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SOURCE IDENTIFICATION PROBLEM FOR DEGENERATE DIFFERENTIAL EQUATIONS

Fadi Awawdeh[1], H.M. Obiedat[2]

This paper deals with an identification problem for degenerate parabolic equations. The problem consists of recovering a source term from the knowledge of an additional observation of the solution by exploiting some accessible measurements. Existence, uniqueness and continuous dependence results are proved for the problem. Applications to the source identification problems for the Poisson-heat equation and Maxwell system are given to illustrate the theory.

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1. Introduction

Using partial differential equations to model physical systems is one of the oldest activities in applied mathematics. A complete model requires certain state inputs in the form of initial and/or boundary data together with what might be called structure inputs such as coefficients or source terms which are related to the physical properties of the system. Obtaining a unique solution for the associated well-posed problem constitutes what we will call solving the direct problem. Solving the direct problem permits the computation of various system outputs of physical interest. On the other hand, when some of the required inputs are not available we may instead be able to determine the missing inputs from outputs that are measured rather than computed by formulating and solving an appropriate inverse problem. In particular, when the missing inputs are one or more unknown coefficients in the partial differential equation, the problem is called a coefficient identification problem and when the source term is missing it is a source identification problem (see [11,15,22,23]).

We point out that the problem of identifying a linear source in nondegenerate parabolic equations is very popular and widely studied in the literature concerning inverse problems for PDEs. The question of uniqueness has been solved in [7,8,14] by a method based on local Carleman estimates. There have been many papers dealing with the Lipschitz stability for parabolic problems (see for instance [13]). All these Lipschitz stability results were obtained using global Carleman estimates,

[1]Hashemite University,Jordan, E-mail: awawdeh@hu.edu.jo
[2] Hashemite University,Jordan, E-mail: hobiedat@hu.edu.jo

which were first introduced to prove observability inequalities and null controllability results. Semigroup theory and fixed point argument are also applied in the field of inverse problems by Prilepko et. al. [24, chapter 7], Orlovsky [20,21], Awawdeh [5,6], Lorenzi [16-19].

In contrast, in both fields of controllability and inverse problems very few results are known for degenerate parabolic equations, even though this class of operators occurs in interesting theoretical and applied problems such as diffusion processes, laminar flow and climatology models (see, e.g., [1-4] and [7,8]).

Let X be a Banach space endowed with the norm $\|\cdot\|$ and let M and L be two single valued closed linear operators in X. Let $z \in X$; let $\phi : X \to \mathbb{R}_+$ be a C^1 functional and let us consider the following identification problem for degenerate equations:

(IP) given $u_0 \in X$ and $g \in C^1([0,\tau];\mathbb{R})$, find $f \in C^1([0,\tau];\mathbb{R})$ and a strict solution $v \in C^1([0,\tau];X)$ of the degenerate problem

$$\begin{cases} \frac{dMv}{dt} = Lv(t) + f(t)z, & 0 \le t \le \tau, \\ Mv(0) = u_0, \end{cases}$$

satisfying the additional condition

$$\phi[Mv(t)] = g(t).$$

The object of this paper can be described as follows: Is it possible to recover a source term f from the knowledge of an additional observation of the solution? We answer this question in the case of the boundary observation $\phi[Mv(t)] = g(t)$.

We note that (IP) represents the mathematical model for various phenomena, such as, for instance, heat transfer obeying Fourier's Law or linear diffusion in a homogeneous three-dimensional body. On the other hand, the set $\{f(\cdot)z;\ z \in X\}$ can be viewed as a family of special feedback laws which modify the instantaneous rate of change, $(Mv)'(t)$, only along the z direction and having the same sense as z, and 'magnitude' $f(t)$. So, the problem (IP) consists in finding the 'right magnitude f' of the feedback, in order for (IP) to have a unique strict solution v having a preassigned mean, i.e. g.

In this work we present some results concerning solutions of problem (IP) using semigroup theory and perturbation theory for linear operators.

The rest of the paper is organized as follows. In the next section, some definitions and preliminary results are introduced. Section 3 presents sufficient conditions for the existence and uniqueness of solutions for problem (IP). Section 4 is devoted to the problem of existence and uniqueness of solution for degenerate systems of Maxwell equations and Poisson-heat equations.

2. **Preliminary Results**

We denote by X a Banach space with norm $\|\cdot\|$ and $A : D(A) \to X$ is the infinitesimal generator of a c_0-semigroup of bounded linear operators $T(t)$, $t \ge 0$, on

X. It is well known that A is closed and its domain $D(A)$ equipped with the graph norm

$$\|x\|_A = \|x\| + \|Ax\|$$

becomes a Banach space, which we shall denote by X_A.

Let L and M be two single valued, closed linear operators in a Banach space X with $D(L) \subset D(M)$. We are concerned with resolvent of the multivalued linear operator LM^{-1}. In order to represent $(\lambda - LM^{-1})^{-1}$ by L and M, we introduce the notion $\rho_M(L)$ of the M resolvent set of L by:

$$\rho_M(L) = \{\lambda \in \mathbb{C}; \lambda M - L \text{ has a single valued and bounded inverse on } X\}$$

and the bounded operator $(\lambda M - L)^{-1}$ is called the M resolvent of L.

Theorem 2.1 ([12])**.** *Let A be a multivalued linear operator on X such that $A - \beta$ is maximal dissipative with some real number β, i.e., A satisfies*

$$\operatorname{Re}(f, u)_X \leq \beta \|u\|_X^2 \qquad \text{for all } \ f \in Au \tag{1}$$

with the range condition

$$R(\lambda_0 - A) = X \qquad \text{for some } \ \lambda_0 > \beta. \tag{2}$$

Then, $\rho(A) \supset (\beta, \infty)$ and A satisfies

$$\left\|(\lambda - A)^{-1}\right\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda - \beta}, \qquad \lambda > \beta.$$

By virtue of Theorem 2.1, if A is a multivalued linear operator on X with a maximal dissipative $A - \beta$, $\beta \in \mathbb{R}$, a semigroup $T(t) = e^{tA}$ is generated by A on the whole space X with the estimate

$$\left\|e^{tA}\right\|_{\mathcal{L}(X)} \leq e^{\beta t}.$$

For more details about the construction of the exponential semigroup, the reader can refer to [12].

Theorem 2.2 ([12])**.** *Let X be a Banach space and A be a multivalued linear operator generating a c_0-semigroup $T(t)$ on X. For any $f \in C^1([0, T]; X)$ and any $u_0 \in D(A)$, the function*

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)ds$$

is the unique solution to the multivalued problem

$$\begin{aligned} &u'(t) \in Au(t) + f(t), \qquad && 0 \leq t \leq T, \\ &u(0) = u_0. && \end{aligned}$$

The following result from perturbation theory for linear operators will be helpful in the sequel.

Theorem 2.3 ([10]). *Let X be a Banach space and let A be the infinitesimal generator of a c_0-semigroup $T(t)$ on X. If $B : X_A \longrightarrow X_A$ is a continuous linear operator, then $A + B$ is the infinitesimal generator of a c_0-semigroup on X.*

3. Main Results

Let us first consider the multivalued identification problem in a Banach space X:

$$\frac{du(t)}{dt} \in Au(t) + f(t)z \;, \qquad 0 \le t \le \tau, \tag{3}$$

$$u(0) = u_0, \tag{4}$$

$$\phi[u(t)] = g(t), \qquad 0 \le t \le \tau, \tag{5}$$

where A is a multivalued linear operator on X, $z, u_0 \in X$, $g \in C^1([0,\tau];\mathbb{R})$, $\phi \in X^*$, X^* being the dual space to X, and $u \in C([0,\tau]; D(A))$ and $f \in C([0,\tau];\mathbb{R})$ are the unknown functions.

We will propose a method coupled with the perturbation theory for linear operators for the solvability of the identification problem (3)-(5).

Theorem 3.1. *Let A be a multivalued linear operator that generates a c_0-semigroup on X, $z \in X$, $u_0 \in D(A)$, $g \in C^1([0,\tau];\mathbb{R})$, $\phi \in X^*$ and $\phi[z] \neq 0$. Then the identification problem (3)-(5) possesses a unique solution in the class of functions*

$$u \in C^1([0,\tau]; D(A)), \qquad f \in C^1([0,\tau];\mathbb{R}).$$

Proof. By applying the linear functional ϕ to both sides of (3) and using (5) we have

$$g'(t) \in \phi[Au(t)] + f(t)\phi[z],$$

and if $\phi[z] \neq 0$, we obtain

$$f(t) \in \frac{1}{\phi[z]}(g'(t) - \phi[Au(t)]). \tag{6}$$

Substituting (6) in (3), we get

$$u'(t) \in Au(t) + \frac{1}{\phi[z]}(g'(t) - \phi[Au(t)])z. \tag{7}$$

By defining the operator

$$Bx = \frac{-1}{\phi[z]}(\phi[Ax])z, \tag{8}$$

then (7) becomes

$$u'(t) \in (A+B)u(t) + \frac{1}{\phi[z]}g'(t)z. \tag{9}$$

The boundedness of the operator B in X_A, follows from the estimate

$$\begin{aligned}\|B\|_A &= \sup_{\|x\|_A=1} \|Bx\| \\ &= \sup_{\|x\|_A=1} \left\| \frac{-1}{\phi[z]}(\phi[A(x)])z \right\| \\ &\leq \sup_{\|x\|_A=1} \frac{1}{|\phi[z]|} \|z\| \|\phi\| \|Ax\| \\ &\leq \frac{1}{|\phi[z]|} \|z\| \|\phi\| .\end{aligned}$$

This proves that B is a bounded linear operator on X_A. By virtue of Theorem 2.3, $A+B$ is the infinitesimal generator of a semigroup $S(t)$, $t \geq 0$. Since $u_0 \in D(A)$, Theorem 2.2 implies that the Cauchy problem (3)-(4) has a unique solution $u(t)$

$$u(t) = S(t)u_0 + \frac{1}{\phi[z]}\int_0^t S(t-s)g'(s)zds, \tag{10}$$

and by (6) and (10), $f(t)$ is uniquely determined. Therefore, the reduced problem (3)-(5) possesses a unique solution (u, f) and the proof is completed. □

Consider now the identification degenerate problem (IP) in the Banach space X where M and L are single valued closed linear operators in X with $D(L) \subset D(M)$, $z \in X$, $u_0 \in D(L)$, $g \in C^1([0,\tau];\mathbb{R})$, $\phi \in X^*$ and the pair $(u, f) \in C([0,\tau]; D(L)) \times C([0,\tau];\mathbb{R})$ is to be determined.

Note that M may have no bounded inverse and so the classical theory of semigroups does not apply here.

We assume the resolvent set $\rho_M(L)$ contains a region

$$\Sigma_\gamma = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda - \gamma) \geq -c(|\mathrm{Im}\,\lambda| \ + 1)^\alpha, \qquad \gamma \in \mathbb{R}, \tag{11}$$

and the M resolvent satisfies

$$\left\| M(\lambda M - L)^{-1} \right\|_{\mathcal{L}(X)} \leq \frac{C}{(|\lambda - \gamma| + 1)^\beta}, \qquad \lambda \in \Sigma_\gamma, \tag{12}$$

with some exponents $0 < \beta \leq \alpha \leq 1$ and constants $c, C > 0$.

We can now prove existence and uniqueness theorem of solutions to the identification problem (IP).

Theorem 3.2. *Let M and L be closed linear operators in the Banach space X with $D(L) \subset D(M)$, $z \in X$, $u_0 \in D(L)$, $M(D(L))$ is dense in X, $g \in C^1([0,\tau];\mathbb{R})$, $\phi \in X^*$, $\phi[z] \neq 0$ and (11) and (12) be satisfied. Then, problem (IP) possesses a unique solution (v, f) such that*

$$Mv \in C^1((0,\tau]; X), \quad Lv \in C((0,\tau]; X), \quad f \in C([0,\tau];\mathbb{R}).$$

Proof. By changing the unknown function to $u(t) = Mv(t)$, we write the identification problem (IP) into the multivalued form

$$\begin{cases} \frac{du(t)}{dt} \in LM^{-1}u(t) + f(t)z\ , & 0 \leq t \leq \tau, \\ u(0) = u_0, & \\ \phi[u(t)] = g(t), & 0 \leq t \leq \tau. \end{cases} \tag{13}$$

In addition, change of the unknown function to $u_\gamma(t) = e^{-\gamma t}u(t)$ yields that (13) is regarded as a multivalued equation of the form (3)-(5) with a coefficient operator $A = LM^{-1} - \gamma$. We can verify that if L and M are any two closed linear operators in X then $\rho_M(L) \subset \rho(LM^{-1})$ and $M(\lambda M - L)^{-1} = (\lambda - LM^{-1})^{-1}$, (see [12]). It follows for $\lambda + \gamma \in \rho_M(L)$ that

$$(\lambda - A)^{-1} = M((\lambda + \gamma)M - L)^{-1}.$$

In this line, (11) and (12) yields directly that:

The resolvent set $\rho(A)$ contains a region

$$\Sigma = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq -c(|\operatorname{Im}\lambda|\ + 1)^\alpha, \qquad \gamma \in \mathbb{R}, \tag{14}$$

and the resolvent $(\lambda - A)^{-1}$ satisfies

$$\left\|(\lambda - A)^{-1}\right\|_{\mathcal{L}(X)} \leq \frac{C}{(|\lambda| + 1)^\beta}, \qquad \lambda \in \Sigma, \tag{15}$$

with some exponents $0 < \beta \leq \alpha \leq 1$ and constants $c, C > 0$. (14) and (15) ensure that the multivalued linear operator A generates an infinitely differentiable semigroup on X, (see [2]). Therefore, the reduced multivalued problem possesses a unique strict solution (u, f). Clearly u_γ is a strict solution to (3)-(5) if and only if v is a strict solution to the identification problem (IP) in the sense

$$Mv \in C^1((0, \tau]; X), \quad Lv \in C((0, \tau]; X).$$

Finally, the uniqueness of the solution v follows from the invertibility of $\gamma M - L$. □

Since $D(A) = D(LM^{-1}) = M(D(L))$ the continuity of $Mv(t)$ at $t = 0$ in the topology of X is obtained as follows:

Theorem 3.3. *Let $u_0 \in \overline{M(D(L)}$ if $\alpha = \beta = 1$ and $u_0 \in M(D(L))$ on the other case ($\alpha \neq 1$ or $\beta \neq 1$). Then for the solution v obtained in Theorem 3.2, $Mv(t)$ is continuous at $t = 0$ in the norm of X, i.e. $Mv \in C([0, \tau]; X)$ with $Mv(0) = u_0$ if $u_0 \in M(D(L))$.*

Next, we consider the identification problem (IP) in X, X being a Hilbert space with the inner product $(\cdot, \cdot)_X$. We assume:

$$\operatorname{Re}(Lv, Mv)_X \leq \beta \left\|Mv\right\|_X^2, \qquad v \in D(L); \tag{16}$$

$$R(\lambda_0 M - L) = X \text{ and } (\lambda_0 M - L)^{-1} \text{ is single valued for some } \lambda_0 > \beta. \tag{17}$$

Then we prove the following theorem:

Theorem 3.4. *Let M and L be linear operators in the Hilbert space X with $D(L) \subset D(M)$, $z \in X$, $u_0 \in D(L)$, $M(D(L))$ is dense in X, $g \in C^1([0,\tau];\mathbb{R})$, $\phi \in X^*$, $\phi[z] \neq 0$ and (16) and (17) be satisfied. Then, for any $u_0 \in M(D(L))$, problem (IP) possesses a unique solution (v, f) such that*

$$Mv \in C^1([0,\tau];X), \quad Lv \in C([0,\tau];X), \quad f \in C^1([0,\tau];\mathbb{R}).$$

Proof. By changing the unknown function $u(t) = Mv(t)$, we write (IP) into a multivalued problem of the form (3)-(5) with the operator coefficient $A = LM^{-1}$. Since M^{-1} is not continuous in general, the operator A is considered as a multivalued linear operator. To show that (1) is satisfied, let $h \in Au$, then $h = LM^{-1}u = Lv$ and $Mv = u$ for some $v \in D(L)$. So that, $(h,u)_X = (Lv, Mv)_X$, which shows that (1) follows from (16). On the other hand, for any $h \in X$, we have by (17) that $h = (\lambda_0 M - L)v$ for some $v \in D(L)$. If we put $u = Mv$, then $u \in M(D(L)) = D(A)$ and $h \in (\lambda_0 - A)u$, i.e. (2) satisfied. According to Theorem 2.1, this proves that $A - \beta$ is maximal dissipative in X and so A is the generator of a c_0-semigroup on X. Clearly, $u_0 \in D(A)$ if and only if $u_0 \in M(D(L))$. □

4. Applications

4.1. Identification Problem of the Poisson-heat Equation

In several applications, when the temperature of a thermal body, subjected to an external supply of heat, is to be determined, the source itself is often unknown or scarcely known. So, we have to face with recovering both the temperature and the unknown source. To compensate for the lack of information, suitable measurements involving the temperature are given, as well as suitable assumptions on the source are made. For instance, it is assumed to depend on one space variable, i.e. on time only or, to be the product of two functions, the first depending on the temperature and the latter on the space variable.

Consider the Poisson-heat equation:

$$\begin{aligned} \frac{\partial m(x)v}{\partial t} &= \Delta v + f(t)h(x), && (x,t) \in \Omega \times (0,T], \\ v &= 0, && (x,t) \in \partial\Omega \times (0,T], \\ m(x)v(x,0) &= u_0(x), && x \in \Omega, \end{aligned} \tag{18}$$

with the supplementary condition

$$\int_\Omega \eta(x)m(x)v(x,t)dx = g(t), \qquad \forall t \in [0,T] \tag{19}$$

in a bounded region $\Omega \subset \mathbb{R}^n$ with a smooth boundary $\partial\Omega$. Here $m(x) \geq 0$ in Ω is a given function in $L^\infty(\Omega)$, $u_0, \eta, h \in H^{-1}(\Omega)$, and g is continuous function on $[0,T]$.

We start by introducing a convenient abstract frame. Let $X = H^{-1}(\Omega)$, then this problem is formulated as a problem of the form (IP) in which $M : L^2(\Omega) \to L^2(\Omega) \subset X$ is the multiplication operator by the function $m(x)$ and $L : H_0^1(\Omega) \to H^{-1}(\Omega)$ is Δ with the Dirichlet boundary conditions.

For $\lambda \in \mathbb{C}$, consider the sesquilinear form

$$a_\lambda(u,v) = \lambda \int_\Omega m(x) u \overline{v} dx + \int_\Omega \nabla u \cdot \nabla \overline{v} dx, \qquad u, v \in H_0^1(\Omega),$$

defined on $H_0^1(\Omega)$. Obviously this form is continuous on $H_0^1(\Omega)$. In addition for $\frac{\pi}{2} < \omega < \pi$ and for suitable $c > 0$, the following estimate

$$|a_\lambda(u,v)| \geq \delta(\|u\|_{H_0^1}^2 + |\lambda| \left\|\sqrt{m}u\right\|_{L^2}^2), \qquad u \in H_0^1(\Omega),$$

is observed to hold for each $\lambda \in \Sigma = \{\lambda \in \mathbb{C} : |\arg \lambda| \leq \omega \ \ or \ \ |\lambda| \leq c\}$ with some uniform $\delta > 0$. Then, since

$$a_\lambda(u,v) = \langle (\lambda M - L)u, v \rangle_{H^{-1} \times H_0^1}, \qquad u, v \in H_0^1(\Omega),$$

the Lax-Milgram Theorem (see [25; p. 92]) yields that $\lambda M - L$, $\lambda \in \Sigma$, has a bounded inverse from $H^{-1}(\Omega)$ to $H_0^1(\Omega)$ with an estimate

$$\delta(\|u\|_{H_0^1}^2 + |\lambda| \left\|\sqrt{m}u\right\|_{L^2}^2) \leq \|\varphi\|_{H^{-1}} \|u\|_{H_0^1} \qquad \text{if} \quad u = (\lambda M - L)^{-1}\varphi.$$

Hence, $\|Lu\|_{H^{-1}} \leq C \|u\|_{H_0^1} \leq C \|\varphi\|_{H^{-1}}$. Moreover, noting the identity

$$\lambda M(\lambda M - L)^{-1} = 1 + L(\lambda M - L)^{-1},$$

we obtain that

$$\left\|M(\lambda M - L)^{-1}\varphi\right\|_{H^{-1}} \leq C |\lambda|^{-1} \|\varphi\|_{H^{-1}}, \qquad \varphi \in H^{-1}(\Omega).$$

This immediately yields that (14) and (15) are valid with $\alpha = \beta = 1$, $\gamma = 0$.

From Theorem 3.2 it follows that for any $u_0 \in H^{-1}(\Omega)$, problem (18)-(19) possesses a unique solution (v, f) such that

$$mv \in C^1((0,T]; H^{-1}(\Omega)), \quad v \in C((0,T]; H^{-1}(\Omega)), \quad f \in C([0,T]; \mathbb{R}).$$

In addition, Theorem 3.3 yields that mv is continuous at $t = 0$ if $u_0 = mv_0$ with some $v_0 \in L^2(\Omega)$.

4.2. The Identification Problem related to the system of Maxwell equations

Consider the system of Maxwell equations in a bounded domain $\Omega \subset \mathbb{R}^3$:

$$\begin{aligned} rot\ E &= -\frac{\partial B}{\partial t}, \\ rot\ H &= \frac{\partial D}{\partial t} + J, \end{aligned} \tag{20}$$

where E is the vector of electric field strength, H is the vector of the magnetic field strength, D and B are designate the electric and magnetic induction vectors, respectively. In what follows we denote by J the current density.

In the sequel we deal with a linear medium in which the vectors of strengths are proportional to those of inductions in accordance with the governing laws:

$$D = \epsilon E, \qquad B = \mu H, \tag{21}$$

and we assume, in addition, that Ohm's law

$$J = \sigma E + I \tag{22}$$

is satisfied in the domain Ω, where ϵ is the dielectric permeability of the medium, μ is the magnetic permeability, σ is the electric conductance and I the density of the extraneous current. Further development is connected with the initial conditions for the vectors of the electric and magnetic inductions:

$$\begin{aligned} D(x,0) &= D_0(x), \\ B(x,0) &= B_0(x). \end{aligned} \tag{23}$$

The direct problem here consists of finding the functions E, D, H, B from the system (20)-(23) for the given functions $\epsilon, \mu, \sigma, I, D_0$ and B_0 involved. The statement of an inverse problem involves the density of the extraneous current as an unknown of the structure

$$I(x,t) = f(t)p(x), \tag{24}$$

here the matrix $p(x)$ of size 3×3 is known for all $x \in \Omega$, while the unknown vector-valued function $f(t)$ is sought. To complete such a setting of the problem, we take the integral overdetermination in the form

$$\int_\Omega E(x,t)w(x) = g(t), \qquad 0 \le t \le T, \tag{25}$$

where the function $w(x)$ is known in advance. The system of equations (20)-(25) is treated as the inverse problem for the Maxwell system related to the unknown functions E, H and f.

By setting

$$v = \begin{pmatrix} E \\ H \end{pmatrix}, \; c(x) = \begin{pmatrix} \epsilon(x) & 0 \\ 0 & \mu(x) \end{pmatrix}, \; b(x) = -\begin{pmatrix} \sigma(x) & 0 \\ 0 & 0 \end{pmatrix}, \; f(x,t) = -\begin{pmatrix} I(x,t) \\ 0 \end{pmatrix} \tag{26}$$

then system (20)-(23) can be written as

$$\frac{\partial c(x)v}{\partial t} = \sum_{i=1}^{3} a_i \frac{\partial v}{\partial x_i} + b(x)v + f(x,t), \qquad (x,t) \in \mathbb{R}^3 \times [0,T], \tag{27}$$

with certain 6×6 matrices a_i, $i = 1,2,3$.

$\epsilon(x), \mu(x)$ and $\sigma(x)$ are assumed to be real matrices and the components of which are bounded measurable functions in $\mathbb{R}^3$. In addition, we assume:

$$\epsilon(x) \text{ is symmetric and } \epsilon(x) \ge 0 \; \text{ for all } x \in \mathbb{R}^3; \tag{28}$$

$$\mu(x) \text{ is symmetric and } \mu(x) \ge \delta, \; \text{ for some } \delta > 0 \text{ uniformly in } x \in \mathbb{R}^3; \tag{29}$$

$$(\{\gamma\epsilon(x) + \sigma(x)\}\xi, \xi) \ge \delta \|\xi\|^2, \; \xi \in \mathbb{R}^3, \text{ for some } \delta > 0 \text{ and } \gamma \ge 0 \text{ uniformly in } x \in \mathbb{R}^3. \tag{30}$$

Further treatment of the system (20)-(25) as an abstract problem is connected with introducing the Lebesgue space

$$X = (L^2(\mathbb{R}^3))^3 \times (L^2(\mathbb{R}^3))^3 = (L^2(\mathbb{R}^3))^6,$$

using the bounded operator M of multiplication by $c(x)$ acting in X (the adjoint operator $M^*$of M satisfies $M^* = M$) and the closed linear operator L accept for each $v \in D(L)$ the set of relations

$$D(L) = \left\{ v \in X : \sum_{i=1}^{3} a_i \frac{\partial v}{\partial x_i} \in X \right\}$$

$$Lv = \sum_{i=1}^{3} a_i \frac{\partial v}{\partial x_i} + b(x)v.$$

In doing so, the symbols $v(t)$ and $f(t)z$ will refer to the same functions $v(x,t)$ and $f(t)p(x)$ but being viewed as abstract functions of the variable t with values in the space X. The symbol u_0 will be used in treating the function $u_0\ (x) = (D_0(x), B_0(x))$ as the element of the space X. With these ingredients, the system (20)-(25) reduces to the inverse problem (IP) in the Banach space X.

Condition (16) is verified as follows. Let $v \in Y = (H^1(\mathbb{R}^3))^6 \subset D(L)$. Then,

$$(Lv, v)_X = -\left(v, \sum_{i=1}^{3} a_i \frac{\partial v}{\partial x_i} + b(x)v \right)_X + (b(x)v, v)_X + (v, b(x)v)_X;$$

so that,

$$\operatorname{Re}(Lv, v)_X = \operatorname{Re}(b(x)v, v)_X = -\operatorname{Re}(\sigma E, E)_{L^2}.$$

In this line, (29) and (30) yield that

$$\begin{aligned} \operatorname{Re}(Lv, v)_X &\le -\delta(\|E\|_{L^2}^2 + \|H\|_{L^2}^2) + \gamma(\epsilon(x)E, E)_{L^2} + (\mu(x)H, H)_{L^2} \\ &\le -\delta \|v\|_X^2 + \lambda((\epsilon(x)E, E)_{L^2} + (\mu(x)H, H)_{L^2}) \\ &= -\delta \|v\|_X^2 + \lambda \|Mv\|_X^2, \end{aligned} \tag{31}$$

where $\lambda \ge \max\{\gamma, 1\}$. This estimate is in fact verified for $v \in D(L)$ also, because there exists a sequence $v_n \in Y$ such that $v_n \to v$ and $Lv_n \to Lv$ in X. Thus, (1) holds with $\beta = \max\{\gamma, 1\}$.

Let us next verify (17). Since (31) yields that

$$\|(\lambda M^*M - L)v\|_X \ge \delta \|v\|_X, \qquad v \in D(L),$$

then $(\lambda M^*M - L)$ is seen to be one-to-one and to have a closed range. Therefore it suffices to verify that $R(\lambda M^*M - L)^\perp = \{0\}$. Let $w \in R(\lambda M^*M - L)^\perp$, then $w \in D(L^*)$ and $(\lambda M^*M - L^*)w = 0$. On the other hand, since the principal part of L is symmetric, $w \in D(L)$. Therefore, (31) yields

$$-\delta \|w\|_X^2 + \lambda \|Mw\|_X^2 \ge \operatorname{Re}(Lw, w)_X = \operatorname{Re}(w, L^*w)_X = \lambda \|Mw\|_X^2,$$

and so $w = 0$.

As a result, if conditions (28)-(30) are satisfied, then a solution E, H, p of the identification problem (20)-(25) exists and it is unique in the class of functions

$$E, H \in C([0,T]; (L^2(\mathbb{R}^3))^3), \quad p \in C([0,T]; \mathbb{R}^3).$$

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