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THE 1-JET GENERALIZED LAGRANGE GEOMETRY INDUCED BY THE RHEONOMIC CHERNOV METRIC

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Lucrarea studiază modelul Lagrange Generalizat pentru metrica Chernov reonomă (cf. [13, 12, 11]*;* [5, 4]*), pe spaţiul jeturilor de ordinul întâi $J^1(\mathbb{R}, M^4)$. Sunt determinate ecuaţiile de câmp gravitaţional şi electromagnetic asociate, evidenţiindu-se relevanţa fizică a acestora.*

The aim of this paper is to develop on the 1*-jet space $J^1(\mathbb{R}, M^4)$, the jet Generalized Lagrange Geometry* ([13, 12]) *for the rheonomic Chernov metric*

$$F_{[3]}(t, y) = \sqrt{h^{11}(t)} \cdot \sqrt[3]{y_1^1 y_1^2 y_1^3 + y_1^1 y_1^2 y_1^4 + y_1^1 y_1^3 y_1^4 + y_1^2 y_1^3 y_1^4}.$$

The associated gravitational and electromagnetic field equations of the associated model are determined, and the Physical relevance is emphasized.

Keywords: rheonomic Chernov metric of order three, canonical nonlinear connection, Cartan canonical connection, $d-$torsions and $d-$curvatures, geometrical Einstein equations.

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1. Introduction

It is obvious that our natural physical intuition distinguishes four dimensions in a natural correspondence with material environment. Consequently, four-dimensionality plays a special role in almost all modern physical theories.

On the other hand, it is an well known fact that, in order to create the Relativity Theory, Einstein was forced to use the Riemannian geometry instead of the classical Euclidean geometry, the Riemannian geometry representing the natural mathematical model for the local *isotropic* space-time. But recent studies of physicists suggest a *non-isotropic* perspective of Space-Time - e.g., the concept of inertial body mass emphasizes the necessity of study of locally non-isotropic spaces ([7]). Among the possible models for the study of non-isotropic physical phenomena, Finsler geometry is an appropriate and effective mathematical framework.

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The works of Russian scholars ([1, 10, 7]) emphasize the importance of the Finsler geometry which is characterized by the complete equivalence of all non-isotropic directions and promote in their works models based on special locally-Minkowski types of $m-$root metrics - e.g., Berwald-Moór or Chernov. Since any of the directions can be related to proper time, such spaces were generically called as having *"multi-dimensional time"* ([15]). In the framework of the 3- and 4-dimensional linear space with Berwald-Moór metric (i.e., having three- and four-dimensional time), Pavlov and his co-workers ([7, 14, 15]) provide new physical model-supporting evidence and geometrical interpretations, such as:

- physical events = points in the multi-dimensional time;
- straight lines = shortest curves;
- intervals = distances between the points along of a straight line;
- light pyramids $\Leftrightarrow$ light cones in a pseudo-Euclidian space;
- surfaces of simultaneity = the surfaces of simultaneous physical events.

An important model of $m-$root type - the Chernov metric ([5, 4]),

$$F : TM \to \mathbb{R}, \qquad F(y) = \sqrt[3]{y^1y^2y^3 + y^1y^2y^4 + y^1y^3y^4 + y^2y^3y^4}, \tag{1}$$

was recently shown to be relevant for Relativity. The larger class of Finsler metrics to which this metric belongs, the $m-$root metrics, have been previously studied by the Japanese geometers Matsumoto and Shimada ([8, 9, 16]).

Considering the former geometrical and physical reasons, the present paper is devoted to the development on the 1-jet space $J^1(\mathbb{R}, M^4)$ of the Finsler-like geometry, applied to geometric gravitational and electromagnetic field theory associated to the natural 1-time rheonomic jet extension of the Chernov metric (1)

$$F_{[3]}(t, y) = \sqrt{h^{11}(t)} \cdot \sqrt[3]{y^1_1y^2_1y^3_1 + y^1_1y^2_1y^4_1 + y^1_1y^3_1y^4_1 + y^2_1y^3_1y^4_1},$$

where $h_{11}(t)$ is a Riemannian metric on $\mathbb{R}$ and $(t, x^1, x^2, x^3, x^4, y^1_1, y^2_1, y^3_1, y^4_1)$ are the coordinates of the 1-jet space $J^1(\mathbb{R}, M^4)$.

The geometry that models gravitational and electromagnetic theories, relying on distinguished $(d-)$connections (and their $d-$torsions and $d-$curvatures), produced by a jet rheonomic Lagrangian function $L : J^1(\mathbb{R}, M^n) \to \mathbb{R}$, was extensively described in [13], where the geometrical ideas are similar, but exhibiting distinct features compared to the ones developed by Miron and Anastasiei in classical Generalized Lagrange Geometry ([11]). The geometrical jet distinguished framework from [13] - generically called as jet geometrical theory of the *rheonomic Lagrange spaces*, was initially stated by Asanov in [2] and developed further in the book [12].

In the sequel, we apply the general geometrical results from [13] to the rheonomic Chernov metric $F_{[3]}$.

2. **Preliminary notations and formulas**

Let $(\mathbb{R}, h_{11}(t))$ be a Riemannian manifold, where $\mathbb{R}$ is the set of real numbers. The Christoffel symbol of the Riemannian metric $h_{11}(t)$ is

$$\varkappa^1_{11} = \frac{h^{11}}{2}\frac{dh_{11}}{dt}, \quad \text{where} \quad h^{11} = \frac{1}{h_{11}} > 0. \tag{2}$$

Let also M^4 be a manifold of dimension four, whose local coordinates are (x^1, x^2, x^3, x^4). Let us consider the 1-jet space $J^1(\mathbb{R}, M^4)$, whose local coordinates are

$$(t, x^1, x^2, x^3, x^4, y_1^1, y_1^2, y_1^3, y_1^4).$$

These transform by the rules (the Einstein convention of summation is used throughout this work):

$$\widetilde{t} = \widetilde{t}(t), \quad \widetilde{x}^p = \widetilde{x}^p(x^q), \quad \widetilde{y}_1^p = \frac{\partial \widetilde{x}^p}{\partial x^q}\frac{dt}{d\widetilde{t}} \cdot y_1^q, \qquad p, q = \overline{1,4}, \tag{3}$$

where $d\widetilde{t}/dt \neq 0$ and rank $(\partial \widetilde{x}^p/\partial x^q) = 4$.

We further consider that the manifold M^4 is endowed with a tensor of kind $(0,3)$, given by the local components $S_{pqr}(x)$, which is totally symmetric in the indices p, q and r. We shall use the notations

$$S_{ij1} = 6S_{ijp}y_1^p, \quad S_{i11} = 3S_{ipq}(x)y_1^p y_1^q, \quad S_{111} = S_{pqr}y_1^p y_1^q y_1^r \tag{4}$$

We assume that the $d-$tensor S_{ij1} is non-degenerate, i.e., there exists the $d-$tensor S^{jk1} on $J^1(\mathbb{R}, M^4)$, such that $S_{ij1}S^{jk1} = \delta_i^k$. In this context, we can consider the *third-root Finsler-like function* ([16], [3]), which is 1-positive homogenous in the variable y,

$$F(t, x, y) = \sqrt[3]{S_{pqr}(x)y_1^p y_1^q y_1^r} \cdot \sqrt{h^{11}(t)} = \sqrt[3]{S_{111}(x, y)} \cdot \sqrt{h^{11}(t)}, \tag{5}$$

where the Finsler function F has as domain of definition all values (t, x, y) which satisfy the condition $S_{111}(x, y) \neq 0$. Then the 3-positive homogeneity of the "y-function" S_{111} (which is a $d-$tensor on the 1-jet space $J^1(\mathbb{R}, M^4)$), leads to the equalities:

$$S_{i11} = \frac{\partial S_{111}}{\partial y_1^i}, \quad S_{i11}y_1^i = 3S_{111}, \quad S_{ij1}y_1^j = 2S_{i11}, \quad S_{ij1} = \frac{\partial S_{i11}}{\partial y_1^j} = \frac{\partial^2 S_{111}}{\partial y_1^i \partial y_1^j},$$

$$S_{ij1}y_1^i y_1^j = 6S_{111}, \quad \frac{\partial S_{ij1}}{\partial y_1^k} = 6S_{ijk}, \quad S_{ijp}y_1^p = \frac{1}{6}S_{ij1}.$$

The *fundamental metrical $d-$tensor* produced by F is given by the formula

$$g_{ij}(t, x, y) = \frac{h_{11}(t)}{2}\frac{\partial^2 F^2}{\partial y_1^i \partial y_1^j}.$$

By direct computations, the fundamental metrical $d-$tensor takes the form

$$g_{ij}(x, y) = \frac{S_{111}^{-1/3}}{3}\left[S_{ij1} - \frac{1}{3S_{111}}S_{i11}S_{j11}\right]. \tag{6}$$

Moreover, since the $d-$tensor S_{ij1} is non-degenerate, the matrix $g = (g_{ij})$ admits an inverse $g^{-1} = (g^{jk})$, whose entries are

$$g^{jk} = 3S_{111}^{1/3}\left[S^{jk1} + \frac{S_1^j S_1^k}{3\left(S_{111} - \mathbf{S}_{111}\right)}\right], \tag{7}$$

where $S_1^j = S^{jp1}S_{p11}$ and $3\mathbf{S}_{111} = S^{pq1}S_{p11}S_{q11}$.

Following the ideas from [13], the *energy action functional*

$$\mathbb{E}(t,x(t))=\int_a^b F^2(t,x(t),y(t))\sqrt{h_{11}(t)}dt=\int_a^b S_{111}^{2/3}\cdot h^{11}\sqrt{h_{11}}dt,$$

where $y(t)=dx/dt$, produces on the 1-jet space $J^1(\mathbb{R},M^4)$, via the Euler-Lagrange equations, the *canonical time dependent spray*

$$\mathcal{S}=\left(H_{(1)1}^{(i)},\ G_{(1)1}^{(i)}\right), \tag{8}$$

where, using the notations (2) and (4), we have $H_{(1)1}^{(i)}=-\dfrac{\varkappa_{11}^1}{2}y_1^i$ and

$$\begin{aligned} G_{(1)1}^{(i)} &= \frac{g^{im}}{6\sqrt[3]{S_{111}}}\left[\frac{\partial S_{m11}}{\partial x^p}y_1^p-\left(1-\varkappa_{11}^1\right)\frac{\partial S_{111}}{\partial x^m}\right]- \\ &\quad -\frac{S_1^i}{6\left(S_{111}-\mathbf{S}_{111}\right)}\left(\frac{\partial S_{111}}{\partial x^p}y_1^p+3\varkappa_{11}^1S_{111}\right). \end{aligned} \tag{9}$$

Remark 2.1. *In the particular case when the components G_{pqr} are independent on the variable x, the expression of (9) simplifies as*

$$G_{(1)1}^{(i)}=-\varkappa_{11}^1\frac{S_{111}}{2\left(S_{111}-\mathbf{S}_{111}\right)}S_1^i. \tag{10}$$

Note that, in this case, the Finsler-like function (5) is of locally-Minkowski type.

It is known ([13]) that the canonical time dependent spray $\mathcal{S}$ given by (8) determines on the 1-jet space $J^1(\mathbb{R},M^4)$ a *canonical nonlinear connection* given by

$$\Gamma=\left(M_{(1)1}^{(i)}=2H_{(1)1}^{(i)}=-\varkappa_{11}^1y_1^i,\ N_{(1)j}^{(i)}=\frac{G_{(1)1}^{(i)}}{\partial y_1^j}\right). \tag{11}$$

3. The rheonomic Chernov metric

Beginning with this Section we will focus only on the *rheonomic Chernov metric*, which is the Finsler-like metric (5) for the particular case

$$S_{pqr}:=S_{[3]pqr}=\begin{cases}\dfrac{1}{3!}, & \{p,q,r\}\text{ - distinct indices}\\ 0, & \text{otherwise.}\end{cases}$$

Consequently, the rheonomic Chernov metric is given by

$$F_{[3]}(t,y)=\sqrt{h^{11}(t)}\cdot\sqrt[3]{y_1^1y_1^2y_1^3+y_1^1y_1^2y_1^4+y_1^1y_1^3y_1^4+y_1^2y_1^3y_1^4}. \tag{12}$$

Moreover, using the preceding notations and formulas, we obtain the following relations:

$$S_{111}:=S_{[3]111}=y_1^1y_1^2y_1^3+y_1^1y_1^2y_1^4+y_1^1y_1^3y_1^4+y_1^2y_1^3y_1^4,$$

$$S_{i11}:=S_{[3]i11}=\frac{\partial S_{[3]111}}{\partial y_1^i}=\frac{S_{[3]111}y_1^i-S_{[4]1111}}{\left(y_1^i\right)^2},$$

$$S_{ij1} := S_{[3]ij1} = \frac{\partial S_{[3]i11}}{\partial y_1^j} = \frac{\partial^2 S_{[3]111}}{\partial y_1^i \partial y_1^j} = \begin{cases} S_{[1]1} - y_1^i - y_1^j, & i \neq j \\ 0, & i = j, \end{cases}$$

where $S_{[4]1111} = y_1^1 y_1^2 y_1^3 y_1^4$ and $S_{[1]1} = y_1^1 + y_1^2 + y_1^3 + y_1^4$. Note that, for $i \neq j$, the following equality holds true as well:

$$S_{[3]i11} \cdot S_{[3]j11} = S_{[3]111}\left(S_{[1]1} - y_1^i - y_1^j\right) + \frac{S_{[4]1111}^2}{\left(y_1^i\right)^2\left(y_1^j\right)^2}.$$

Because we have $0 \neq \det\left(S_{ij1}\right)_{i,j=\overline{1,4}} = 4\left[4S_{[4]1111} - S_{[1]1}S_{[3]111}\right] := \mathbf{D}_{1111}$, we find

$$S^{jk1} := S_{[3]}^{jk1} = \begin{cases} \dfrac{-2}{\mathbf{D}_{1111}}\left(y_1^j + y_1^k\right)\left[y_1^j y_1^k + \dfrac{S_{[4]1111}}{y_1^j y_1^k}\right], & j \neq k \\ \dfrac{1}{\mathbf{D}_{1111}} \cdot \dfrac{1}{y_1^j}\left[\prod\limits_{l=1}^{4}\left(y_1^j + y_1^l\right)\right], & j = k. \end{cases}$$

Further, laborious computations lead to:

$$\begin{aligned} S_1^j &:= S_{[3]1}^j = S_{[3]}^{jp1} S_{[3]p11} = \frac{1}{2} y_1^j, \\ \mathbf{S}_{111} &:= \mathbf{S}_{[3]111} = S_{[3]}^{pq1} S_{[3]p11} S_{[3]q11} = \frac{1}{2} S_{[3]111}. \end{aligned} \tag{13}$$

Replacing now the above computed entities into the formulas (6) and (7), we get $g_{ij} := g_{[3]ij} =$

$$= \begin{cases} \dfrac{S_{[3]111}^{-1/3}}{9}\left[2\left(S_{[1]1} - y_1^i - y_1^j\right) - \dfrac{S_{[4]1111}^2}{S_{[3]111}} \cdot \dfrac{1}{\left(y_1^i\right)^2\left(y_1^j\right)^2}\right], & i \neq j \\ \dfrac{-S_{[3]111}^{-4/3}}{9} \cdot S_{[3]i11}^2, & i = j \end{cases} \tag{14}$$

and

$$g^{jk} := g_{[3]}^{jk} = 3S_{[3]111}^{1/3}\left[S^{jk1} + \frac{1}{6S_{[3]111}} y_1^j y_1^k\right]. \tag{15}$$

Consequently, using (10) and (13), we find the following geometrical result:

Proposition 3.1. *For the rheonomic Chernov metric (12), the energy action functional*

$$\mathbb{E}_{[3]}(t, x(t)) = \int_a^b \sqrt[3]{\left(y_1^1 y_1^2 y_1^3 + y_1^1 y_1^2 y_1^4 + y_1^1 y_1^3 y_1^4 + y_1^2 y_1^3 y_1^4\right)^2} \cdot h^{11}\sqrt{h_{11}}dt$$

produces on the 1*-jet space* $J^1(\mathbb{R}, M^4)$ *the canonical time dependent spray*

$$\mathcal{S}_{[3]} = \left(H_{(1)1}^{(i)} = -\frac{\varkappa_{11}^1}{2} y_1^i,\ G_{(1)1}^{(i)} = -\frac{\varkappa_{11}^1}{2} y_1^i\right). \tag{16}$$

Moreover, the formulas (11) and (16) imply

Corollary 3.1. *The canonical nonlinear connection on the* 1*-jet space* $J^1(\mathbb{R}, M^4)$ *produced by the rheonomic Chernov metric (12) is*

$$\Gamma_{[3]} = \left(M^{(i)}_{(1)1} = -\varkappa^1_{11} y^i_1,\ N^{(i)}_{(1)j} = -\frac{\varkappa^1_{11}}{2}\delta^i_j \right), \tag{17}$$

where δ^i_j *is the Kronecker symbol.*

4. Cartan canonical connection. $d-$torsions and curvatures

The importance of the nonlinear connection (17) is coming from the possibility of construction of the dual *local adapted bases*: of $d-$vector fields

$$\left\{ \frac{\delta}{\delta t} = \frac{\partial}{\partial t} + \varkappa^1_{11} y^p_1 \frac{\partial}{\partial y^p_1} \ ;\ \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} + \frac{\varkappa^1_{11}}{2}\frac{\partial}{\partial y^i_1} \ ;\ \frac{\partial}{\partial y^i_1} \right\} \subset \mathcal{X}(E) \tag{18}$$

and of $d-$covector fields

$$\left\{ dt \ ;\ dx^i \ ;\ \delta y^i_1 = dy^i_1 - \varkappa^1_{11} y^i_1 dt - \frac{\varkappa^1_{11}}{2} dx^i \right\} \subset \mathcal{X}^*(E), \tag{19}$$

where $E = J^1(\mathbb{R}, M^4)$. Note that, under a change of coordinates (3), the elements of the adapted bases (18) and (19) transform as classical tensors. Consequently, all subsequent geometrical objects on the 1-jet space $J^1(\mathbb{R}, M^4)$ (as Cartan canonical connection, torsion, curvature etc.) will be described in local adapted components. We emphasize that the definition of local components of connections, torsion and curvature, obey the formalism used in the works [11, 12, 13, 3].

Using a general result from [13], by direct computations, we can give the following important geometrical result:

Theorem 4.1. *The Cartan canonical* $\Gamma_{[3]}$*-linear connection, produced by the rheonomic Chernov metric (12), has the following adapted local components:*

$$C\Gamma_{[3]} = \left(\varkappa^1_{11},\ G^k_{j1} = 0,\ L^i_{jk} = \frac{\varkappa^1_{11}}{2} C^{i(1)}_{j(k)},\ C^{i(1)}_{j(k)} \right),$$

where

$$\begin{aligned} C^{i(1)}_{j(k)} = \ & 3S^{im1}_{[3]} S_{[3]jkm} + \frac{1}{9}\frac{1}{S^2_{[3]111}} S_{[3]j11} S_{[3]k11} y^i_1 - \\ & - \frac{1}{6}\frac{1}{S_{[3]111}} \left[S_{[3]jk1}\frac{y^i_1}{2} + \delta^i_j S_{[3]k11} + \delta^i_k S_{[3]j11} \right]. \end{aligned} \tag{20}$$

Proof. Using the Chernov derivative operators (18) and (19), together with the relations (14) and (15), we apply the general formulas which give the adapted components of the Cartan canonical connection, namely [13]

$$G^k_{j1} = \frac{g^{km}_{[3]}}{2}\frac{\delta g_{[3]mj}}{\delta t}, \quad L^i_{jk} = \frac{g^{im}_{[3]}}{2}\left(\frac{\delta g_{[3]jm}}{\delta x^k} + \frac{\delta g_{[3]km}}{\delta x^j} - \frac{\delta g_{[3]jk}}{\delta x^m} \right),$$

$$C^{i(1)}_{j(k)} = \frac{g^{im}_{[3]}}{2}\left(\frac{\partial g_{[3]jm}}{\partial y^k_1} + \frac{\partial g_{[3]km}}{\partial y^j_1} - \frac{\partial g_{[3]jk}}{\partial y^m_1} \right) = \frac{g^{im}_{[3]}}{2}\frac{\partial g_{[3]jk}}{\partial y^m_1},$$

where, by computations, we have

$$\frac{\partial g_{[3]jk}}{\partial y_1^m} = 2S_{[3]111}^{-1/3}S_{[3]jkm} + \frac{4}{27}S_{[3]111}^{-7/3}S_{[3]j11}S_{[3]k11}S_{[3]m11} - \\ -\frac{1}{9}S_{[3]111}^{-4/3}\left\{S_{[3]jk1}S_{[3]m11} + S_{[3]km1}S_{[3]j11} + S_{[3]mj1}S_{[3]k11}\right\}.$$

For details, we refer to [16] and [3]. □

Remark 4.1. *The following properties of the d−tensor* $C_{j(k)}^{i(1)}$ *hold true:*

$$C_{j(k)}^{i(1)} = C_{k(j)}^{i(1)}, \quad C_{j(m)}^{i(1)}y_1^m = 0.$$

Theorem 4.2. *The Cartan canonical connection* $C\Gamma_{[3]}$ *of the rheonomic Chernov metric (12) has* ***three*** *effective local torsion d-tensors:*

$$P_{(1)i(j)}^{(k)\ (1)} = -\frac{1}{2}\varkappa_{11}^1 C_{i(j)}^{k(1)}, \quad P_{i(j)}^{k(1)} = C_{i(j)}^{k(1)}, \quad R_{(1)1j}^{(k)} = \frac{1}{2}\left(\frac{d\varkappa_{11}^1}{dt} - \varkappa_{11}^1\varkappa_{11}^1\right)\delta_j^k.$$

Proof. A general h-normal Γ-linear connection on the 1-jet space $J^1(\mathbb{R}, M^4)$ is characterized by *eight* effective $d-$tensors of torsion (cf. [13]). For our Cartan canonical connection $C\Gamma_{[3]}$ these reduce to the following *three* (the other five cancel):

$$P_{(1)i(j)}^{(k)\ (1)} = \frac{\partial N_{(1)i}^{(k)}}{\partial y_1^j} - L_{ji}^k, \quad R_{(1)1j}^{(k)} = \frac{\delta M_{(1)1}^{(k)}}{\delta x^j} - \frac{\delta N_{(1)j}^{(k)}}{\delta t}, \quad P_{i(j)}^{k(1)} = C_{i(j)}^{k(1)}. \qquad \square$$

Theorem 4.3. *The Cartan canonical connection* $C\Gamma_{[3]}$ *of the rheonomic Chernov metric (12) has* ***three*** *effective local curvature d−tensors:*

$$R_{ijk}^l = \frac{1}{4}\varkappa_{11}^1\varkappa_{11}^1 S_{i(j)(k)}^{l(1)(1)}, \quad P_{ij(k)}^{l\ (1)} = \frac{1}{2}\varkappa_{11}^1 S_{i(j)(k)}^{l(1)(1)},$$

$$S_{i(j)(k)}^{l(1)(1)} = \frac{\partial C_{i(j)}^{l(1)}}{\partial y_1^k} - \frac{\partial C_{i(k)}^{l(1)}}{\partial y_1^j} + C_{i(j)}^{m(1)}C_{m(k)}^{l(1)} - C_{i(k)}^{m(1)}C_{m(j)}^{l(1)}.$$

Proof. A general h-normal Γ-linear connection on the 1-jet space $J^1(\mathbb{R}, M^4)$ is characterized by *five* effective $d-$tensors of curvature (cf. [13]). For our Cartan canonical connection $C\Gamma_{[3]}$ these reduce to the following *three* (the other two cancel):

$$R_{ijk}^l = \frac{\delta L_{ij}^l}{\delta x^k} - \frac{\delta L_{ik}^l}{\delta x^j} + L_{ij}^m L_{mk}^l - L_{ik}^m L_{mj}^l,$$

$$P_{ij(k)}^{l\ (1)} = \frac{\partial L_{ij}^l}{\partial y_1^k} - C_{i(k)|j}^{l(1)} + C_{i(m)}^{l(1)}P_{(1)j(k)}^{(m)\ (1)},$$

$$S_{i(j)(k)}^{l(1)(1)} = \frac{\partial C_{i(j)}^{l(1)}}{\partial y_1^k} - \frac{\partial C_{i(k)}^{l(1)}}{\partial y_1^j} + C_{i(j)}^{m(1)}C_{m(k)}^{l(1)} - C_{i(k)}^{m(1)}C_{m(j)}^{l(1)},$$

where $C_{i(k)|j}^{l(1)} = \frac{\delta C_{i(k)}^{l(1)}}{\delta x^j} + C_{i(k)}^{m(1)}L_{mj}^l - C_{m(k)}^{l(1)}L_{ij}^m - C_{i(m)}^{l(1)}L_{kj}^m$. □

Remark 4.2. *We have denoted by* $/1$, $|i$ *and* $\big|_{(i)}^{(1)}$ *the Cartan covariant derivatives with respect to the corresponding* $\mathbb{R}-$*horizontal (temporal),* $M-$*horizontal and vertical vector fields of the basis* (18).

5. Applications of the rheonomic Chernov metric

5.1. Geometrical gravitational theory

From a physical point of view, on the 1-jet space $J^1(\mathbb{R}, M^4)$, the rheonomic Chernov metric (12) produces the adapted metrical $d-$tensor

$$\mathbb{G}_{[3]} = h_{11}dt \otimes dt + g_{[3]ij}dx^i \otimes dx^j + h^{11}g_{[3]ij}\delta y_1^i \otimes \delta y_1^j, \tag{21}$$

where $g_{[3]ij}$ is given by (14). This may be regarded as a *"non-isotropic gravitational potential"*. In such a physical context, the nonlinear connection $\Gamma_{[3]}$ (used in the construction of the distinguished 1-forms δy_1^i) prescribes, most likely, a sort of *"interaction"* between (t)-, (x)- and (y)-fields.

We postulate that the non-isotropic gravitational potential $\mathbb{G}_{[3]}$ is governed by the *geometrical Einstein equations*

$$\text{Ric}\ \left(C\Gamma_{[3]}\right) - \frac{\text{Sc}\ \left(C\Gamma_{[3]}\right)}{2}\mathbb{G}_{[3]} = \mathcal{K}\mathcal{T}, \tag{22}$$

where $\text{Ric}\ \left(C\Gamma_{[3]}\right)$ is the *Ricci d-tensor* associated to the Cartan canonical connection $C\Gamma_{[3]}$ (in Riemannian sense and using adapted bases), $\text{Sc}\ \left(C\Gamma_{[3]}\right)$ is the *scalar curvature*, $\mathcal{K}$ is the *Einstein constant* and $\mathcal{T}$ is the intrinsic *stress-energy* $d-$tensor of matter.

In this way, working with the adapted basis of vector fields (18), we can find the local geometrical Einstein equations for the rheonomic Chernov metric (12). Firstly, by direct computations, we find:

Theorem 5.1. *The Ricci $d-$tensor of the Cartan canonical connection $C\Gamma_{[3]}$ of the rheonomic Chernov metric (12) has the following effective local Ricci $d-$tensor components:*

$$\begin{aligned}
R_{ij} := R^r_{ijr} &= \frac{1}{4}\varkappa^1_{11}\varkappa^1_{11}\mathbb{S}^{(1)(1)}_{(i)(j)},\\
P^{\ (1)}_{i(j)} = P^{(1)}_{(i)j} := P^{r\ (1)}_{ij(r)} &= \frac{1}{2}\varkappa^1_{11}\mathbb{S}^{(1)(1)}_{(i)(j)},\\
\mathbb{S}^{(1)(1)}_{(i)(j)} &= -9S^{pq1}_{[3]}S^{rm1}_{[3]}\left(S_{[3]ijp}S_{[3]qrm} - S_{[3]ipr}S_{[3]jqm}\right) +\\
&\quad + \frac{1}{12}\frac{1}{S_{[3]111}}S_{[3]ij1} - \frac{1}{18}\frac{1}{S^2_{[3]111}}S_{[3]i11}S_{[3]j11},
\end{aligned}$$

where $\mathbb{S}^{(1)(1)}_{(i)(j)} = S^{r(1)(1)}_{i(j)(r)}$ is the vertical Ricci $d-$tensor field.

Proof. Using the equality (20), by laborious direct computations, we obtain the following equalities (we assume implicit summation by r and m):

$$\begin{aligned}
\frac{\partial C^{r(1)}_{i(j)}}{\partial y^r_1} &= 3\frac{\partial S^{rm1}_{[3]}}{\partial y^r_1}S_{[3]ijm} - \frac{1}{2}\frac{1}{S_{[3]111}}S_{[3]ij1} + \frac{5}{9}\frac{1}{S^2_{[3]111}}S_{[3]i11}S_{[3]j11},\\
\frac{\partial C^{r(1)}_{i(r)}}{\partial y^j_1} &= 3\frac{\partial S^{rm1}_{[3]}}{\partial y^j_1}S_{[3]irm} - \frac{2}{3}\frac{1}{S_{[3]111}}S_{[3]ij1} + \frac{2}{3}\frac{1}{S^2_{[3]111}}S_{[3]i11}S_{[3]j11},
\end{aligned}$$

$$C^{m(1)}_{i(j)}C^{r(1)}_{m(r)} = 9S^{mp1}_{[3]}S^{rq1}_{[3]}S_{[3]ijp}S_{[3]mrq} + \frac{2}{9}\frac{1}{S^2_{[3]111}}S_{[3]i11}S_{[3]j11} -$$
$$-\frac{1}{2}\frac{1}{S_{[3]111}}S^{rq1}_{[3]}\left\{S_{[3]irq}S_{[3]j11} + S_{[3]jrq}S_{[3]i11}\right\} - \frac{1}{6}\frac{1}{S_{[3]111}}S_{[3]ij1},$$
$$C^{m(1)}_{i(r)}C^{r(1)}_{m(j)} = 9S^{mp1}_{[3]}S^{rq1}_{[3]}S_{[3]irp}S_{[3]mjq} + \frac{1}{6}\frac{1}{S^2_{[3]111}}S_{[3]i11}S_{[3]j11} -$$
$$-\frac{1}{2}\frac{1}{S_{[3]111}}S^{rq1}_{[3]}\left\{S_{[3]irq}S_{[3]j11} + S_{[3]jrq}S_{[3]i11}\right\} - \frac{1}{12}\frac{1}{S_{[3]111}}S_{[3]ij1}.$$

Finally, taking into account that we have

$$\mathbb{S}^{(1)(1)}_{(i)(j)} = S^{r(1)(1)}_{i(j)(r)} = \frac{\partial C^{r(1)}_{i(j)}}{\partial y^r_1} - \frac{\partial C^{r(1)}_{i(r)}}{\partial y^j_1} + C^{m(1)}_{i(j)}C^{r(1)}_{m(r)} - C^{m(1)}_{i(r)}C^{r(1)}_{m(j)},$$

and using the equalities

$$\frac{\partial S^{rm1}_{[3]}}{\partial y^r_1}S_{[3]ijm} = -6S^{mp1}_{[3]}S^{rq1}_{[3]}S_{[3]ijp}S_{[3]mrq},$$
$$\frac{\partial S^{rm1}_{[3]}}{\partial y^j_1}S_{[3]irm} = -6S^{mp1}_{[3]}S^{rq1}_{[3]}S_{[3]irp}S_{[3]jmq},$$

we obtain the required result. □

Remark 5.1. *The vertical Ricci d−tensor* $\mathbb{S}^{(1)(1)}_{(i)(j)}$ *has the following property of symmetry:* $\mathbb{S}^{(1)(1)}_{(i)(j)} = \mathbb{S}^{(1)(1)}_{(j)(i)}$.

Proposition 5.1. *The scalar curvature of the Cartan canonical connection* $C\Gamma_{[3]}$ *of the rheonomic Chernov metric (12) is given by*

$$Sc\ \left(C\Gamma_{[3]}\right) = \frac{4h_{11} + \varkappa^1_{11}\varkappa^1_{11}}{4}\cdot\mathbb{S}^{11}, \quad \text{where} \quad \mathbb{S}^{11} = g^{pq}_{[3]}\mathbb{S}^{(1)(1)}_{(p)(q)}.$$

Proof. The general formula for the scalar curvature of a Cartan connection is (cf. [13]) $\mathrm{Sc}\ \left(C\Gamma_{[3]}\right) = g^{pq}_{[3]}R_{pq} + h_{11}g^{pq}_{[3]}\mathbb{S}^{(1)(1)}_{(p)(q)}$. □

Describing the global geometrical Einstein equations (22) in the adapted basis of vector fields (18), it is known the following important geometrical and physical result (cf. [13]):

Theorem 5.2. *The local* ***geometrical Einstein equations*** *that govern the non-isotropic gravitational potential (21) (produced by the rheonomic Chernov metric (12)) are given by:*

$$\begin{cases} \xi_{11}\mathbb{S}^{11}h_{11} = \mathcal{T}_{11} \\ \dfrac{\varkappa^1_{11}\varkappa^1_{11}}{4\mathcal{K}}\mathbb{S}^{(1)(1)}_{(i)(j)} + \xi_{11}\mathbb{S}^{11}g_{ij} = \mathcal{T}_{ij} \\ \dfrac{1}{\mathcal{K}}\mathbb{S}^{(1)(1)}_{(i)(j)} + \xi_{11}\mathbb{S}^{11}h^{11}g_{ij} = \mathcal{T}^{(1)(1)}_{(i)(j)} \end{cases} \tag{23}$$

$$\begin{cases} 0=\mathcal{T}_{1i}, \quad 0=\mathcal{T}_{i1}, & 0=\mathcal{T}^{(1)}_{(i)1}, \\ 0=\mathcal{T}^{\ (1)}_{1(i)}, \quad \dfrac{\varkappa^1_{11}}{2\mathcal{K}}\mathbb{S}^{(1)(1)}_{(i)(j)}=\mathcal{T}^{\ (1)}_{i(j)}, & \dfrac{\varkappa^1_{11}}{2\mathcal{K}}\mathbb{S}^{(1)(1)}_{(i)(j)}=\mathcal{T}^{(1)}_{(i)j}, \end{cases} \tag{24}$$

where $\xi_{11}=-\dfrac{4h_{11}+\varkappa^1_{11}\varkappa^1_{11}}{8\mathcal{K}}$.

Remark 5.2. *The local geometrical Einstein equations (23) and (24) impose as the stress-energy d−tensor of matter $\mathcal{T}$ to be symmetrical. In other words, the stress-energy d−tensor of matter $\mathcal{T}$ must satisfy the local symmetry conditions*

$$\mathcal{T}_{AB}=\mathcal{T}_{BA}, \quad \forall\ A,B\in\left\{1,\ i,\ {}^{(1)}_{(i)}\right\}.$$

5.2. Geometrical electromagnetic theory

In the paper [13], using only a given Lagrangian function L on the 1-jet space $J^1(\mathbb{R},M^4)$, a geometrical theory for electromagnetism was also constructed. In this background of jet relativistic rheonomic Lagrange geometry, we work with an *electromagnetic distinguished 2-form* $\mathbb{F}=F^{(1)}_{(i)j}\delta y^i_1\wedge dx^j$, where

$$F^{(1)}_{(i)j}=\frac{h^{11}}{2}\left[g_{jm}N^{(m)}_{(1)i}-g_{im}N^{(m)}_{(1)j}+\left(g_{ir}L^r_{jm}-g_{jr}L^r_{im}\right)y^m_1\right],$$

which is characterized by the following *geometrical Maxwell equations* [13]

$$\begin{cases} F^{(1)}_{(i)j/1} & =\dfrac{1}{2}\mathcal{A}_{\{i,j\}}\Big\{\overline{D}^{(1)}_{(i)1|j}-D^{(1)}_{(i)m}G^m_{j1}+d^{(1)(1)}_{(i)(m)}R^{(m)}_{(1)1j}- \\ & \quad -\left[C^{p(1)}_{j(m)}R^{(m)}_{(1)1i}-G^p_{i1|j}\right]h^{11}g_{pq}y^q_1\Big\}, \\ \displaystyle\sum_{\{i,j,k\}}F^{(1)}_{(i)j\ |\ k} & =-\dfrac{1}{8}\displaystyle\sum_{\{i,j,k\}}\frac{\partial^3L}{\partial y^i_1\partial y^p_1\partial y^m_1}\left[\frac{\delta N^{(m)}_{(1)j}}{\delta x^k}-\frac{\delta N^{(m)}_{(1)k}}{\delta x^j}\right]y^p_1, \\ \displaystyle\sum_{\{i,j,k\}}F^{(1)}_{(i)j}\Big|^{(1)}_{(k)} & =0, \end{cases}$$

where $\mathcal{A}_{\{i,j\}}$ denotes an alternate sum, $\sum_{\{i,j,k\}}$ means a cyclic sum and we have

$$\begin{cases} \overline{D}^{(1)}_{(i)1}=\dfrac{h^{11}}{2}\dfrac{\delta g_{ip}}{\delta t}y^p_1, \quad D^{(1)}_{(i)j}=h^{11}g_{ip}\left[-N^{(p)}_{(1)j}+L^p_{jm}y^m_1\right], \\ d^{(1)(1)}_{(i)(j)}=h^{11}\left[g_{ij}+g_{ip}C^{p(1)}_{m(j)}y^m_1\right], \quad \overline{D}^{(1)}_{(i)1|j}=\dfrac{\delta\overline{D}^{(1)}_{(i)1}}{\delta x^j}-\overline{D}^{(1)}_{(m)1}L^m_{ij}, \end{cases}$$

$$G^k_{i1|j}=\frac{\delta G^k_{i1}}{\delta x^j}+G^m_{i1}L^k_{mj}-G^k_{m1}L^m_{ij},$$

$$\begin{cases} F^{(1)}_{(i)j/1} = \dfrac{\delta F^{(1)}_{(i)j}}{\delta t} + F^{(1)}_{(i)j}\varkappa^1_{11} - F^{(1)}_{(m)j}G^m_{i1} - F^{(1)}_{(i)m}G^m_{j1}, \\ F^{(1)}_{(i)j\,|\,k} = \dfrac{\delta F^{(1)}_{(i)j}}{\delta x^k} - F^{(1)}_{(m)j}L^m_{ik} - F^{(1)}_{(i)m}L^m_{jk}, \\ F^{(1)}_{(i)j}\,\big|\,{}^{(1)}_{(k)} = \dfrac{\partial F^{(1)}_{(i)j}}{\partial y^k_1} - F^{(1)}_{(m)j}C^{m(1)}_{i(k)} - F^{(1)}_{(i)m}C^{m(1)}_{j(k)}. \end{cases}$$

For the rheonomic Chernov metric (12) we have $L = F^2_{[3]}$ and, consequently, we obtain the electromagnetic 2-form $\mathbb{F} := \mathbb{F}_{[3]} = 0$.

In conclusion, the locally-Minkowski rheonomic Chernov geometrical electromagnetic theory is trivial. In our opinion, this fact suggests that the metric (12) has rather gravitational connotations than electromagnetic ones in its $h-$flat ($x-$independent) version, which leads to the need of considering $x-$dependent conformal deformations of the structure (as, e.g., recently proposed by Garas'ko in [6]).

6. **Conclusion**

In recent physical and geometrical studies ([1, 7, 14, 15]), an important role is played by the Finslerian metric

$$F_{[2]}(t,y) = \sqrt{h^{11}(t)} \cdot \sqrt{y^1_1y^2_1 + y^1_1y^3_1 + y^1_1y^4_1 + y^2_1y^3_1 + y^2_1y^4_1 + y^3_1y^4_1} \tag{25}$$

which produces the fundamental metrical $d-$tensor

$$g_{ij} := g_{[2]ij} = \frac{h_{11}(t)}{2}\frac{\partial^2 F^2_{[2]}}{\partial y^i_1 \partial y^j_1} = \frac{1}{2}\left(1-\delta_{ij}\right) \Rightarrow g^{jk} := g^{jk}_{[2]} = \frac{2}{3}\left(1 - 3\delta^{jk}\right).$$

The Finslerian metric (25) generates the jet canonical nonlinear connection

$$\Gamma_{[2]} = \left(M^{(i)}_{(1)1} = -\varkappa^1_{11}y^i_1,\ N^{(i)}_{(1)j} = -\frac{\varkappa^1_{11}}{2}\delta^i_j\right)$$

and the Cartan $\Gamma_{[2]}$-linear connection

$$C\Gamma_{[2]} = \left(\varkappa^1_{11},\ G^k_{j1} = 0,\ L^i_{jk} = 0,\ C^{i(1)}_{j(k)} = 0\right).$$

For the Cartan connection $C\Gamma_{[2]}$ all torsion $d-$tensors vanish, except

$$R^{(k)}_{(1)1j} = \frac{1}{2}\left[\frac{d\varkappa^1_{11}}{dt} - \varkappa^1_{11}\varkappa^1_{11}\right]\delta^k_j,$$

and all curvature $d-$tensors are zero. Consequently, all Ricci $d-$tensors vanish and the scalar curvature cancels. The geometrical Einstein equations (22) produced by the Finslerian metric (25) become trivial, namely

$$0 = \mathcal{T}_{AB}, \quad \forall\ A, B \in \left\{1,\ i,\ {}^{(1)}_{(i)}\right\}.$$

At the same time, the electromagnetic 2-form associated to the Finslerian metric (25) has the trivial form $\mathbb{F} := \mathbb{F}_{[2]} = 0$. In conclusion, both the metric-tensor based geometrical (gravitational and electromagnetic) theories are shown to be trivial for the

case of the rheonomic locally Finslerian Chernov metric (25). Hence, for developing a non-trivial $h-$model, one may need to consider other closely related alternatives offered by $h-$conformally-deformed models or by $x-$dependent rheonomic Finsler metrics of $m-$root type.

REFERENCES

[1] *G.S. Asanov*, "Finslerian Extension of General Relativity", Reidel, Dordrecht, 1984.

[2] *G.S. Asanov*, "Jet extension of Finslerian gauge approach", Fortschritte der Physik **38**, No. 8 (1990), 571–610.

[3] *Gh. Atanasiu, M. Neagu*, "On Cartan spaces with the m-th root metric $K(x,p) = \sqrt[m]{a^{i_1 i_2 ... i_m}(x) p_{i_1} p_{i_2} ... p_{i_m}}$", Hypercomplex Numbers in Geometry and Physics (2010), in press; http://arXiv.org/math.DG/0802.2887v4 (2008).

[4] *V. Balan*, "Numerical multilinear algebra of symmetric m-root structures. Spectral properties and applications", In Symmetry: Culture and Science, Symmetry Festival 2009, Symmetry in the History of Science, Art and Technology; Part 2; Geometric Approaches to Symmetry 2010; Budapest, Hungary, **21** (1–3) (2010), 119–131.

[5] *V.M. Chernov*, "On defining equations for the elements of associative and commutative algebras and on associated metric forms", In: "Space-Time Structure. Algebra and Geometry" (D.G. Pavlov, Gh. Atanasiu, V. Balan Eds.), Russian Hypercomplex Society, Lilia Print, Moscow, 2007, 189–209.

[6] *G.I. Garas'ko*, "Foundations of Finsler Geometry for Physicists" (in Russian), Tetru Eds, Moscow, 2009.

[7] *G.I. Garas'ko, D.G. Pavlov*, "The notions of distance and velocity modulus in the linear Finsler spaces", "Space-Time Structure. Algebra and Geometry" (D.G. Pavlov, Gh. Atanasiu, V. Balan Eds.), Russian Hypercomplex Society, Russian Hypercomplex Society, Lilia Print, Moscow, 2007, 104–117.

[8] *M. Matsumoto*, "On Finsler spaces with curvature tensors of some special forms", Tensor N. S. **22** (1971), 201–204.

[9] *M. Matsumoto, H. Shimada*, "On Finsler spaces with 1-form metric. II. Berwald-Moór's metric $L = \left(y^1 y^2 ... y^n\right)^{1/n}$", Tensor N. S. **32** (1978), 275–278.

[10] *R.V. Mikhailov*, "On some questions of four dimensional topology: a survey of modern research", Hypercomplex Numbers in Geometry and Physics 1 (1), Vol. **1** (2004), 100–103.

[11] *R. Miron, M. Anastasiei*, "The Geometry of Lagrange Spaces: Theory and Applications", Kluwer Academic Publishers, 1994.

[12] *M. Neagu*, "Riemann-Lagrange Geometry on 1-Jet Spaces", Matrix Rom, Bucharest, 2005.

[13] *M. Neagu*, "The geometry of relativistic rheonomic Lagrange spaces", BSG Proc. **5**, Geometry Balkan Press, Bucharest, 2001, 142–169.

[14] *D.G. Pavlov*, "Chronometry of three-dimensional time", Hypercomplex Numbers in Geometry and Physics **1** (**1**), Vol. **1** (2004), 19–30.

[15] *D.G. Pavlov*, "Four-dimensional time", Hypercomplex Numbers in Geometry and Physics 1 (1), Vol. **1** (2004), 31–39.

[16] *H. Shimada*, "On Finsler spaces with the metric $L(x,y) = \sqrt[m]{a_{i_1 i_2 ... i_m}(x) y^{i_1} y^{i_2} ... y^{i_m}}$", Tensor N. S. **33** (1979), 365–372.