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IMPLICIT TYPE FIXED POINT THEOREMS FOR BOUNDED MULTIMAPS

Muhammad Usman ALI[1,2], Tayyab KAMRAN[3,2], Quanita KIRAN[4]

In this paper, we prove some implicit type fixed point theorems for bounded multimaps in a partial metric space by introducing a new family of mappings from $\left(\mathbb{R}^+\right)^4$ *to* $\mathbb{R}^+$*. Our results generalize some existing fixed point theorems. We also construct some examples to establish the generality of our results.*

Keywords: Partial metric spaces, Fixed point, Φ_ψ -family.

1. Introduction and Preliminaries

Matthews [1] introduced the notion of a partial metric space by using the idea that in many problems the distance between same points is not always zero. This notion has many applications not only in mathematics but also in computer sciences [1]. Several authors compliment the work of Matthews and proved several fixed point results in this setting, see for example [2-15]. We collect some basis definitions and results to work with a partial metric space.

Definition 1.1 *[1] Let* X *be a nonempty set. A mapping* $p : X \times X \to \mathbb{R}^+ = [0,\infty)$ *is a partial metric on* X *, if for all* $x, y, z \in X$ *, we have*

(P1) $p(x,x) = p(y,y) = p(x,y)$ if and only if $x = y$;

(P2) $p(x,x) \le p(x,y)$;

(P3) $p(x,y) = p(y,x)$;

(P4) $p(x,z) \le p(x,y) + p(y,z) - p(y,y)$.

[1] Department of Sciences & Humanities, National University of Computer and Emerging Sciences-(FAST), H-11/4, Islamabad Pakistan, e-mail: muh_usman_ali@yahoo.com (M. U. Ali).
[2] Department of Mathematics, School of Natural Sciences, National University of Sciences and Technology H-12, Islamabad Pakistan.
[3] Department of Mathematics, Quaid-i-Azam University, Islamabad Pakistan, e-mail: tayyabkamran@gmail.com (T. Kamran).
[4] School of Electrical Engineering and Computer Science, National University of Sciences and Technology H-12, Islamabad Pakistan, e-mail: quanita.kiran@seecs.edu.pk (Q. Kiran).

Remark 1.2 *([1]) If* $p(x,y)=0$ *then* $(P1)$ *and* $(P2)$ *implies* $x=y$ *but converse is not true in general.*

Lemma 1.3 *[1] Every metric space is a partial metric space.*

Example 1.4 *[1] Let* $X=\{[a,b]:a,b\in\mathbb{R},a\le b\}$. *Define a function* $p:X\times X\to\mathbb{R}^+$ *by,* $p([a,b],[c,d])=\max\{b,d\}-\min\{a,c\}$. *Then* (X,p) *is a partial metric space. Note that* p *is not a metric on* X.

Remark 1.5 *[1] Every partial metric* p *on* X *generates a* T_0 *topology* τ_p *on* X *with base consisting of open balls* $(p-\text{balls})$ $\{B_p(x,\varepsilon):x\in X,\varepsilon>0\}$, *where*

$$B_p(x,\varepsilon)=\{y\in X:p(x,y)<p(x,x)+\varepsilon\}.$$

Definition 1.6 *[1] Let* (X,p) *be a partial metric space. Then,*

(a) A sequence $\{x_n\}$ *in* (X,p) *is said to be convergent to a point* $x\in X$ *with respect to* τ_p *if and only if* $p(x,x)=\lim_{n\to\infty}p(x,x_n)$.

(b) A sequence $\{x_n\}$ *in* X *is said to be a Cauchy sequence if and only if* $\lim_{n,m\to\infty}p(x_n,x_m)$ *exists and is finite.*

(c) A partial metric space (X,p) *is called a complete partial metric space if every Cauchy sequence* $\{x_n\}$ *in* X *converges with respect to* τ_p *to a point* $x\in X$.

Remark 1.7 *[1] For a partial metric space* p *on* X, *the function*

$$d_p(x,y)=2p(x,y)-p(x,x)-p(y,y).$$

defines a metric on X.

Lemma 1.8 *[1] Let* (X,p) *be a partial metric space. A sequence* $\{x_n\}$ *in* (X,d_p) *is said to be convergent to a point* $x\in X$ *if and only if*

$$p(x,x)=\lim_{n\to\infty}p(x,x_n)=\lim_{n,m\to\infty}p(x_n,x_m).$$

Lemma 1.9 *[1] Let* (X,p) *be a partial metric space. Then*

(a) A sequence $\{x_n\}$ *in* X *is Cauchy in* (X,p) *if and only if it is Cauchy in* (X,d_p).

(b) A partial metric space (X,p) *is complete if and only if the metric space* (X,d_p) *is complete.*

By Aydi *et al*. [16], a subset A of a partial metric space (X,p) is said to be bounded, if there exist $x_0\in X$ and $M\ge 0$ such that for all $a\in A$ we have $a\in B_p(x_0,M)$, that is, $p(x_0,a)<p(x_0,x_0)+M$. A subset A is closed, if closedness is taken from (X,τ_p) where τ_p is the topology induced by p. Let $B_p(X)$ be the family of all nonempty bounded subsets of the partial metric space

(X,p). Now we recollect some important notions from [16]. Let $A,B \in B_p(X)$, then

$p(x,A) = \inf\{p(x,a) : a \in A\}$,

$p(A,B) = \inf\{p(x,y) : x \in A, y \in B\}$,

the function $\delta_p : B_p(X) \times B_p(X) \to \mathsf{R}^+$ is defined as

$$\delta_p(A,B) = \sup\{p(a,b) : a \in A, b \in B\}.$$

Remark 1.10 *[16] If* $d_p(x,A) = \inf\{d_p(x,a) : a \in A\}$, *then it is easy to prove that* $p(x,A) = 0$ *implies that* $d_p(x,A) = 0$.

Matthews [1] proved the Banach contraction principle in the setting of partial metric spaces as follows:

Theorem 1.13 *[1] Let* (X,p) *be a complete partial metric space and let* $T : X \to X$ *be a mapping such that there exists* $\alpha \in [0,1)$ *satisfying*

$$p(Tx,Ty) \le \alpha p(x,y)$$

for each $x, y \in X$ *. Then* T *has a unique fixed point.*

In this paper, we introduce a new class of implict type mappings, then by using it we prove some implicit type fixed point theorems for bounded multivalued mappings in partial metric space, as well as in partial metric space endowed with a partial order relation. Haghi *et al*. [8] showed that the fixed point theorem for mapping satisfying the given contractive condition in 0-complete partial metric space may follows from the corresponding fixed point theorem for mapping satisfying the same contractive condition in a complete metric space. It is worth mentioning that the contractive condition we will use is new even in the setting of metric space.

2. Main result

Let $\psi : [0,\infty) \to [0,\infty)$ be a nondecreasing mapping such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t \ge 0$ and $\psi(t) < t$ for all $t > 0$. By Φ_ψ we denote the family of functions $\phi : (\mathsf{R}^+)^4 \to \mathsf{R}^+ = [0,\infty)$, satisfying the following conditions:

(i) ϕ is continuous and nondecreasing in each coordinate;

(ii) let $u_1, u_2 \in \mathbb{R}^+$ such that if $u_1 < u_2$ and $u_1 \le \phi(u_2,u_2,u_1,u_2)$, then $u_1 \le \psi(u_2)$. If $u_1 \ge u_2$ and $u_1 \le \phi(u_1,u_2,u_1,u_1)$, then $u_1 = 0$;

(iii) if $u \in \mathbb{R}^+$ such that $u \le \phi\left(0,0,u,\frac{1}{2}u\right)$, then $u = 0$.

Following are some examples of $\phi \in \Phi_\psi$:

• Let $\phi_1(u_1,u_2,u_3,u_4)=\alpha\max\{u_1,u_2,u_3,u_4\}$ with $\psi(t)=\alpha t$, where $\alpha\in[0,1)$.
• Let $\phi_2(u_1,u_2,u_3,u_4)=\alpha u_4$ with $\psi(t)=\alpha t$, where $\alpha\in[0,1)$.
• Let $\phi_3(u_1,u_2,u_3,u_4)=\alpha\max\{u_1,u_2,u_3\}$ with $\psi(t)=\alpha t$, where $\alpha\in[0,1)$.
• Let $\phi_4(u_1,u_2,u_3,u_4)=\alpha\max\{u_2,u_3\}$ with $\psi(t)=\alpha t$, where $\alpha\in[0,1)$.
• Let $\phi_5(u_1,u_2,u_3,u_4)=\alpha u_1$ with $\psi(t)=\alpha t$, where $\alpha\in[0,1)$.
• Let $\phi_6(u_1,u_2,u_3,u_4)=\frac{\alpha}{2}(u_2+u_3)$ with $\psi(t)=\frac{\alpha}{2}t$, where $\alpha\in[0,1)$.
• Let $\phi_7(u_1,u_2,u_3,u_4)=\alpha\max\left\{u_1,\frac{1}{2}(u_2+u_3),u_4\right\}$ with $\psi(t)=\alpha t$, where $\alpha\in[0,1)$.
• Let $\phi_8(u_1,u_2,u_3,u_4)=au_1+b(u_2+u_3)+cu_4$ with $\psi(t)=(a+2b+c)t$, where a,b,c are nonnegative real numbers such that $a+2b+c\in[0,1)$.
• Let $\phi_9(u_1,u_2,u_3,u_4)=au_2+bu_3+cu_1$ with $\psi(t)=(a+b+c)t$, where a,b,c are nonnegative real numbers such that $a+b+c\in[0,1)$.

Theorem 2.1 *Let (X,p) be a complete partial metric space and $T:X\to B_p(X)$ be a mapping such that for each $x,y\in X$, there exist $\phi\in\Phi_\psi$ and $L\ge 0$ with*

$$\delta_p(Tx,Ty)\le\phi\left(p(x,y),p(x,Tx),p(y,Ty),\frac{1}{2}(p(y,Tx)+p(x,Ty))\right)+L(p(y,Tx)-p(y,y)). \quad (1)$$

Then T has a fixed point.

Proof. Let $\{x_n\}$ be a sequence in X such that $x_{n+1}\in Tx_n$ for each $n\in\mathbb{N}\cup\{0\}$. If $x_{N+1}=x_N$ for some $N\in\mathbb{N}\cup\{0\}$. Then x_N is a fixed point of *T*. Suppose $x_{n+1}\ne x_n$ for each $n\in\mathbb{N}\cup\{0\}$. From (1), we have

$$\begin{aligned}p(x_{n+1},x_{n+2})&\le\delta_p(Tx_n,Tx_{n+1})\\
&\le\phi\left(p(x_n,x_{n+1}),p(x_n,Tx_n),p(x_{n+1},Tx_{n+1}),\frac{1}{2}(p(x_{n+1},Tx_n)+p(x_n,Tx_{n+1}))\right)\\
&\quad+L(p(x_{n+1},Tx_n)-p(x_{n+1},x_{n+1}))\\
&\le\phi\left(p(x_n,x_{n+1}),p(x_n,x_{n+1}),p(x_{n+1},x_{n+2}),\frac{1}{2}(p(x_{n+1},x_{n+1})+p(x_n,x_{n+2}))\right)\\
&\quad+L(0)\\
&\le\phi\left(p(x_n,x_{n+1}),p(x_n,x_{n+1}),p(x_{n+1},x_{n+2}),\frac{1}{2}(p(x_n,x_{n+1})+p(x_{n+1},x_{n+2}))\right). \quad (2)\end{aligned}$$

We claim that $p(x_{n+1},x_{n+2}) < p(x_n,x_{n+1})$ for each $n\in\mathbb{N}\cup\{0\}$. Suppose on contrary that $p(x_{n+1},x_{n+2}) \geq p(x_n,x_{n+1})$ for some n. Since ϕ is nondecreasing, by using this in (2), we have

$$p(x_{n+1},x_{n+2}) \leq \phi(p(x_{n+1},x_{n+2}),\, p(x_n,x_{n+1}),\, p(x_{n+1},x_{n+2}),\, p(x_{n+1},x_{n+2})). \tag{3}$$

By (3) and property (ii) of Φ_ψ, we have

$$p(x_{n+1},x_{n+2}) = 0.$$

Which is a contradiction to our assumption, i.e. $x_{n+1}\neq x_n$ for each $n\in\mathbb{N}\cup\{0\}$. Thus $p(x_{n+1},x_{n+2}) < p(x_n,x_{n+1})$ for all n. From (2), we have

$$p(x_{n+1},x_{n+2}) \leq \phi(p(x_n,x_{n+1}),\, p(x_n,x_{n+1}),\, p(x_{n+1},x_{n+2}),\, p(x_n,x_{n+1})). \tag{4}$$

By (4) and property (ii) of Φ_ψ, we have

$$p(x_{n+1},x_{n+2}) \leq \psi(p(x_n,x_{n+1}))\ \forall\, n\in\mathbb{N}\cup\{0\}.$$

Consequently, we get

$$p(x_{n+1},x_{n+2}) \leq \psi^{n+1}(p(x_0,x_1)),\ \textit{forall}\ n\in\mathbb{N}\cup\{0\}. \tag{5}$$

Let $n>m$, we have

$$\begin{aligned} p(x_m,x_n) &\leq p(x_m,x_{m+1}) + p(x_{m+1},x_{m+2}) + \cdots + p(x_{n-1},x_n) - \sum_{i=m+1}^{n-1} p(x_i,x_i) \\ &\leq p(x_m,x_{m+1}) + p(x_{m+1},x_{m+2}) + \cdots + p(x_{n-1},x_n) \\ &\leq \psi^m(p(x_0,x_1)) + \psi^{m+1}(p(x_0,x_1)) + \cdots + \psi^{n-1}(p(x_0,x_1)) \\ &= \sum_{i=m}^{n-1}\psi^i(p(x_0,x_1)). \end{aligned}$$

Thus, we have

$$d_p(x_m,x_n) \leq 2p(x_m,x_n) \leq 2\sum_{i=m}^{n-1}\psi^i(p(x_0,x_1)) < \infty.$$

Therefore, we conclude that $\{x_n\}$ is a Cauchy sequence in (X,d_p). Since (X,p) is a complete partial metric space, by Lemma 1.9-(b), (X,d_p) is a complete metric space. Then there exists $x^*\in X$ such that $x_n\to x^*\in X$ with respect to d_p, as $n\to\infty$. By Lemma 1.8, we have

$$p(x^*,x^*) = \lim_{n\to\infty} p(x_n,x^*) = \lim_{n,m\to\infty} p(x_n,x_m) = 0. \tag{6}$$

From (1), we have

$$\begin{aligned} \delta_p(x_{n+1},Tx^*) &\leq \delta_p(Tx_n,Tx^*) \\ &\leq \phi\left(p(x_n,x^*),\, p(x_n,Tx_n),\, p(x^*,Tx^*),\, \frac{1}{2}(p(x^*,Tx_n) + p(x_n,Tx^*)) \right) \end{aligned}$$

$$
\begin{aligned}
&+L(p(x^*,Tx_n)-p(x^*,x^*))\\
&\le \phi\left(p(x_n,x^*),p(x_n,x_{n+1}),p(x^*,Tx^*),\frac{1}{2}(p(x^*,x_{n+1})+p(x_n,Tx^*))\right)\\
&+L(p(x^*,x_{n+1})-p(x^*,x^*)).
\end{aligned}
$$

By using the triangular inequality, it further implies that

$$
\begin{aligned}
\delta_p(x^*,Tx^*) &\le p(x^*,x_{n+1})+\delta_p(x_{n+1},Tx^*)-p(x_{n+1},x_{n+1})\\
&\le p(x^*,x_{n+1})+\delta_p(x_{n+1},Tx^*)\\
&\le p(x^*,x_{n+1})+\phi(p(x_n,x^*),p(x_n,x_{n+1}),p(x^*,Tx^*),\\
&\quad \frac{1}{2}(p(x^*,x_{n+1})+p(x_n,x^*)+p(x^*,Tx^*)))\\
&\quad +L(p(x^*,x_{n+1})-p(x^*,x^*)). \qquad (7)
\end{aligned}
$$

Letting $n\to\infty$ in (7), we have

$$
\begin{aligned}
\delta_p(x^*,Tx^*) &\le \phi\left(0,0,p(x^*,Tx^*),\frac{1}{2}(0+p(x^*,Tx^*))\right)+L(0)\\
&\le \phi\left(0,0,\delta_p(x^*,Tx^*),\frac{1}{2}\delta_p(x^*,Tx^*)\right). \qquad (8)
\end{aligned}
$$

By (8) and property (iii) of Φ_ψ, we have $\delta_p(x^*,Tx^*)=0$. Hence $Tx^*=\{x^*\}$.

Corollary 2.2 *Let* (X,p) *be a complete partial metric space and* $T:X\to B_p(X)$ *be a mapping such that for each* $x,y\in X$ *, we have*

$$
\delta_p(Tx,Ty)\le \alpha\max\left\{p(x,y),p(x,Tx),p(y,Ty),\frac{1}{2}(p(y,Tx)+p(x,Ty))\right\}
$$
$$
+L(p(y,Tx)-p(y,y))
$$

where $\alpha\in[0,1)$ *and* $L\ge 0$. *Then* T *has a fixed point.*

Proof. Let $\phi(u_1,u_2,u_3,u_4)=\phi_1(u_1,u_2,u_3,u_4)=\alpha\max\{u_1,u_2,u_3,u_4\}$ with $\psi(t)=\alpha t$, where $\alpha\in[0,1)$. From (1), we have

$$
\delta_p(Tx,Ty)\le \alpha\max\left\{p(x,y),p(x,Tx),p(y,Ty),\frac{1}{2}(p(y,Tx)+p(x,Ty))\right\}
$$
$$
+L(p(y,Tx)-p(y,y)),
$$

for all $x,y\in X$. Therefore by Theorem 2.1, T has a fixed point.

Example 2.3 *Let* $X=\left\{(0,0),\left(0,\frac{1}{4}\right),\left(0,\frac{1}{2}\right),(0,1)\right\}$ *be endowed with partial metric* p *define by* $p(x,y)=\max\{x_1,y_1\}+\max\{x_2,y_2\}$ *for each* $x,y\in X$. *Define* $T:X\to B_p(X)$ *by*

$$Tx=\begin{cases}\{(0,0)\} & if\ \ x\in\left\{(0,0),\left(0,\frac{1}{4}\right)\right\}\\ \left\{(0,0),\left(0,\frac{1}{4}\right)\right\} & if\ \ x=\left(0,\frac{1}{2}\right)\\ \left\{\left(0,\frac{1}{2}\right)\right\} & if\ \ x=(0,1).\end{cases}$$

Consider $\phi(u_1,u_2,u_3,u_4)=\frac{1}{2}\max\{u_1,u_2,u_3,u_4\}$ with $\psi(t)=\frac{1}{2}t$ and $L=0$. Now, it can be easy to see that (1) holds. Hence by Theorem 2.1, T has a fixed point.

Theorem 2.4 *Let* (X,p) *be a complete partial metric space and* $T:X\to X$ *be a mapping such that for each* $x,y\in X$, *there exist* $\phi\in\Phi_\psi$ *and* $L\geq 0$ *with*

$$p(Tx,Ty)\leq\phi\left(p(x,y),p(x,Tx),p(y,Ty),\frac{1}{2}(p(y,Tx)+p(x,Ty))\right)$$
$$+L(p(y,Tx)-p(y,y)).$$

Then T *has a fixed point.*

Proof. Proof of this theorem follows on same lines as of Theorem 2.1.

Let (X,p) be a partial metric space endowed with ordering $\preccurlyeq$ and $N(X)$ is a set of all nonempty subsets of X. For $A,B\in N(X)$, we have following relations [17]:

- $A\prec_1 B$, if for each $a\in A$ there exists $b\in B$ such that $a\preccurlyeq b$.
- $A\prec_2 B$, if for each $b\in B$ there exists $a\in A$ such that $a\preccurlyeq b$.

Note that $\prec_1$ and $\prec_2$ are not partial orders on $N(X)$ for detail see [18, Remark 3.5].

Theorem 2.5 *Let* (X,p) *be a complete partial metric space endowed with partial ordering* $\preccurlyeq$. *Let* $T:X\to B_p(X)$ *be a mapping satisfying the following conditions:*

(i) there exists $x_0\in X$ *such that* $\{x_0\}\prec_1 Tx_0$;

(ii) for $x,y\in X$, $x\preccurlyeq y$ *implies* $Tx\prec_1 Ty$;

(iii) if $\{x_n\}$ *is a nondecreasing sequence in* X *such that* $x_n\to x$, *then* $x_n\preccurlyeq x$

for each $n \in \mathbb{N} \cup \{0\}$*;*

(iv) there exist $\phi \in \Phi_\psi$ *and* $L \geq 0$ *such that*

$$\delta_p(Tx,Ty) \leq \phi\left(p(x,y), p(x,Tx), p(y,Ty), \frac{1}{2}(p(y,Tx)+p(x,Ty))\right) + L(p(y,Tx)-p(y,y)), \quad (9)$$

for each $x, y \in X$ *with* $x \preccurlyeq y$.

Then T *has a fixed point.*

Proof. By hypothesis (i) there exists $x_0 \in X$ such that $\{x_0\} \prec_1 Tx_0$. Then there exists $x_1 \in Tx_0$ such that $x_0 \preccurlyeq x_1$. By hypothesis (ii) we have $Tx_0 \prec_1 Tx_1$. Then for $x_1 \in Tx_0$ there exists $x_2 \in Tx_1$ such that $x_1 \preccurlyeq x_2$. Continuing in this way we have a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$ for each $n \in \mathbb{N} \cup \{0\}$ and

$$x_0 \preccurlyeq x_1 \preccurlyeq x_2 \preccurlyeq x_3 \preccurlyeq \cdots \preccurlyeq x_n \preccurlyeq x_{n+1} \preccurlyeq \cdots \quad (10)$$

If there exists some $N \in \mathbb{N} \cup \{0\}$ such that $x_N = x_{N+1}$, then x_N is a fixed point of T. Suppose $x_N \neq x_{N+1}$ for each $N \in \mathbb{N} \cup \{0\}$. As $x_n \preccurlyeq x_{n+1}$ for each n. From (9), we have

$$p(x_{n+1}, x_{n+2}) \leq \delta_p(Tx_n, Tx_{n+1})$$

$$\leq \phi\left(p(x_n, x_{n+1}), p(x_n, Tx_n), p(x_{n+1}, Tx_{n+1}), \frac{1}{2}(p(x_{n+1}, Tx_n) + p(x_n, Tx_{n+1}))\right) + L(p(x_{n+1}, Tx_n) - p(x_{n+1}, x_{n+1}))$$

$$\leq \phi\left(p(x_n, x_{n+1}), p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2}), \frac{1}{2}(p(x_{n+1}, x_{n+1}) + p(x_n, x_{n+2}))\right) + L(0)$$

$$\leq \phi\left(p(x_n, x_{n+1}), p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2}), \frac{1}{2}(p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}))\right). \quad (11)$$

We claim that $p(x_{n+1}, x_{n+2}) < p(x_n, x_{n+1})$ for each $n \in \mathbb{N} \cup \{0\}$. Suppose on contrary that $p(x_{n+1}, x_{n+2}) \geq p(x_n, x_{n+1})$ for some n. Since ϕ is nondecreasing, by using this in (11), we have

$$p(x_{n+1}, x_{n+2}) \leq \phi(p(x_{n+1}, x_{n+2}), p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2}), p(x_{n+1}, x_{n+2})). \quad (12)$$

By (12) and property (ii) of Φ_ψ, we have $p(x_{n+1}, x_{n+2}) = 0$, i.e., $x_{n+1} = x_{n+2}$. Which is a contradiction to assumption that $x_n \neq x_{n+1}$ for each $n \in \mathbb{N} \cup \{0\}$. Thus $p(x_{n+1}, x_{n+2}) < p(x_n, x_{n+1})$ for each $n \in \mathbb{N} \cup \{0\}$. By using it in (11), we have

$$p(x_{n+1}, x_{n+2}) \leq \phi(p(x_n, x_{n+1}), p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2}), p(x_n, x_{n+1})). \tag{13}$$

By (13) and property (ii) of Φ_ψ, we have

$$p(x_{n+1}, x_{n+2}) \leq \psi(p(x_n, x_{n+1})) \forall\, n \in \mathbb{N} \cup \{0\}.$$

Consequently, we get

$$p(x_{n+1}, x_{n+2}) \leq \psi^{n+1}(p(x_0, x_1)) \forall\, n \in \mathbb{N} \cup \{0\}. \tag{14}$$

Let $n > m$, we have

$$\begin{aligned} p(x_m, x_n) &\leq p(x_m, x_{m+1}) + p(x_{m+1}, x_{m+2}) + \cdots + p(x_{n-1}, x_n) - \sum_{i=m+1}^{n-1} p(x_i, x_i) \\ &\leq p(x_m, x_{m+1}) + p(x_{m+1}, x_{m+2}) + \cdots + p(x_{n-1}, x_n) \\ &\leq \psi^m(p(x_0, x_1)) + \psi^{m+1}(p(x_0, x_1)) + \cdots + \psi^{n-1}(p(x_0, x_1)) \\ &= \sum_{i=m}^{n-1} \psi^i(p(x_0, x_1)). \end{aligned}$$

Thus, we have

$$d_p(x_m, x_n) \leq 2p(x_m, x_n) \leq 2\sum_{i=m}^{n-1} \psi^i(p(x_0, x_1)) < \infty.$$

Therefore, we conclude that $\{x_n\}$ is a Cauchy sequence in (X, d_p). Since (X, p) is a complete partial metric space, by Lemma 1.9-(b), (X, d_p) is a complete metric space. Then there exists $x^* \in X$ such that $x_n \to x^* \in X$ with respect to d_p, as $n \to \infty$. By Lemma 1.8, we have

$$p(x^*, x^*) = \lim_{n\to\infty} p(x_n, x^*) = \lim_{n,m\to\infty} p(x_n, x_m) = 0. \tag{15}$$

By hypothesis (iii), we have $x_n \preccurlyeq x^*$ for each $n \in \mathbb{N} \cup \{0\}$. Thus, from (9), we have

$$\begin{aligned} \delta_p(x_{n+1}, Tx^*) &\leq \delta_p(Tx_n, Tx^*) \\ &\leq \phi\left(p(x_n, x^*), p(x_n, Tx_n), p(x^*, Tx^*), \frac{1}{2}(p(x^*, Tx_n) + p(x_n, Tx^*))\right) \\ &\quad + L(p(x^*, Tx_n) - p(x^*, x^*)) \\ &\leq \phi\left(p(x_n, x^*), p(x_n, x_{n+1}), p(x^*, Tx^*), \frac{1}{2}(p(x^*, x_{n+1}) + p(x_n, Tx^*))\right) \\ &\quad + L(p(x^*, x_{n+1}) - p(x^*, x^*)). \end{aligned} \tag{16}$$

By using the triangular inequality, it further implies that

$$\begin{aligned}\delta_p(x^*,Tx^*) &\leq p(x^*,x_{n+1})+\delta_p(x_{n+1},Tx^*)-p(x_{n+1},x_{n+1})\\ &\leq p(x^*,x_{n+1})+\delta_p(x_{n+1},Tx^*)\\ &\leq p(x^*,x_{n+1})+\phi(p(x_n,x^*),\,p(x_n,x_{n+1}),\,p(x^*,Tx^*),\\ &\frac{1}{2}(p(x^*,x_{n+1})+p(x_n,x^*)+p(x^*,Tx^*)))+L(p(x^*,x_{n+1})-p(x^*,x^*)).\end{aligned} \tag{17}$$

Letting $n\to\infty$ in (17), we have

$$\begin{aligned}\delta_p(x^*,Tx^*) &\leq \phi\left(0,0,p(x^*,Tx^*),\frac{1}{2}p(x^*,Tx^*)\right)+L(0)\\ &\leq \phi\left(0,0,\delta_p(x^*,Tx^*),\frac{1}{2}\delta_p(x^*,Tx^*)\right).\end{aligned} \tag{18}$$

By (18) and property (ii) of Φ_ψ, we have $\delta_p(x^*,Tx^*)=0$. Hence $Tx^*=\{x^*\}$. Moreover x^* is a fixed point.

Corollary 2.6 *Let (X,p) be a complete partial metric space endowed with partial ordering $\preccurlyeq$. Let $T:X\to B_p(X)$ be a mapping satisfying the following conditions:*

(i) there exists $x_0\in X$ such that $\{x_0\}\prec_1 Tx_0$;

(ii) for $x,y\in X$, $x\preccurlyeq y$ implies $Tx\prec_1 Ty$;

(iii) if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n\to x$, then $x_n\preccurlyeq x$ for each $n\in\mathbb{N}\cup\{0\}$;

(iv) there exist $\alpha\in[0,1)$ and $L\geq 0$ such that

$$\delta_p(Tx,Ty)\leq \alpha\max\left\{p(x,y),p(x,Tx),p(y,Ty),\frac{1}{2}(p(y,Tx)+p(x,Ty))\right\}+L(p(y,Tx)-p(y,y))$$

for all $x,y\in X$ with $x\preccurlyeq y$.

Then T has a fixed point.

Proof. Let $\phi(u_1,u_2,u_3,u_4)=\phi_1(u_1,u_2,u_3,u_4)=\alpha\max\{u_1,u_2,u_3,u_4\}$ with $\psi(t)=\alpha t$, where $\alpha\in[0,1)$. From (9), we have

$$\delta_p(Tx,Ty)\leq \alpha\max\left\{p(x,y),p(x,Tx),p(y,Ty),\frac{1}{2}(p(y,Tx)+p(x,Ty))\right\}+L(p(y,Tx)-p(y,y))$$

for all $x, y \in X$ with $x \preccurlyeq y$. Therefore by Theorem 2.5, T has a fixed point.

Example 2.7 *Let* $X = \left\{(0,0), \left(0, \frac{1}{4}\right), \left(0, \frac{1}{2}\right), (0,1), (1,0)\right\}$ *be endowed with partial metric* p *define by* $p(x, y) = \max\{x_1, y_1\} + \max\{x_2, y_2\}$ *for each* $x, y \in X$ *and partial ordering is defined as follows*

$$(x_1, x_2) \preccurlyeq (y_1, y_2) \Leftrightarrow x_1 = y_1 \text{ and } x_2 \le y_2.$$

Define $T : X \to B_p(X)$ by

$$Tx = \begin{cases} \{(0,0)\} & if \;\; x \in \left\{(0,0), \left(0, \frac{1}{4}\right)\right\} \\ \left\{(0,0), \left(0, \frac{1}{4}\right)\right\} & if \;\; x = \left(0, \frac{1}{2}\right) \\ \left(0, \frac{1}{2}\right) & if \;\; x = (0,1) \\ \{(1,0), (0,1)\} & if \;\; x = (1,0) \end{cases}$$

Consider $\phi(u_1, u_2, u_3, u_4) = \frac{1}{2}\max\{u_1, u_2, u_3, u_4\}$ with $\psi(t) = \frac{1}{2}t$ and $L = 0$. Now, it can be easy to see that (9) holds. For $x_0 = (1,0)$ we have $\{x_0\} \prec_1 Tx_0$. Also, if $x \preccurlyeq y$, then $Tx \prec_1 Ty$. Hence all conditions of Theorem 2.5 holds. Therefore, T has a fixed point.

Remark 2.8 *Note that Theorem 2.1 is not applicable on above example, when* $L = 0$. *To see consider* $x = (0,0)$ *and* $y = (1,0)$.

Theorem 2.9 *Let* (X, p) *be a complete partial metric space endowed with partial ordering* $\preccurlyeq$. *Let* $T : X \to B_p(X)$ *be a mapping satisfying the following conditions:*

(i) there exists $x_0 \in X$ such that $Tx_0 \prec_2 \{x_0\}$;

(ii) for $x, y \in X$, $x \preccurlyeq y$ implies $Tx \prec_2 Ty$;

(iii) if $\{x_n\}$ is a nonincreasing sequence in X such that $x_n \to x$, then $x_n \succcurlyeq x$ for each $n \in \mathbb{N} \cup \{0\}$;

(iv) there exist $\phi \in \Phi_\psi$ and $L \ge 0$ such that

$$\delta_p(Tx, Ty) \le \phi\left(p(x, y), p(x, Tx), p(y, Ty), \frac{1}{2}(p(y, Tx) + p(x, Ty))\right)$$

$$+L(p(y,Tx)-p(y,y))$$

for each $x, y \in X$ with $x \succcurlyeq y$.
Then T has a fixed point.

Proof. The proof follows on the same lines as in Theorem 2.5.

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