lines with the same number of points, choosing one of them is a subjective problem. A bigger distance is also not convenient because the line we'll obtain depends on the first two points which defined it in the first step.

Finally, we accepted that the more accuracy results are obtain by determination of the slope with higher frequency: the method suppose that each two points on the log-log curve are bind together by a line, each such line have a slope, the slopes histogram is represented and the fractal dimension will be the slope with the higher frequency. For the example considered above we'll have:

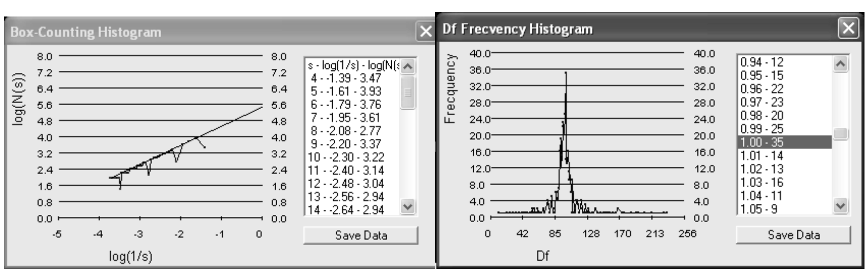


Fig. 6 The log-log curve (left) and the frequency histogram (right) proves that the line have the dimension 1.

Although a line is not a fractal, also proven by its integer dimension, we used it to tests our methods.

For the log-log curve in case of Koch Snowflake, the histogram of slopes frequency indicates the value 1.30:

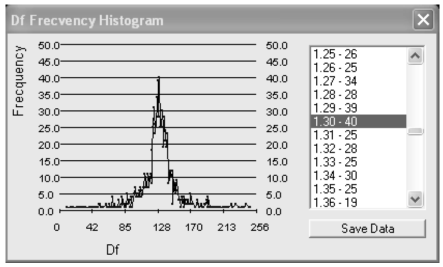


Fig. 7. The histogram of slopes frequency indicates that the fractal dimension of Koch Snowflake is around the value 1.30. (analytic dimension is 1.2618)

2.2. The fractal dimension spectrum for grey-level images

In many problems, the analyzed images are not black and white, but grey-level or color images. Thus, it is necessary to imagine a way to evaluate the complexity of these images. In this paper we propose a technique derived by the box-counting method which allows processing 24-bit color images [4][6]. The algorithm supposes the next five steps:

1. the area of interest is selected, using a mobile cursor. The size area can be 64X64, 128X128, 256X256 or 512X512.
2. the color image is converted into 256-grey levels image, using the formula:

$$I=0,299R+0,587G+0,114B$$

where R/G/B are the red/green/blue components which defines the color of every pixel.

3. the image is binarized using a threshold between 1-255 grey level: all pixels whose grey level is greater or equal to the threshold will be transformed in white, the rest will become black. At this point, the forms inside the image are white on a black background.
4. tracing the contour: once the image is binarized, the next step is to trace an outline of the white areas: all the white pixels which have at least one neighbor black will become part of the contour (in our analysis we considered that one pixel has 8 neighbors: N, NE, E, SE, S, SV, V, NV). The rest of pixels will be transformed in black.
5. the resulted outline can now be analyzed by estimating its global fractal dimension, using the box-counting algorithm described earlier.

Using the above algorithm we obtain a fractal dimension spectrum, where a box-counting dimension is related to every grey level contained in the image.

As an example, we consider the Koch Snowflake composed by 256 levels of grey. We process the image with the algorithm presented above and we obtain the fractal dimension spectrum. As we observe, the fractal spectrum is constant for almost grey levels, fact which proves that the Koch Snowflake is preserving its structure for any grey level we choose for binarization. The fractal dimension is inside the interval 1.2-1.3, we notice that the analytic dimension for the Koch Snowflake is *1.2618.*

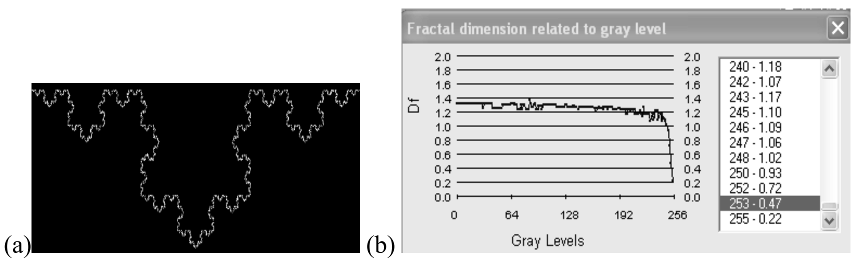


Fig. 8. (a)The 256-grey level Koch Snowflake. (b) Fractal dimension spectrum proves that the snowflake is preserving its structure for almost grey level: Df≈1.2-1.3 (analytic: 1.2618)

2.3. Local Fractal Dimension

The box-counting method is indicated in cases of homogenous structures, supplying global, average information of analyzed objects. The method ignores the heterogeneous nature of images. Such images, with different textures may have the same box-counting global dimension box. The existence, inside an image, of different regions with different fractal dimension requires alternative methods witch will extend the fractal to multi-fractal notion.

In order to describe the heterogeneous nature of an object, we may compute, for every single point in the image, a local dimension (box-counting dimension, by example), limited to a neighborhood of the central pixel. Thus, instead of a single value meant to characterize the whole image, we'll have a set of values, one for each point in the analyzed object. The values will be represented into a histogram in order to give emphasis to the distribution of the local irregularities into the image.

The algorithm we tested was imaged for medical researches [5] and successfully applied [3][5]. It consists in the next four steps:

1. Consider the current point P;
2. Mark all the points connected with P within a growing s-size window centered at P (s is under a fixed smax value which may not be modified during the analysis of the whole image: 32, for example). Notice that every point has eight neighbor pixels (N, NW, V, SW, S, SE, E, NE).

3. Count every time how many points N(s) of the analyzed object are within the window;

4. Using the least square method, compute the slope of the log-log curve composed by the (log(N(s),log(s)) points.

For example, suppose that we want to measure an image containing a combination of the two objects discussed above: the fractal Koch Snowflake traversed by a Euclidian shape – a line, having the Euclidian dimension 1.

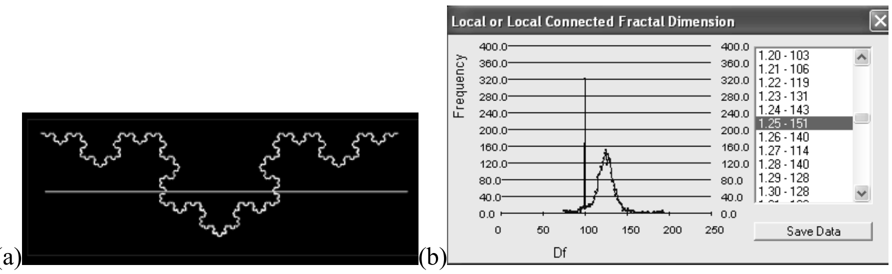
(a) (b)


Fig. 9. Image containing two shapes with different dimension: the Koch Snowflake (analytic dimension 1.26) and a line (euclidian dimension 1)

As we can notice in the Local Fractal Dimension window, the histogram of dimensions have two peeks: one for the 1-value, characterizing the Euclidian line, and the second for 1.25 value, characterizing the Koch Snowflake, proving the existence inside the image of two shapes with different fractal dimension.

An important disadvantage of this method for determination the local fractal dimensions is the presence of a supplementary parameter: the size of the maximum covering square. Tests shows that in multifractal cases, this value influents significant the results.

If the image analyzed is a single fractal, such as Koch Snowflake, then the global fractal dimension (computed with box-counting algorithm) is closed to the local dimension with the higher frequency:

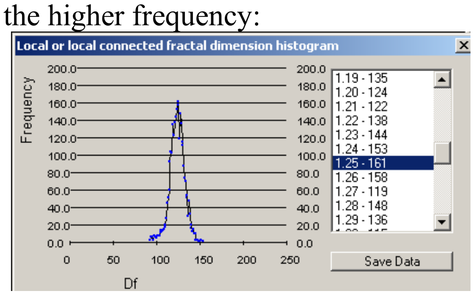


Fig. 10. The most frequent local dimension for the Koch Snowflake is 1.25

As we notice in the histogram of local dimension, the most frequent dimension is 1.25. The box-counting dimension was between 1.25-1.26, the Koch Snowflake having the analytic dimension is 1.2618.

Conclusions

In this paper we investigated several ways to evaluate the fractal dimension for black and white, respectively grey-level images. The methods were tested on two categories of images: on one hand we used euclidian images with known euclidian dimension and on the other hand we used self-similar fractals with known fractal dimension. The self-similar fractals were generated using their IFS (Iterated Transforms System) with another original software application. The fractals were generated into two variants: binary and 256 - grey level images.

As a measure of fractality for homogenous images, we tested the extended box-counting method. Determination of the log-log curve's slope is an unsolved problem; we've tested several methods and we accepted that the histogram of slopes frequencies yields to the more accurate results. For grey-level images we propose a fractal spectrum which also uses the box-counting method. The local dimension is an alternative way to characterize the binary images; it is also indicated in case of multifractal structures.

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