ON THE UNIQUE RECOVERY OF TIME-DEPENDENT COEFFICIENT IN A HYPERBOLIC EQUATION FROM NONLOCAL DATA

Elvin I. Azizbayov[1,2]

The purpose of this paper is to discuss the unique restoration of coefficients in a hyperbolic equation from some data. First, the problem reduces to an equivalent system by applying the Fourier method. Then with reference to the Banach fixed point principle, the existence and uniqueness of a solution to this system were demonstrated. Further, on the basis of the equivalency of these problems the existence and uniqueness theorem for the classical solution of the inverse coefficient problem is proved for the smaller value of time.

Keywords: Inverse problem, restoration coefficient, hyperbolic equation, overdetermination condition, classical solution, existence, uniqueness.

MSC2010: Primary 35R30, 35L10, 35L70; Secondary 35A01, 35A02, 35A09.

1. Introduction

Let $T > 0$ be a fixed time moment and let D_T denotes the rectangular region in the xt-plane defined by the inequalities $0 \le x \le 1$, $0 \le t \le T$. We further assume that $f(x,t)$, $\varphi(x)$, $\psi(x)$, $P_i(t)$ $(i = 1,2)$, and $h(t)$ are given functions of $x \in [0,1]$ and $t \in [0,T]$. Consider the one-dimensional inverse problem of identifying an unknown pair of functions $(u(x,t), a(t))$ for the equation

$$u_{tt}(x,t) - u_{xx}(x,t) = a(t)u(x,t) + f(x,t),\ (x,t) \in D_T, \tag{1}$$

with the nonlocal initial conditions

$$\begin{aligned} u(x,0) &= \int_0^T P_1(t)u(x,t)dt + \varphi(x), \\ u_t(x,0) &= \int_0^T P_2(t)u(x,t)dt + \psi(x),\ 0 \le x \le 1, \end{aligned} \tag{2}$$

periodic boundary conditions

$$u(0,t) = u(1,t),\ 0 \le t \le T, \tag{3}$$

integral boundary conditions

$$\int_0^1 u(x,t)dx = 0,\ 0 \le t \le T, \tag{4}$$

[1]Department of Computational Mathematics, Baku State University, Baku, Azerbaijan
[2]Department of Mathematics, Khazar University, Baku, Azerbaijan.
e-mail: eazizbayov@bsu.edu.az

and the overdetermination condition

$$u(x_0, t) = h(t), \ 0 \le t \le T, \tag{5}$$

where $x_0 \in (0, 1)$ is some fixed point.

In many practical situations, it is required to determine the coefficients in an ordinary or partial differential equation from known functionals of its solution. Problems of these types are often called inverse problems of mathematical physics and may be contrasted with problems in which an equation is given and its solution is sought under initial and boundary conditions.

Inverse problems arise in many areas of mathematical physics, and applications are rapidly spreading to areas such as geophysics, chemistry, medicine, and engineering, etc. In other words, the study of inverse problems is of vital interest to many areas of science and technology, like geophysical exploration, system identification, nondestructive testing, seismology, mineral exploration, ultrasonic tomography, and so forth.

In the past few decades, a great deal of interest has been to the inverse coefficient problems. The general theory of inverse problems for partial differential equations is well described in the works of many authors. More detailed expositions and classification of works related to the investigation of inverse problems for partial differential equations can be found in many books and monographs (see for example, [6], [8], [12], [16]–[18], [20], [27]–[29], and the references therein). The existence and uniqueness problems of solutions of inverse hyperbolic problems with various measurements were investigated in [1], [2], [10], [11], [13], [19], [24]–[26], [30], [32]. Moreover, many studies are devoted to the analysis of the of numerical aspects of the coefficient identification problem for one-dimensional hyperbolic equations (see, e.g., [4], [7], [9], [14], [15], [31], and the references given therein).

In the present paper, we consider an inverse problem for identifying the time-dependent coefficient in a one-dimensional hyperbolic equation. It will be noted also that the statement of the problem and the proof technique used in this paper differ from those of the above articles, and the conditions in the theorems differ significantly from those therein. A distinctive feature of this article is the consideration of a hyperbolic equation with both spatial and time nonlocal conditions.

Definition 1.1. *A pair of functions $(u(x, t), a(t)) \in C^2(D_T) \times C[0, T], \ t \in [0, T]$ is called a solution of problem (1)-(5) if each statement of the problem (1)-(5) is satisfied by these functions in the classical (usual) sense.*

Now, to study problem (1) - (5), we first consider the following problem:

$$y''(t) = a(t)y(t), \ 0 \le t \le T, \tag{6}$$

$$y(0) = \int_0^T P_1(t)y(t)dt, \quad y'(0) = \int_0^T P_2(t)y(t)dt, \tag{7}$$

where $P_1(t), \ P_2(t), \ a(t) \in C[0, T]$ are given functions of $t \in [0, T]$ and $y = y(t)$ is unknown function. By this we mean that $y(t)$ is a function belonging to the set $C^2[0, T]$ and satisfying the problem (6), (7) in the usual sense.

We state the following lemma about the uniqueness of solution of problem (6), (7) without proof.

Lemma 1.1. *(see* [21]*) Let* $P_1(t),\ P_2(t),\ a(t) \in C[0,T]$*, and*

$$\left(T\,\|P_2(t)\|_{C[0,T]} + \|P_1(t)\|_{C[0,T]} + \frac{T}{2}\,\|a(t)\|_{C[0,T]}\right)T < 1$$

holds. Then problem (6), (7) has only the trivial solution.

Now along with the inverse boundary-value problem (1) - (5), we consider the following auxiliary inverse problem: It is required to determine a pair $(u(x,t), a(t))$ of functions $u(x,t) \in C^2(D_T)$ and $a(t) \in C[0,T]$ from relations (1)-(3), and

$$u_x(0,t) = u_x(1,t),\ 0 \le t \le T, \tag{8}$$

$$h''(t) - u_{xx}(x_0,t) = a(t)h(t) + f(x_0,t),\ 0 \le t \le T. \tag{9}$$

The following theorem holds true.

Theorem 1.1. *Suppose* $\varphi(x),\ \psi(x) \in C[0,1],\ P_i(t) \in C[0,T],\ i = 1,2,\ h(t) \in C^2[0,T],\ h(t) \neq 0,\ f(x,t) \in C(D_T),\ \int_0^1 f(x,t)dx = 0,\ 0 \le t \le T$*, and the following consistency conditions are fulfilled:*

$$\int_0^1 \varphi(x)dx = 0,\ \int_0^1 \psi(x)dx = 0, \tag{10}$$

$$h(0) = \int_0^T P_1(t)h(t)dt + \varphi(x_0), \quad h'(0) = \int_0^T P_2(t)h(t)dt + \psi(x_0). \tag{11}$$

Then the following statements are true:

- (i) *each classical solution* $(u(x,t), a(t))$ *of the problem (1)-(5) is a solution of problem (1)-(3), (8), (9), as well;*
- (ii) *each solution* $(u(x,t), a(t))$ *of the problem (1)-(3), (8), (9) by virtue of*

$$\left(T\,\|P_2(t)\|_{C[0,T]} + \|P_1(t)\|_{C[0,T]} + \frac{T}{2}\,\|a(t)\|_{C[0,T]}\right)T < 1, \tag{12}$$

is a classical solution of problem (1)-(5).

Proof. Let $(u(x,t), a(t))$ be the classical solution of problem (1) - (5). Integrating both sides of Equation (1) from 0 to 1 gives

$$\frac{d^2}{dt^2}\int_0^1 u(x,t)dx - (u_x(1,t) - u_x(0,t))$$

$$= a(t)\int_0^1 u(x,t)dx + \int_0^1 f(x,t)dx,\ 0 \le t \le T. \tag{13}$$

Using the fact that $\int_0^1 f(x,t)dx = 0,\ 0 \le t \le T$, and the boundary condition (3), we conclude that the equality (8) is true.

Letting $x = x_0$ in Equation (1), we get

$$u_{tt}(x_0,t) - u_{xx}(x_0,t) = a(t)u(x_0,t) + f(x_0,t),\ 0 \le t \le T. \tag{14}$$

Further, assuming that $h(t) \in C^2[0,T]$, and differentiating (5) twice, we have

$$u_{tt}(x_0,t) = h''(t),\ 0 \le t \le T, \tag{15}$$

Taking into account the condition (5) and the relation (15) in (14) we obtain (9).

Now, suppose that $(u(x,t), a(t))$ is a solution to problem (1) - (3), (8), (9). Then from (13), allowing for $\int_0^1 f(x,t)dx = 0,\ 0 \le t \le T$, (3), and (8), we find:

$$\frac{d^2}{dt^2}\int_0^1 u(x,t)dx = a(t)\int_0^1 u(x,t)dx,\ 0 \le t \le T. \tag{16}$$

By using the initial conditions (2) and the compatibility conditions (10), we may write

$$\begin{gathered}\int_0^1 u(x,0)dx - \int_0^T P_1(t)\left(\int_0^1 u(x,t)dx\right)dt = \int_0^1 \varphi(x)dx = 0,\\ \int_0^1 u_t(x,0)dx - \int_0^T P_2(t)\left(\int_0^1 u(x,t)dx\right)dt = \int_0^1 \psi(x)dx = 0.\end{gathered} \tag{17}$$

Lemma 1.1 enables us to conclude that the problem (16), (17) has only a trivial solution. Then, $\int_0^1 u(x,t)dx = 0,\ 0 \le t \le T$, i.e., the condition (4) is satisfied.

Further, it follows from (9) and (14) that

$$\frac{d^2}{dt^2}(u(x_0,t) - h(t)) = a(t)(u(x_0,t) - h(t)),\ 0 \le t \le T. \tag{18}$$

Exploiting the initial conditions (2) and the compatibility conditions (11), we have

$$u(x_0,0) - h(0) - \int_0^T P_1(t)(u(x_0,t) - h(t))dt = \varphi(x_0) - \left(h(0) - \int_0^T P_1(t)h(t)dt\right) = 0,$$

$$u_t(x_0,0) - h'(0) - \int_0^T P_2(t)(u(x_0,t) - h(t))dt$$

$$= \psi(x_0) - \left(h'(0) - \int_0^T P_2(t)h(t)dt\right) = 0. \tag{19}$$

Hence by Lemma 1.1 and relations (18), (19), we conclude that condition (5) is satisfied. □

2. **Existence and uniqueness of the solution of the inverse problem on the interval** $[0,1]$

Consider the following system of functions on the interval :

$$1, \cos\lambda_1 x, \sin\lambda_1 x, ..., \cos\lambda_k x, \sin\lambda_k x, ... \tag{20}$$

The system of functions (20) forms an orthonormal basis, of the space $L_2(0,1)$ for $\lambda_k = 2k\pi,\ k = 0,1,....$ Then the first component of classical solution $(u(x,t), a(t))$ of the problem (1)-(3), (8), (9) has the form:

$$u(x,t) = \sum_{k=0}^{\infty} u_{1k}(t)\cos\lambda_k x + \sum_{k=1}^{\infty} u_{2k}(t)\sin\lambda_k x,\ \ \lambda_k = 2\pi k, \tag{21}$$

where

$$u_{10}(t) = \int_0^1 u(x,t)dx, \quad u_{1k}(t) = 2\int_0^1 u(x,t)\cos\lambda_k x dx,\ \ k = 1,2,...,$$

$$u_{2k}(t) = 2\int_0^1 u(x,t)\sin\lambda_k x dx,\ \ k = 1,2,....$$

Applying the formal scheme of the Fourier method and using (1) and (2) we get

$$u''_{10}(t) = F_{10}(t;u,a),\ \ 0 \le t \le T, \tag{22}$$

$$u''_{ik}(t) + \lambda_k^2 u_{ik}(t) = F_{ik}(t;u,a),\ \ i = 1,2;\ \ k = 1,2,...,\ \ 0 \le t \le T, \tag{23}$$

$$u_{10}(0) = \varphi_{10} + \int_0^T P_1(t)u_{10}(t)dt, u'_{10}(0) = \psi_{10} + \int_0^T P_2(t)u_{10}(t)dt, \tag{24}$$

$$u_{ik}(0) = \varphi_{ik} + \int_0^T P_1(t)u_{ik}(t)dt, u'_{ik}(0) = \psi_{ik} + \int_0^T P_2(t)u_{ik}(t)dt,\ \ i = 1,2;\ \ k \in N, \tag{25}$$

where

$$F_{1k}(t) = a(t)u_{1k}(t) + f_{1k}(t),\ \ k = 0,1,...,$$

$$f_{10}(t) = \int_0^1 f(x,t)dx, \quad f_{1k}(t) = 2\int_0^1 f(x,t)\cos\lambda_k x dx,\ \ k = 1,2,...,$$

$$\varphi_{10} = \int_0^1 \varphi(x)dx,\ \ \psi_{10} = 2\int_0^1 \psi(x)dx,$$

$$\varphi_{1k} = 2\int_0^1 \varphi(x)\cos\lambda_k x dx, \quad \psi_{1k} = 2\int_0^1 \psi(x)\cos\lambda_k x dx,\ \ k = 0,1,...,$$

$$F_{2k}(t) = a(t)u_{2k}(t) + f_{2k}(t), \quad f_{2k}(t) = 2\int_0^1 f(x,t)\sin\lambda_k x dx,\ \ k = 1,2,...,$$

$$\varphi_{2k} = 2\int_0^1 \varphi(x)\sin\lambda_k x dx, \quad \psi_{2k} = 2\int_0^1 \psi(x)\sin\lambda_k x dx,\ \ k = 1,2,....$$

Further, from (22) - (25) we find:

$$u_0(t)=\varphi_0+\int_0^T P_1(t)u_0(t)dt$$

$$+t\left(\psi_0+\int_0^T P_2(t)u_0(t)dt\right)+\int_0^t (t-\tau)F_0(\tau;u,a)d\tau, \tag{26}$$

$$u_{ik}(t)=\left(\varphi_{ik}+\int_0^T P_1(t)u_{ik}(t)dt\right)\cos\lambda_k t+\frac{1}{\lambda_k}\left(\psi_{ik}+\int_0^T P_2(t)u_{ik}(t)dt\right)sin\lambda_k t$$

$$+\frac{1}{\lambda_k}\int_0^t F_{ik}(\tau;u,a)\sin\lambda_k(t-\tau)d\tau, i=1,2; k=1,2,...,0\le t\le T. \tag{27}$$

To determine the component $u(x,t)$ of the solution $(u(x,t),a(t))$ of problem (1) - (3), (8), (9), we substitute $u_{1k}(t),\ k=0,1,...$ and $u_{2k}(t),\ k=1,2,...$ in (21), obtain

$$u(x,t)=\left(\varphi_0+\int_0^T P_1(t)u_0(t)dt\right)+t\left(\psi_0+\int_0^T P_2(t)u_0(t)dt\right)$$

$$+\int_0^t (t-\tau)F_0(\tau;u,a)d\tau+\sum_{k=1}^{\infty}\left\{\left(\varphi_{1k}+\int_0^T P_1(t)u_{1k}(t)dt\right)\cos\lambda_k t\right.$$

$$\left.+\frac{1}{\lambda_k}\left(\psi_{1k}+\int_0^T P_2(t)u_{1k}(t)dt\right)sin\lambda_k t+\frac{1}{\lambda_k}\int_0^t F_{1k}(\tau;u,a)\sin\lambda_k(t-\tau)d\tau\right\}\cos\lambda_k x$$

$$+\sum_{k=1}^{\infty}\left\{\left(\varphi_{2k}+\int_0^T P_1(t)u_{2k}(t)dt\right)\cos\lambda_k t+\frac{1}{\lambda_k}\left(\psi_{2k}+\int_0^T P_2(t)u_{2k}(t)dt\right)sin\lambda_k t\right.$$

$$\left.+\frac{1}{\lambda_k}\int_0^t F_{2k}(\tau;u,a)\sin\lambda_k(t-\tau)d\tau\right\}\sin\lambda_k x. \tag{28}$$

It follows from (9) and (21) that

$$a(t)=[h(t)]^{-1}\{h''(t)-f(x_0,t)$$

$$+\sum_{k=1}^{\infty}\lambda_k^2 u_{1k}(t)\cos\lambda_k x_0+\sum_{k=1}^{\infty}\lambda_k^2 u_{2k}(t)\sin\lambda_k x_0\}. \tag{29}$$

In order to obtain the expression for the second component of the solution of problem (1) - (3), (8), (9), we substitute (27) in (29), we get

$$a(t)=[h(t)]^{-1}\{h''(t)-f(x_0,t)+\sum_{k=1}^{\infty}\lambda_k^2\left[\left(\varphi_{1k}+\int_0^T P_1(t)u_{1k}(t)dt\right)\cos\lambda_k t\right.$$

$$+\frac{1}{\lambda_k}\left(\psi_{1k}+\int\limits_0^T P_2(t)u_{1k}(t)dt\right) sin\lambda_k t +\frac{1}{\lambda_k}\int\limits_0^t F_{1k}(\tau;u,a)\sin\lambda_k(t-\tau)d\tau\Bigg]\Bigg\}\cos\lambda_k x_0$$

$$+\sum_{k=1}^{\infty}\lambda_k^2\Bigg[\left(\varphi_{2k}+\int\limits_0^T P_1(t)u_{2k}(t)dt\right)\cos\lambda_k t+\frac{1}{\lambda_k}\left(\psi_{2k}+\int\limits_0^T P_2(t)u_{2k}(t)dt\right) sin\lambda_k t$$

$$+\frac{1}{\lambda_k}\int\limits_0^t F_{2k}(\tau;u,a)\sin\lambda_k(t-\tau)d\tau\Bigg]\Bigg\}\sin\lambda_k x_0. \qquad (30)$$

Thus, the solution of problem (1) - (3), (8), (9) was reduced to the solution of system (28), (30) with respect to unknown functions $u(x,t)$ and $a(t)$.

We have the following lemma.

Lemma 2.1. [23] *If $(u(x,t),a(t))$ is any solution to problem (1)-(3), (8), (9), then the functions*

$$u_{10}(t)=\int\limits_0^1 u(x,t)dx,$$

$$u_{1k}(t)=2\int\limits_0^1 u(x,t)\cos\lambda_k x dx,\quad u_{2k}(t)=2\int\limits_0^1 u(x,t)\sin\lambda_k x dx,\ k=1,2,...,$$

satisfy the system of equations (26), (27) on the interval $[0,T]$.

It follows from Lemma 2.1 that

Corollary 2.1. *Assume that the system (28), (30) has a unique solution. Then problem (1)-(3), (8), (9) has at most one solution. In other words, if the problem (1)-(3), (8), (9) has a solution, then it is unique.*

With the purpose to study the problem (1) - (3), (8), (9), we consider the following functional spaces.

By $B_{2,T}^3$ [5], we denote a set of all functions of the form

$$u(x,t)=\sum_{k=0}^{\infty}u_{1k}(t)\cos\lambda_k x+\sum_{k=1}^{\infty}u_{2k}(t)\sin\lambda_k x,\ \lambda_k=2\pi k,$$

in the region D_T, where each of the function $u_k(t)$ $(k=0,1,2,...)$ is continuous over an interval $[0,T]$ and satisfies the following condition:

$$J(u)\equiv\|u_0(t)\|_{C[0,T]}+\left\{\sum_{k=1}^{\infty}\left(\lambda_k^3\|u_{1k}(t)\|_{C[0,T]}\right)^2\right\}^{\frac{1}{2}}$$

$$+\left\{\sum_{k=1}^{\infty}\left(\lambda_k^3\|u_{2k}(t)\|_{C[0,T]}\right)^2\right\}^{\frac{1}{2}}<+\infty.$$

The norm in the space $B_{2,T}^3$ is defined as follows

$$\|u(x,t)\|_{B_{2,T}^3}=J(u).$$

It is known that the space $B^3_{2,T}$ is a Banach space [22]. Obviously, $E^3_T = B^3_{2,T} \times C[0,T]$ is also Banach space with the norm

$$\|z(x,t)\|_{E^3_T} = \|u(x,t)\|_{B^3_{2,T}} + \|a(t)\|_{C[0,T]}.$$

Now consider the operator

$$\Phi(u,a) = \{\Phi_1(u,a),\ \Phi_2(u,a)\},$$

in the space E^3_T, where

$$\Phi_1(u,a) = \tilde{u}(x,t) \equiv \sum_{k=0}^{\infty} \tilde{u}_{1k}(t)\cos\lambda_k x + \sum_{k=1}^{\infty} \tilde{u}_{2k}(t)\sin\lambda_k x, \quad \Phi_2(u,a) = \tilde{a}(t),$$

and the functions $\tilde{u}_{10}(t)$, $\tilde{u}_{ik}(t)$, $i = 1,2$; $k = 1,2,...$, and $\tilde{a}(t)$ are equal to the right-hand sides of (26), (27), and (30), respectively.

Hence we get

$$\begin{aligned} \|\tilde{u}_{10}(t)\|_{C[0,T]} &\le |\varphi_{10}| + T\,|\psi_{10}| \\ &+T(\|P_1(t)\|_{C[0,T]} + T\,\|P_2(t)\|_{C[0,T]})\,\|u_{10}(t)\|_{C[0,T]} + |\psi_{10}| \\ &+T\sqrt{T}\left(\int_0^T |f_{10}(\tau)|^2\,d\tau\right)^{\frac{1}{2}} + T^2\,\|a(t)\|_{C[0,T]}\,\|u_{10}(t)\|_{C[0,T]}, \end{aligned} \tag{31}$$

$$\begin{aligned} \left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|\tilde{u}_{ik}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} &\le \sqrt{6}\left(\sum_{k=1}^{\infty}(\lambda_k^3\,|\varphi_{ik}|)^2\right)^{\frac{1}{2}} \\ &+\sqrt{6}\left(\sum_{k=1}^{\infty}(\lambda_k^2\,|\psi_{ik}|)^2\right)^{\frac{1}{2}} + \sqrt{6}(\|P_1(t)\|_{C[0,T]} \\ &+\|P_2(t)\|_{C[0,T]})T\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_{ik}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} \\ &+\sqrt{6T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2\,|f_{ik}(\tau)|)^2 d\tau\right)^{\frac{1}{2}} \\ &+\sqrt{6}T\,\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_{ik}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}, \end{aligned} \tag{32}$$

$$\begin{aligned} \|\tilde{a}(t)\|_{C[0,T]} &\le \left\|[h(t)]^{-1}\right\|_{C[0,T]}\{\left\|h''(t) - f(x_0,t)\right\|_{C[0,T]} + \left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}} \\ &\times\sum_{i=1}^{2}\left[\left(\sum_{k=1}^{\infty}(\lambda_k^3\,|\varphi_{ik}|)^2\right)^{\frac{1}{2}} + T\,\|P_1(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_{ik}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right. \\ &+\left(\sum_{k=1}^{\infty}(\lambda_k^2\,|\psi_{ik}|)^2\right)^{\frac{1}{2}} + T\,\|P_2(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_{ik}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} \end{aligned}$$

$$+\sqrt{T}\left(\int_0^T \sum_{k=1}^{\infty}(\lambda_k^2\,|f_{ik}(\tau)|)^2 d\tau\right)^{\frac{1}{2}}$$

$$+T\,\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_{ik}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\Bigg]\Bigg\}. \qquad (33)$$

We impose the following conditions on the function $\varphi, \psi, f, P_1, P_2$, and h:

(A) $\varphi(x) \in C^2[0,1],\ \varphi'''(x) \in L_2(0,1),\ \varphi(0) = \varphi(1),\ \varphi'(0) = \varphi'(1),$
$\varphi''(0) = \varphi''(1);$

(B) $\psi(x) \in C^1[0,1],\ \psi''(x) \in L_2(0,1),\ \psi(0) = \psi(1),\ \psi'(0) = \psi'(1);$

(C) $f(x,t), f_x(x,t) \in C(D_T),\ f_{xx}(x,t) \in L_2(D_T),$
$f(0,t) = f(1,t),\ f_x(0,t) = f_x(1,t),\ 0 \le t \le T;$

(D) $P_1(t), P_2(t) \in C[0,T],\ h(t) \in C^2[0,T],\ h(t) \ne 0,\ 0 \le t \le T.$

Then from (31) - (33) we find, respectively, that

$$\|\tilde{u}_0(t)\|_{C[0,T]} \le \|\varphi(x)\|_{L_2(0,1)} + T\,\|\psi(x)\|_{C[0,T]} + T\sqrt{T}\,\|f(x,t)\|_{L_2(0,1)}$$
$$+T(\|P_2(t)\|_{C[0,T]} + T\,\|P_1(t)\|_{C[0,T]} + T\,\|a(t)\|_{C[0,T]})\,\|u_0(t)\|_{C[0,T]}, \qquad (34)$$

$$\left\{\sum_{k=1}^{\infty}\left(\lambda_k^3\,\|\tilde{u}_{ik}(t)\|_{C[0,T]}\right)^2\right\}^{\frac{1}{2}} \le \sqrt{6}\,\|\varphi'''(x)\|_{L_2(0,1)} + \sqrt{6}\,\|\psi''(x)\|_{L_2(0,1)}$$
$$+\sqrt{6T}\,\|f_{xx}(x,t)\|_{L_2(D_T)} + \sqrt{6}T(\|P_2(t)\|_{C[0,T]} + T\,\|P_1(t)\|_{C[0,T]}$$
$$+T\,\|a(t)\|_{C[0,T]})\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_{ik}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}, \qquad (35)$$

$$\|\tilde{a}(t)\|_{C[0,T]} \le \left\|[h(t)]^{-1}\right\|_{C[0,T]}\{\left\|h''(t) - f(x_0,t)\right\|_{C[0,T]} + 2\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}$$
$$\times[\left\|\varphi'''(x)\right\|_{L_2(0,1)} + \left\|\psi''(x)\right\|_{L_2(0,1)} + \sqrt{T}\,\|f_{xx}(x,t)\|_{L_2(D_T)} + T(\|P_2(t)\|_{C[0,T]}$$
$$+T\,\|P_1(t)\|_{C[0,T]} + \|a(t)\|_{C[0,T]})\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\Bigg]\Bigg\}. \qquad (36)$$

The inequalities (34) and (35) yields

$$\|\tilde{u}(x,t)\|_{B^3_{2,T}} \le A_1(T) + B_1(T)$$
$$\times\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^3_{2,T}} + C_1(T)\,\|u(x,t)\|_{B^3_{2,T}}, \qquad (37)$$

where

$$A_1(T) = \|\varphi(x)\|_{L_2(0,1)} + T\,\|\psi(x)\|_{L_2(0,1)} + T\sqrt{T}\,\|f(x,t)\|_{L_2(D_T)}$$
$$+2\sqrt{6}\,\|\varphi'''(x)\|_{L_2(0,1)} + 2\sqrt{6}\,\|\psi''(x)\|_{L_2(0,1)} + 2\sqrt{6T}\,\|f_x(x,t)\|_{L_2(D_T)},$$
$$B_1(T) = T^2 + 2\sqrt{6}T,$$
$$C_1(T) = T(1 + 2\sqrt{6})\,\|P_1(t)\|_{C[0,T]} + T(T + 2\sqrt{6})\,\|P_2(t)\|_{C[0,T]}.$$

Also, from (36) we obtain:

$$\|\tilde{a}(t)\|_{C[0,T]} \le A_2(T) + B_2(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^3_{2,T}} + C_2(T)\,\|u(x,t)\|_{B^3_{2,T}}\,, \tag{38}$$

where

$$A_2(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]} \{\left\|h''(t) - f(x_0,t))\right\|_{C[0,T]} + 2\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}$$

$$\times[\left\|\varphi'''(x)\right\|_{L_2(0,1)} + \left\|\psi''(x)\right\|_{L_2(0,1)} + \sqrt{T}\,\|f_x(x,t)\|_{L_2(D_T)}]\},$$

$$B_2(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}} T,$$

$$C_2(T) = 2\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}} T(\|P_2(t)\|_{C[0,T]} + T\,\|P_1(t)\|_{C[0,T]}).$$

From inequalities (37) and (38) we arrive at the following estimation

$$\|\tilde{u}(x,t)\|_{B^3_{2,T}} + \|\tilde{a}(t)\|_{C[0,T]} \le A(T) + B(T)\,\|a(t)\|_{C[0,T]}$$

$$\times\|u(x,t)\|_{B^3_{2,T}} + C(T)\,\|u(x,t)\|_{B^3_{2,T}}\,, \tag{39}$$

where

$$A(T) = A_1(T) + A_2(T),\ B(T) = B_1(T) + B_2(T),\ C(T) = C_1(T) + C_2(T).$$

Let K_R denote the closed ball of radius $R = A(T) + 2$ centered at zero in E^3_T.

Theorem 2.1. *Let that hypotheses (A)-(D) be satisfied and suppose that*

$$(B(T)(A(T)+2) + C(T))(A(T)+2) < 1. \tag{40}$$

Then problem (1)-(3), (8), (9) has a unique solution in the ball $K = K_R$.

Proof. In the space E^3_T we consider the following operator equation

$$z = \Phi z, \tag{41}$$

where $z = \{u,a\}$ and the components $\Phi_i(u,a)$, $i = 1,2$, of the operator $\Phi(u,a)$ are defined from the right sides of equations (28) and (30), respectively.

Similar to (37) we obtain that for any $z, z_1, z_2 \in K_R$ the following inequalities hold

$$\|\Phi z\|_{E^3_T} \le A(T) + B(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^3_{2,T}} + C(T)\,\|u(x,t)\|_{B^3_{2,T}}$$

$$\le A(T) + (B(T)(A(T)+2) + C(T))(A(T)+2), \tag{42}$$

$$\|\Phi z_1 - \Phi z_2\|_{E^3_T} \le B(T)R(\|u_1(x,t) - u_2(x,t)\|_{B^3_{2,T}} +$$

$$+\|a_1(t) - a_2(t)\|_{C[0,T]}) + C(T)\,\|u_1(x,t) - u_2(x,t)\|_{B^3_{2,T}}\,. \tag{43}$$

Then by virtue of (40) it follows from estimations (42) and (43) that the operator Φ acts in the ball $K = K_R$, and is contractive. Therefore, the operator Φ has a unique fixed point $\{z\} = \{u,a\}$ that is a unique solution of equation (39), i.e., it is a unique solution of system (28) and (30) in the ball $K = K_R$. Then the function $u(x,t)$ as an element of space $B^3_{2,T}$ is continuous and has continuous derivatives $u_x(x,t)$ and $u_{xx}(x,t)$ in D_T.

Equation (23) now gives

$$\left(\sum_{k=1}^{\infty}(\lambda_k \left\|u''_{ik}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}} \le \sqrt{2}\left(\sum_{k=1}^{\infty}(\lambda_k^3 \left\|u_k(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$+\sqrt{2}\left\|\left\|f_x(x,t)+a(t)u_x(x,t)\right\|_{C[0,T]}\right\|_{L(0,1)}, i=1,2.$$

It follows from the last inequality that the function $u_{tt}(x,t)$ is continuous in the region D_T.

Thus, it is easy to verify that equation (1) and conditions (2), (3), (8), (9) are satisfied in the usual sense. Therefore, $(u(x,t),a(t))$ is a solution of problem (1)-(3), (8), (9), and it is unique by virtue of Lemma 2.1. □

Finally, Theorem 1.1 and Theorem 2.1 straightforward implies the unique solvability of the original problem (1) - (5).

Theorem 2.2. *Suppose that all assumptions of Theorem 2.1, and the conditions*

$$\int_0^1 \varphi(x)dx = 0, \ \int_0^1 \psi(x)dx = 0, \ \int_0^1 f(x,t)dx = 0, \ 0 \le t \le T,$$

$$h(0) = \int_0^T P_1(t)h(t)dt + \varphi(x_0), \quad h'(0) = \int_0^T P_2(t)h(t)dt + \psi(x_0),$$

$$\left(T\left\|P_2(t)\right\|_{C[0,T]} + \left\|P_1(t)\right\|_{C[0,T]} + \frac{T}{2}(A(T)+2)\right)T < 1,$$

hold. Then problem (1) - (5) has a unique classical solution in the ball $K = K_R(\|z\|_{E_T^3} \le A(T)+2)$ *of the space* E_T^3.

REFERENCES

[1] D.S. Anikonov and D.S. Konovalova, *Direct and inverse problems for a wave equation with discontinuous coefficients*, St Petersburg Polytechnic University Journal-Physics and Mathematics, **11**(2018), 61-72.

[2] Y.E. Anikonov, *Uniqueness theorem for solutions to an inverse problem for wave-equation*, Mathematical Notes of the Academy of Sciences of the USSR, **19**(1976), 127-128.

[3] A. Ashyralyev and F. Emharab, *Identification hyperbolic problems with the Neumann boundary condition*, Bulletin of the Karaganda University-Mathematics, **91**(2018), 89-98.

[4] A. Ashyralyev and F. Emharab, *Source identification problems for hyperbolic differential and difference equations*, Journal of Inverse and Ill-posed Problems, **27**(2019), 301-315.

[5] E. Azizbayov and Y. Mehraliyev, *Inverse problem for a parabolic equation in a rectangle domain with integral conditions*, European Journal of Pure and Applied Mathematics, **10**(2017), 981-994.

[6] M.I. Belishev and A.S. Blagovestchenskii, *Dynamical inverse problems of the wave propagation theory*, St.-Petersburg State University, St.-Petersburg, 1999.

[7] M. Bellassoued and O. Ben Fraj, *Stable recovery of time dependent coefficient from arbitrary measurements for wave equation*, Journal of Mathematical Analysis and Applications, **482**(2020), 123533.

[8] Yu.Ya. Belov, *Inverse problems for partial differential equations*, Inverse and ill-posed problems series, VSP, Utrecht, 2002.

[9] I. Ben Aicha, *Stability estimate for a hyperbolic inverse problem with time-dependent coefficient*, Inverse Problems, **31**(2015), 125010.

[10] J.R. Cannon and P.Duchateau, *An inverse problem for an unknown source term in a wave equation*, SIAM Journal on Applied Mathematics, **43**(1983), 553-564.

[11] A.M. Denisov, *Existence of a solution of the inverse coefficient problem for a quasilinear hyperbolic equation*, Computational Mathematics and Mathematical Physics, **59**(2019), 550-558.

[12] A.M. Denisov, *Introduction to the theory of inverse problems*, Moscow University Press, Moscow, 1994. (in Russian)

[13] A.M. Denisov, *Problems of determining the unknown source in parabolic and hyperbolic equations*, Computational Mathematics and Mathematical Physics, **55**(2015), 829-833.

[14] S.O. Hussein and D. Lesnic, *Determination of forcing functions in the wave equation: the space-dependent case*, Journal of Engineering Mathematics, **96**(2016), 115-133.

[15] S.O. Hussein and D. Lesnic, *Determination of forcing functions in the wave equation: the time-dependent case*, Journal of Engineering Mathematics, **96**(2016), 135-153.

[16] V. Isakov, *Inverse source problems*, Mathematical Surveys and Monographs series, American Mathematical Society, 1990.

[17] V.K. Ivanov, V.V. Vasin, and V.P. Tanana, *Theory of linear ill-posed problems and its applications*, VSP, Utrecht-Boston-Koln-Tokyo, 2002.

[18] S.I. Kabanikhin, Inverse and ill-posed problems, Theory and applications, De Gruyter, Berlin, 2012.

[19] M. Kirane and N. Al-Salti, *Inverse problems for a nonlocal wave equation with an involution perturbation*, Journal of Nonlinear Sciences and Applications, **9**(2016), 1243-1251.

[20] M.M. Lavrent'ev, V.G. Romanov, and S.T. Shishatskii, *Ill-posed problems of mathematical physics and analysis*, Moscow, 1980.

[21] Ya.T. Megraliev and F.Kh. Alizade, *Inverse boundary value problem for a Boussinesq type equation of fourth order with nonlocal time integral conditions of the second kind*, Vestn. Udmurtsk. Univ. Mat. Mekh. Komp. Nauki, **26**(2016),503-514. (in Russian)

[22] Ya.T. Megraliev, *On one nonlocal boundary value problem for the inverse hyperbolic equation of second order*, Herald of Tver State University, (2013), 27-38. (in Russian)

[23] Y.T. Mehraliyev, *On an inverse boundary value problem for a second-order elliptic equation with the integral condition*, Visnyk of the Lviv University, series: Mechanics and Mathematics, (2012), 145-156.

[24] Ya.T. Mehraliev and U.S. Alizade, *On the problem of identifying a linear source for the third-order hyperbolic equation with integral condition*, PFMT, **40**(2019), 80-87. (in Russian)

[25] G. Nakamura and M. Watanabe, *An inverse boundary value problem for a nonlinear wave equation*, Inverse Problems and Imaging, **2**(2008), 121-131.

[26] T.E. Oussaeif and A. Bouziani, *Inverse problem of a hyperbolic equation with an integral overdetermination condition*, Electronic Journal of Differential Equations, **2016**(2016), 1-7.

[27] A.I. Prilepko, D.G. Orlovsky, and I.A. Vasin, *Methods for solving inverse problems in mathematical physics*, Marcel Dekker, New York, 1999.

[28] A.G. Ramm, *Inverse problems*, Springer, 2005.

[29] V.G. Romanov, *Some inverse problems for hyperbolic equations*, Nauka, Novosibirsk, 1972. (in Russian)

[30] A.Yu. Shcheglov, *Iterative method for recovery a nonlinear source in a hyperbolic equation with final overdetermination*, Journal of Inverse and Ill-posed Problems, **10**(2013), 629-641.

[31] I. Tekin, *Determination of a time-dependent coefficient in a wave equation with unusual boundary condition*, Filomat, **33**(2019), 2653-2665.

[32] I. Tekin, Y.T. Mehraliyev, M.I. Ismailov, *Existence and uniqueness of an inverse problem for nonlinear Klein-Gordon equation*, Mathematical Methods in the Applied Sciences, **42**(2019), 3739-3753.