U.P.B. Sci. Bull., Series A, Vol. 72, Iss. 2, 2010 ISSN 1223-7027

MULTIPLE (n,m)-HYBRID LAPLACE TRANSFORMATION AND APPLICATIONS TO MULTIDIMENSIONAL HYBRID SYSTEMS. PART I

Valeriu PREPELIȚĂ[1]

Este prezentat un spațiu de funcții original care sunt continue în raport cu n variabile şi discrete în raport cu m variabile. Pe această mulțime se defineşte o transformare hibridă multiplă de tip Laplace şi z. Se studiază principalele proprietăți ale acestei transformări, printre care teoremele de liniaritate, întârziere, deplasare, derivare şi diferență a originalului, derivare a imaginii. Aceste proprietăți vor fi utilizate într-o lucrare ulterioară pentru a rezolva ecuații multiple diferențiale cu diferențe şi ecuații integrale multiple şi pentru obținerea reprezentării sistemelor de comandă hibride multiple în domeniul frecvență.

A space of original functions which are continuous with respect to n variables and discrete with respect to m variables is presented. A multiple hybrid Laplace and z type transformation is defined on this set. Its main properties are studied, including linearity, time-delay, translation, differentiation and difference of the original, differentiation of the image. Other theorems such as integration and sum of the original, convolution, initial and final values etc. will be presented in a subsequent paper, as well as some methods to determine the originals.

These properties will be used to solve multiple differential-difference and multiple integral equations and to obtain the frequency-domain representations of multidimensional hybrid control systems.

Key words: original functions, multiple hybrid Laplace transformation, continuous-discrete $(n\mathrm{D})$ systems

1. Introduction

In the last three decades the theory of multidimensional $(n\mathrm{D})$ control systems knew a strong development, due to its applications in various important domains as image processing, computer tomography, geophysics, seismology etc.

A distinct branch of this theory is represented by the continuous-discrete $n\mathrm{D}$ systems which appear as models in many problems, for instance in the study of linear repetitive processes [1], [2], [10] or in the iterative learning control synthesis [6]. Such hybrid systems were studied in [3], [4], [5], [7], [8].

[1] Prof., Department of Mathematics I, University POLITEHNICA of Bucharest, Romania, e-mail: vprepelita@mathem.pub.ro

In the theory of "classical" 1D systems the frequency domain methods, based on Laplace transformation in the continuous case or z-transformation in the discrete-time case, play a very important role. In order to extend the frequency domain methods to multiple hybrid systems one needs a generalization of these transformation.

The aim of this paper is to give a complete analysis of a suitable hybrid Laplace-z type transformation and to emphasize its applications in the study of multidimensional continuous-discrete systems or for solving multiple hybrid equations. In section 2 the continuous-discrete original functions are defined and it is shown that their set is a complex commutative linear algebra with unity. A multiple hybrid Laplace transformation is defined as a linear operator defined on this algebra and taking values in the set of multivariable functions which are analytic over a suitable domain.

The main properties of this transformation are stated and proved, including linearity, homothety, two time-delay theorems, translation, differentiation and difference of the original, differentiation of the image, integration and sum of the original, integration of the image, convolution, product of originals, initial and final values. Some results are generalizations of the properties of the 2D continuous-discrete transformation studied in [9]. Other properties and some methods for determining the original will be presented in Part II.

Some applications of the (n,m)-hybrid Laplace transform will be provided in Part III to solve multiple differential-difference and integral equations as well as for the study of multidimensional continuous-discrete systems.

2. Multiple (n,m)-hybrid Laplace transformation

We denote by $\langle n\rangle$ the set $\{1,2,\ldots,n\}$.

Definition 2.1. A function $f:\mathbf{R}^n\times\mathbf{Z}^m\to\mathbf{C}$ is said to be a *continuous-discrete original function* (or simply an *original*) if f has the following properties:

(i) $f(t_1,\ldots,t_n;k_1,\ldots,k_m)=0$ if $t_i<0$ or $k_j<0$ for some $i\in\langle n\rangle$ or $j\in\langle m\rangle$.

(ii) $f(\cdot,\ldots,\cdot;k_1,\ldots,k_m)$ is piecewise smooth on $\mathbf{R}_+^n$ for any $(k_1,\ldots,k_m)\in\mathbf{Z}_+^m$.

(iii) $\exists\, M_j>0$, $\sigma_{fi}\in\mathbf{R}$, $i\in\langle n\rangle$, $R_{fj}>0$, $j\in\langle m\rangle$ such that

$$|f(t_1,\ldots,t_n;k_1,\ldots,k_m)|\le M_f\exp\left(\sum_{j=1}^{n}\sigma_{fi}t_i\right)\prod_{j=1}^{m}R_{fj}^{k_j} \tag{2.1}$$

$\forall\, t_i > 0$, $i \in \langle n \rangle$, $\forall\, k_j \geq 0$, $j \in \langle m \rangle$.

The constants σ_{fi}, R_{fj} will be also denoted by σ_i, R_j. The smallest such constants are called respectively the *indices of the order of growth* and *the radii of convergence* of the original function f.

We denote by $O_{n,m}$ the set of original functions $f:\mathbf{R}^n \times \mathbf{Z}^m \to \mathbf{C}$. Sometimes we shall denote by $f(t;k)$ the value of f at $t=(t_1,\ldots,t_n)$, $(k_1,\ldots,k_m)$. The structure of $O_{n,m}$ is established in the Propositions 2.2 and 2.4 below, whose proofs are omitted for lack of space.

Proposition 2.2. *The set $O_{n,m}$ with the addition, multiplication and scalar multiplication is a complex commutative linear algebra with unity, where the unit element is the "unit step function" u, $u(t_1,\ldots,t_n;k_1,\ldots,k_m)=1$ on $\mathbf{R}_+^n \times \mathbf{Z}_+^m$ and equal to 0 otherwise*.

Definition 2.3. Given $f, g \in O_{n,m}$, the (n,m)*-hybrid convolution* of f and g is the function denoted by $f * g$ defined by

$$(f*g)(t_1,\ldots,t_n;k_1,\ldots,k_m) = \int_0^{t_1} \ldots \int_0^{t_n} \sum_{l_1=0}^{k_1} \ldots \sum_{l_m=0}^{k_m} f(u_1,\ldots,u_n;l_1,\ldots,l_m)\cdot \tag{2.2}$$
$$\cdot\, g(t_1-u_1,\ldots,t_n-u_n;k_1-l_1,\ldots,k_m-l_m)du_1\ldots du_n$$

for $(t_1,\ldots,t_n;k_1,\ldots,k_m) \in \mathbf{R}_+^n \times \mathbf{Z}_+^m$ and equal to 0 otherwise.

Proposition 2.4. *$O_{n,m}$ is closed under the (n,m)-hybrid convolution*.

Proposition 2.5. *$O_{n,m}$ with the addition, the (n,m)-hybrid convolution and the scalar multiplication by complex numbers is a commutative linear algebra with unity.*

Proof. By Propositions 2.2 and 2.4, $O_{n,m}$ is closed under the three mentioned operations. One can verify that the (n,m)-hybrid convolution is distributive, commutative and $\alpha(f*g)=(\alpha f)*g=f*(\alpha g)$, $\forall\, \alpha \in \mathbf{C}$, $f*\delta_{n,m}=\delta_{n,m}*f=f$ where $\delta_{n,m} \in O_{n,m}$ is defined by

$$\delta_{n,m}(t_1,\ldots,t_n;k_1,\ldots,k_m) = \begin{cases} \prod_{i=1}^{n} \delta(t_i) & \text{if} \quad k_1=\ldots=k_m=0 \\ 0 & \text{otherwise,} \end{cases}$$

where δ is the Dirac distribution.

□

Definition 2.6. For any original f, the function

$$F(s_1,\ldots,s_n;z_1,\ldots,z_m)=\int_0^\infty\ldots\int_0^\infty\sum_{k_1=0}^{\infty}\ldots\sum_{k_m=0}^{\infty}f(t_1,\ldots,t_n;k_1,\ldots,k_m)\cdot \\ \cdot e^{-s_1t_1}\cdots e^{-s_nt_n}z_1^{k_1}\cdots z_m^{-k_m}dt_1\ldots dt_n \tag{2.3}$$

is called the (n,m)-*hybrid Laplace transform* (or the *image*) of f.
We shall also use the notation $F(s;z)$ for $F(s_1,\ldots,s_n;z_1,\ldots,z_m)$.

Proposition 2.7. *The multiple improper integral and the multivariable Taylor series in (2.3) are absolutely convergent in the domain*

$$D(f)=\{(s_1,\ldots,s_n;z_1,\ldots,z_m)\in C^{n+m}\mid \operatorname{Re}s_i>\sigma_{fi}, i\in\langle n\rangle;\ |z_j|>R_{fj},\ j\in\langle m\rangle\} \tag{2.4}$$

and uniformly convergent on any domain

$$D'(f)=\{(s_1,\ldots,s_n;z_1,\ldots,z_m)\in C^{n+m}\mid \operatorname{Re}s_i\ge\sigma_i',\quad i\in\langle n\rangle;\ |z_j|\ge R_j',\ j\in\langle m\rangle\}$$

with $\sigma_i'>\sigma_{fi}$, $i\in\langle n\rangle$ *and* $R_j'>R_{fj}$, $j\in\langle m\rangle$.

Proof. Since $\operatorname{Re}s_i>\sigma_{fi}$ and $|z_j|>R_{fj}$, $\lim\limits_{t_i\to\infty}e^{-(\operatorname{Re}s_i-\sigma_{fi})t_i}=0$, $\forall\, i\in\langle n\rangle$ and the geometric series $\sum\limits_{k_j=0}^{\infty}\left(\dfrac{R_{fj}}{|z_j|}\right)^{k_j}$ are convergent and have the sum $\dfrac{|z_j|}{|z_j|-R_{fj}}$, $j\in\langle m\rangle$. Then, by (2.3) and (iii) we obtain

$$|F(s_1,\ldots,s_n;z_1,\ldots,z_m)|\le\frac{M_f\prod_{j=1}^{m}|z_j|}{\left(\prod_{i=1}^{n}(\operatorname{Re}s_i-\sigma_{fi})\right)\left(\prod_{j=1}^{m}(|z_j|-R_{fj})\right)}<\infty. \tag{2.5}$$

Analogously, for $\operatorname{Re}s_i\ge\sigma_i'>\sigma_{fi}$, $i\in\langle n\rangle$ and $|z_j|>R_j'>R_{fj}$, $j\in\langle m\rangle$, one obtains

$$|F(s_1,\ldots,s_n;z_1,\ldots,z_m)|\le\left(\prod_{i=1}^{n}\ \left(\int_0^\infty\ e^{-(\sigma_i'-\sigma_{fi})t_i}dt_i\right)\right)\left(\prod_{j=1}^{m}\ \frac{R_{fj}}{R_j'}\right)=$$

$$= \frac{M_f \prod_{j=1}^{m} R'_j}{\left(\prod_{i=1}^{n} (\sigma'_i - \sigma_{fi})\right)\left(\prod_{j=1}^{m} (R'_j - R_{fj})\right)} < \infty .$$

and the uniform convergence results as a consequence of the Weierstrass criterion.

□

Corollary 2.8. *The* (n,m)*-hybrid Laplace transform* $F(s;z)$ *is analytic in the domain* $D(f)$ (2.4).

Remark 2.9. Assume that $s_i \to \infty$ and $\operatorname{Arg} s_i \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for some $i \in \langle n \rangle$; then $\operatorname{Re} s_i \to \infty$ and by (2.5) $\lim_{s_i \to \infty} | F(s_1,\ldots,s_n;z_1,\ldots,z_m)| = 0$.

Let us denote by $\mathcal{A}_{n,m}$ the complex linear algebra of the functions $F : \mathbf{C}^{n+m} \to \mathbf{C}$ which are analytic in a domain of the form (2.4).

Definition 2.10. The operator $\mathcal{L}_{n,m} : O_{n,m} \to \mathcal{A}_{n,m}$ defined by (2.3) is called the (n,m)*-hybrid Laplace transformation* ((n,m)-HLT).

Now, we shall emphasize the principal properties of the operator $\mathcal{L}_{n,m}$.

Theorem 2.11 (Linearity). *For any* $f, g \in O_{n,m}$ *and* $\alpha, \beta \in \mathbf{C}$,

$$\mathcal{L}_{n,m}[\alpha f + \beta g] = \alpha \mathcal{L}_{n,m}[f] + \beta \mathcal{L}_{n,m}[g]. \tag{2.6}$$

Proof. Equality (2.6) is an immediate consequence of (2.3) and Proposition 2.2 and it holds in $D(f) \cap D(g)$.

□

Theorem 2.12 (Homothety). *If there exists* $(b_1,\ldots,b_m) \in (\mathbf{N}^*)^m$ *such that* $f(t_1,\ldots,t_n;k_1,\ldots,k_m) = 0$ *for any* $(t_1,\ldots,t_n;k_1,\ldots,k_m) \in \mathbf{R}_+^n \times \mathbf{Z}_+^m$ *with* $(k_1,\ldots,k_m) \neq (b_1,\ldots,b_m)$*, then*

$$\mathcal{L}_{n,m}[f(a_1t_1,\ldots,a_nt_n;b_1k_1,\ldots,b_mk_m)] = \\ = \left(\prod_{i=1}^{n} a_i^{-1}\right) F(a_1^{-1}s_1,\ldots,a_n^{-1}s_n; z_1^{b_1^{-1}},\ldots,z_m^{b_m^{-1}}) \tag{2.7}$$

for any $a_i > 0$, $s_i \in \mathbf{C}$ with $\operatorname{Re} s_i > a_i \sigma_{fi}$, $i \in \langle n \rangle$ and any $z_j \in \mathbf{C}$ with $|z_j| > R_{fj}^{b_j}$, $j \in \langle m \rangle$.

Proof. By using the change of the variables of integration $a_i t_i = x_i$, $i \in \langle n \rangle$ and the change of the indices of summation $b_j k_j = l_j$, $j \in \langle m \rangle$, we get

$$\mathcal{L}_{n,m}[f(a_1t_1,\ldots,a_nt_n;b_1k_1,\ldots,b_mk_m)] =$$

$$\int_0^\infty \ldots \int_0^\infty \sum_{k_1=0}^\infty \ldots \sum_{k_m=0}^\infty f(a_1t_1,\ldots,a_nt_n;b_1k_1,\ldots,b_mk_m)\cdot$$

$$\cdot \exp\left(-\sum_{i=1}^{\infty} s_i t_i\right)\left(\prod_{j=1}^{m} z_j^{-k_j}\right) dt_1 \ldots dt_n =$$

$$= \left(\prod_{i=1}^{n} a_i^{-1}\right)\int_0^\infty \ldots \int_0^\infty \sum_{l_1=0}^\infty \ldots \sum_{l_m=0}^\infty f(x_1,\ldots,x_n;l_1,\ldots,l_m)\cdot$$

$$\cdot \exp\left(-\sum_{i=1}^{n} a_i^{-1} s_i t_i\right)\left(\prod_{j=1}^{m} z_j^{-b_j^{-1} l_j}\right) dx_1 \ldots dx_n =$$

$$= \left(\prod_{i=1}^{n} a_i^{-1}\right) F(a_1^{-1}s_1,\ldots,a_n^{-1}s_n;z_1^{b_1^{-1}},\ldots,z_m^{b_m^{-1}}).$$

□

Theorem 2.13 (First time-delay theorem). *For any* $(a_1,\ldots,a_n) \in \mathbf{R}_+^n$, $(b_1,\ldots,b_m) \in Z_+^m$,

$$\mathcal{L}_{n,m}[f(t_1-a_1,\ldots,t_n-a_n;k_1-b_1,\ldots,k_m-b_m)] =$$

$$= \exp\left(-\sum_{i=1}^{n} a_i s_i\right)\left(\prod_{j=1}^{m} z_j^{-b_j}\right) F(s_1,\ldots,s_n;z_1,\ldots,z_m). \tag{2.8}$$

Proof. By the change of variables $t_1 - a_i = x_i$, $i \in \langle n \rangle$, $k_j - b_j = l_j$, $j \in \langle m \rangle$ and by taking into account condition (i) in Definition 2.1, we obtain:

$$\int_0^\infty \dots \int_0^\infty \sum_{k_1=0}^\infty \dots \sum_{k_m=0}^\infty f(t_1-a_1,\dots,t_n-a_n;k_1-b_1,\dots,k_m-b_m)\cdot$$

$$\cdot \exp\left(-\sum_{i=1}^n s_i t_i\right)\left(\prod_{j=1}^m z_j^{-k_j}\right)dt_1\dots dt_n =$$

$$= \exp\left(-\sum_{i=1}^n a_i s_i\right)\left(\prod_{j=1}^m z_j^{-b_j}\right)\int_0^\infty \dots \int_0^\infty \sum_{l_1=0}^\infty \dots \sum_{l_m=0}^\infty f(x_1,\dots,x_n;l_1,\dots,l_m)$$

$$= \exp\left(-\sum_{i=1}^n s_i x_i\right)\left(\prod_{i=1}^m z_j^{-l_j}\right)dx_1,\dots,dx_n$$

equality which proves (2.8).

□

We shall use the following notations : for some sets $\alpha=\{i_1,\dots,i_p\}\subset\langle n\rangle$ and $\beta=\{j_1,\dots,j_q\}\subset\langle m\rangle$, $|\alpha|=p$ (the cardinality of α), $|\beta|=q$, $E_\alpha=\{\varepsilon \mid \varepsilon\subset\alpha \text{ or } \varepsilon=\varnothing\}$, $E'_\beta=\{\delta \mid \delta\subset\beta \text{ or } \delta=\varnothing\}$; for $\varepsilon\in E_\alpha$ and $\delta\in E'_\beta$, $\bar\varepsilon=\langle n\rangle\setminus\varepsilon$, $\bar\delta=\langle m\rangle\setminus\delta$. If $\varepsilon=\{i\}$ or $\delta=\{j\}$, $\bar i,\bar j$ denote $\bar\varepsilon$ and $\bar\delta$ respectively.

For $a=(a_i)_{i\in\alpha}\in\mathbf{R}_+^{|\alpha|}$ with $a_i>0$, $\forall i\in\alpha$ and $b=(b_j)_{j\in\beta}\in\mathbf{Z}_+^{|\beta|}$, with $b_j>0$, $\forall j\in\beta$ and for $\varepsilon=(\varepsilon_1,\dots,\varepsilon_\gamma)\in E_\alpha$ and $\delta=(\delta_1,\dots,\delta_\mu)\in E'_\beta$ we denote by $D_{a,\varepsilon}$ and $D'_{b,\delta}$ the sets $D_{a,\varepsilon}=\prod_{i\in\varepsilon}[0,a_i]$ and $D'_{b,\delta}=\prod_{j\in\delta}\{0,1,\dots,b_j-1\}$ and by $\int_{D_{a,\varepsilon}}$ and $\sum_{D'_{b,\delta}}$ the multiple integral $\int_0^{a_{\varepsilon_1}}\dots\int_0^{a_{\varepsilon_\gamma}}$, respectively the multiple sum $\sum_{k_{\delta_1}=0}^{b_{\delta_1}-1}\dots\sum_{k_{\delta_\mu}=0}^{b_{\delta_\mu}-1}$; if $\varepsilon=\varnothing$ or $\delta=\varnothing$ the corresponding multiple integral or sum lack; $f(t+a;k+b)$ denotes

$$f(t_1,\dots,t_{i_1-1},t_{i_1}+a_{i_1},t_{i_1+1},\dots,t_{i_p-1},t_{i_p}+a_{i_p},t_{i_p+1},\dots,t_n;$$
$$k_1,\dots,k_{j_1-1},k_{j_1}+b_{j_1},k_{j_1+1},\dots,k_{j_q-1},k_{j_q}+b_{j_q},k_{j_q+1},\dots,k_m).$$

Definition 2.14. For $\alpha=\{i_1,\dots,i_p\}\subset\langle n\rangle$ and $\beta=\{j_1,\dots,j_q\}\subset\langle m\rangle$, the (α,β)-*partial* (n,m)-*hybrid transform* of the original f is defined by

$$\mathcal{L}_{n,m}^{\alpha,\beta}[f(t,k)]=\int_0^\infty\ldots\int_0^\infty\sum_{k_{j_1}=0}^{\infty}\ldots\sum_{k_{j_q}=0}^{\infty}f(t_1,\ldots,t_n;k_1,\ldots,k_m)\cdot$$
$$\cdot\exp\left(-\sum_{i\in\alpha}^{n}s_it_i\right)\left(\prod_{j\in\beta}z_j^{-k_j}\right)dt_{i_1}\ldots dt_{i_p}. \tag{2.9}$$

If $\alpha=\langle n\rangle$ and $\beta=\langle m\rangle$, $\mathcal{L}_{n,m}^{\alpha,\beta}=\mathcal{L}_{n,m}$; if $\beta=\varnothing$, $\mathcal{L}_{n,m}^{\alpha,\varnothing}=\mathcal{L}_p$ (the multiple Laplace transformation) ; if $\alpha=\varnothing$, $\mathcal{L}_{n,m}^{\varnothing,\beta}=\mathcal{Z}_q$ (the multiple z-transformation) ; if $\alpha=\beta=\varnothing$, $\mathcal{L}_{n,m}^{\varnothing,\varnothing}[f]=f$.

Theorem 2.15. (Second delay theorem). *For any* $a=(a_i)_{i\in\alpha}\in\mathbf{R}_+^{|\alpha|}$ *and* $b=(b_j)_{j\in\beta}\in\mathbf{Z}_+^{|\beta|}$

$$\mathcal{L}_{n,m}[f(t+a;k+b)]=\exp\left(\sum_{i\in\alpha}a_is_i\right)\left(\prod_{j\in\beta}z_j^{b_j}\right)[F(s;z)+ \tag{2.10}$$
$$+\sum_{\varepsilon\in E_\alpha}\sum_{\delta\in E_\beta}(-1)^{|\varepsilon|+|\delta|}\int_{D_{a,\varepsilon}}\sum_{D'_{b,\gamma}}\mathcal{L}_{n,m}^{\bar\varepsilon,\bar\delta}[f(t,k)]\cdot\exp\left(-\sum_{i\in\alpha}s_it_i\right)\left(\prod_{j\in\delta}z_j^{-k_j}\right)\prod_{i\in\varepsilon}dt_i.$$

Proof. By the change of variables $t_i+a_i=x_i$, $i\in\alpha$, $k_j+b_j=l_j$, $j\in\beta$, one obtains the (n,m)-hybrid transform in (2.10) in the form:

$$\int_0^\infty\ldots\int_0^\infty\sum_{k_1=0}^{\infty}\ldots\sum_{k_m=0}^{\infty}f(t+a;k+b)\exp\left(-\sum_{i=1}^{n}s_it_i\right)\left(\prod_{j=1}^{m}z_j^{-k_j}\right)dt_1\ldots dt_n=$$
$$=\exp\left(\sum_{i\in\alpha}a_is_i\right)\left(\prod_{j\in\beta}z_j^{b_j}\right)\int_0^\infty\ldots\int_{a_{i_1}}^\infty\ldots\int_{a_{i_p}}^\infty\ldots\int_0^\infty\sum_{k_1=0}^{\infty}\ldots\sum_{l_{j_1}=b_{j_1}}^{\infty}\ldots\sum_{l_{j_2}=b_{j_q}}^{\infty}\ldots\sum_{k_m=0}^{\infty}\cdot$$
$$\cdot f(t_1,\ldots,x_{i_1},\ldots,x_{i_p},\ldots,x_n;k_1,\ldots,l_{j_1},\ldots,l_{j_q},\ldots,k_n)dt_1\ldots dx_{i_1}\ldots dx_{i_p}\ldots dt_n.$$

By replacing succesively $\int_{a_i}^\infty=\int_0^\infty-\int_0^{a_i}$ and $\sum_{l_l=b_j}^{\infty}=\sum_{l_j=0}^{\infty}-\sum_{l_j=0}^{b_j-1}$, a long but straightforward calculus gives (2.10).

Theorem 2.16. (Translation). *For any* $a_i\in\mathbf{C}$, $i\in\langle n\rangle$, $b_j\in\mathbf{C}\setminus\{0\}$, $j\in\langle m\rangle$,

$$\mathcal{L}_{n,m}\left[\exp\left(\sum_{i=1}^{n} a_i t_i\right)\left(\prod_{j=1}^{m} b_j^{k_j}\right) f(t,\ldots,t_n;k_1,\ldots,k_m)\right] = \\ = F(s_1-a_1,\ldots,s_n-a_n;z_1b_1^{-1},\ldots,z_mb_m^{-1}) \tag{2.11}$$

where $\operatorname{Re} s_i > \operatorname{Re} a_i + \sigma_{ji}$, $i \in \langle n\rangle$ *and* $|z_j| > |b_j| R_{fj}$, $j \in \langle m\rangle$.

Proof. The (n,m)-hybrid Laplace transform in (2.11) has the expression

$$\int_0^\infty \ldots \int_0^\infty \sum_{k_1=0}^{\infty} \ldots \sum_{k_m=0}^{\infty} \exp\left(\sum_{i=1}^{n} a_i t_i\right)\left(\prod_{j=1}^{m} b_j^{k_j}\right) f(t_1,\ldots,t_n;k_1,\ldots,k_m)\cdot$$

$$\cdot\exp\left(-\sum_{i=1}^{n} s_i t_i\right)\left(\prod_{j=1}^{m} z_j^{-k_j}\right) dt_1\ldots dt_n = \int_0^\infty \ldots \int_0^\infty \sum_{k_1=0}^{\infty} \ldots \sum_{k_m=0}^{\infty} f(t_1,\ldots,t_n;t_1,\ldots,t_m)\cdot$$

$$\cdot\exp\left(-\sum_{i=1}^{n} (s_1-a_i)t_i\right)\left(\prod_{j=1}^{m} (z_jb_j^{-1})^{-k_j}\right) dt_1,\ldots,dt_n =$$

$$= F(s_1-a_1,\ldots,s_n-a_n;z_1b_1^{-1},\ldots,z_mb_m^{-1}).$$

□

We introduce the following notations: given $\alpha = \{i_1,\ldots,i_p\} \subset \langle n\rangle$, a p-tuple $(\gamma_{i_1},\ldots,\gamma_{i_p}) \in \mathbf{N}^p$ is denoted by γ_α or simply by γ and $\dfrac{\partial^\gamma f}{\partial t^\gamma} = \dfrac{\partial^{\gamma_{i_1}+\cdots+\gamma_{i_p}}}{\partial t_{i_1}^{\gamma_{i_1}}\cdots\partial t_{i_p}^{\gamma_{i_p}}}$, $s^\gamma = s_{i_1}^{\gamma_{i_1}}\cdots s_{i_p}^{\gamma_{i_p}}$. The family of all unvoid subsets ε of α is denoted by E_γ^α or E_γ. For $\varepsilon \in E_\gamma^\alpha$, $\bar\varepsilon = \alpha \setminus \varepsilon$, $s_\varepsilon^{\gamma_\varepsilon} = \prod_{i\in\varepsilon} s_i^{\gamma_i}$ and $s_{\bar\varepsilon}^{\gamma_{\bar\varepsilon}} = 1$ if $\varepsilon = \alpha$; if $\varepsilon = \{\iota_1,\ldots.\iota_p\}$ and $\eta_\varepsilon = (\eta_{\iota_1},\ldots,\eta_{\iota_p}) \in \mathrm{N}^p$, $\eta_\varepsilon \le \gamma_\varepsilon$ means $\eta_i \le \gamma_i$, $\forall i \in \varepsilon$; $f(0_\varepsilon^+;k)$ denotes the limit from the right

$$f(t_1,\ldots,t_{\iota_1-1},0+,t_{\iota_1+1},\ldots,t_{\iota_p-1},0+,t_{\iota_p+1},\ldots,t_n;k_1,\ldots,k_m);$$

if $\varepsilon = \{i\}$ then $f(0_\varepsilon^+;k)$ is denoted by $f(0_i^+;k)$. Similarly $f(t;k_1,\ldots,k_{j-1},0,k_{j+1},\ldots,k_m)$ is denoted $f(t;0_j)$ and we can use the notation $f(0_i^+;0_j)$ which combines these notations.

Theorem 2.17 (Differentiation of the original). *For any* $i \in \langle n\rangle$

$$\mathcal{L}_{n,m}\left[\frac{\partial f}{\partial t_i}(t;k)\right]=s_iF(s;z)-\mathcal{L}_{n,m}^{\bar{i},\langle m\rangle}[f(0_i^+;k)] \tag{2.12i}$$

$$\mathcal{L}_{n,m}\left[\frac{\partial^{\gamma} f}{\partial t^{\gamma}}(t;k)\right]=s^{\gamma}F(s;z)+ + \sum_{\varepsilon\in E_{\gamma}}(-1)^{|\varepsilon|}s_{\bar{\varepsilon}}^{\gamma_{\bar{\varepsilon}}}\sum_{\eta_{\varepsilon}\le\gamma_{\varepsilon}-1}s_{\varepsilon}^{\gamma_{\varepsilon}-\eta_{\varepsilon}-1}\mathcal{Z}_{n,m}^{\bar{\varepsilon},\langle m\rangle}\left[\frac{\partial^{\eta_{\varepsilon}} f}{\partial t^{\eta_{\varepsilon}}}(0_{\varepsilon}^+;k)\right]. \tag{2.12ii}$$

Proof. By Definition 2.1 we have

$$|f(t;k)\exp(-s_it_i)|\le M\exp\left(\sum_{l\ne i}\sigma_l t_l\right)\left(\prod_{j=1}^{m}R_j^{k_j}\right)\exp(-(\operatorname{Re}s_i-\sigma_i)t_i)\,.$$

Since $\operatorname{Re}s_i>\sigma_i$, $\lim\limits_{t_i\to\infty}\exp(-(\operatorname{Re}s_i-\sigma_i)t_i)=0$, hence $\lim\limits_{t_i\to\infty}f(t,k)\exp(-s_it_i)=0$.

We integrate by parts the i-th integral in $\mathcal{L}_{n,m}\left[\frac{\partial f}{\partial t_i}(t;k)\right]$ and we get

$$\int_0^{\infty}\frac{\partial f}{\partial t_i}(t;k)\exp(-s_it_i)dt_i=f(t;k)e^{-s_it_i}\,\Big|_0^{\infty}-\int_0^{\infty}f(t;k)(-s_i)\exp(-s_it_i)dt_i=$$

$$=s_i\int_0^{\infty}f(t;k)\exp(-s_it_i)dt_i-f(0_i^+;k).$$

By applying the other $n-1$ integrals and the m series of Definition 2.6 we obtain

$$\mathcal{L}_{n,m}\left[\frac{\partial f}{\partial t_i}(t;k)\right]=s_i\underbrace{\int_0^{\infty}\ldots\int_0^{\infty}}_{n}\sum_{k_1=0}^{\infty}\ldots\sum_{k_m=0}^{\infty}f(t_1,\ldots,t_n;k_1,\ldots,k_m)\cdot$$

$$\cdot\exp\left(-\sum_{l=1}^{n}s_lt_l\right)\left(\prod_{j=1}^{m}z_j^{-k_j}\right)dt_1\ldots dt_n-$$

$$-\underbrace{\int_0^{\infty}\ldots\int_0^{\infty}}_{n}\sum_{k_1=0}^{\infty}\ldots\sum_{k_m=0}^{\infty}f(t_1,\ldots,t_{i-1},0+,t_{i+1},\ldots,t_n;k_1,\ldots,k_m)\cdot$$

$$\cdot\exp\left(-\sum_{\substack{l=1\\l\ne i}}^{n}s_lt_l\right)\left(\prod_{j=1}^{m}z_j^{-k_j}\right)dt_1\ldots dt_{i-1}dt_{i+1}\ldots dt_n$$

i.e. (2.12i). It results that (2.12ii) is true for $\gamma=(\gamma_i)$ where $\gamma_i=1$, $\alpha=\{i\}$, $E=\{\{i\}\}$, $\varepsilon=\{i\}$, $\overline{\varepsilon}=\varnothing$. Assume that (2.12ii) is true for $\gamma=(\gamma_1,\gamma_2,\ldots,\gamma_p)$ and we shall prove (2.12ii) for $\gamma'=(\gamma_1+1,\gamma_2,\ldots,\gamma_p)$ and for $\gamma''=(\gamma_1,\gamma_2,\ldots,\gamma_{p+1})$ with $\gamma_{p+1}=1$. By (2.12i) we have

$$\begin{aligned}&\mathcal{L}_{n,m}\left[\frac{\partial^{\gamma'}}{\partial t^{\gamma'}}(t;k)\right]=\mathcal{L}_{n,m}\left[\frac{\partial}{\partial t_1}\left(\frac{\partial^{\gamma}f}{\partial t^{\gamma}}\right)\right]=\\&=s_1\mathcal{L}_{n,m}\left[\frac{\partial f}{\partial t^{\gamma}}(t;k)\right]-\mathcal{L}_{n,m}^{\overline{1},\langle m\rangle}\left[\frac{\partial^{\gamma}f}{\partial t^{\gamma}}(0_1^+;k)\right].\end{aligned}\tag{2.13}$$

By (2.11ii)

$$\begin{aligned}&s_1\mathcal{L}_{n,m}\left[\frac{\partial^{\gamma}f}{\partial t^{\gamma}}(t;k)\right]=s_1s^{\gamma}F(s;z)+\\&+\sum_{\varepsilon\in E_\gamma}(-1)^{|\varepsilon|}s_1s_{\overline{\varepsilon}}^{\gamma_{\overline{\varepsilon}}}\sum_{\eta_\varepsilon\le\gamma_\varepsilon-1}s_\varepsilon^{\gamma_\varepsilon-\eta_\varepsilon-1}\mathcal{L}_{n,m}^{\overline{\varepsilon},\langle m\rangle}\left[\frac{\partial^{\eta_\varepsilon}f}{\partial t^{\eta_\varepsilon}}(0_\varepsilon^+;k)\right]=\\&=s^{\gamma'}F(s;z)+\sum_{\varepsilon\in E_{\gamma'}^1}(-1)^{|\varepsilon|}s_{\overline{\varepsilon}}^{\gamma'_{\overline{\varepsilon}}}\sum_{\eta_\varepsilon\le\gamma'_\varepsilon-1}^{*}\mathcal{L}_{n,m}^{\overline{\varepsilon},\langle m\rangle}\left[\frac{\partial^{\eta_\varepsilon}f}{\partial t^{\eta_\varepsilon}}(0_\varepsilon^+;k)\right]+\\&+\sum_{\varepsilon\in E_{\gamma'}^2}(-1)^{|\varepsilon|}s_{\overline{\varepsilon}}^{\gamma'_{\overline{\varepsilon}}}\sum_{\eta_\varepsilon\le\gamma'_\varepsilon-1}^{**}s_\varepsilon^{\gamma'_\varepsilon-\eta_\varepsilon-1}\mathcal{L}_{n,m}^{\overline{\varepsilon},\langle m\rangle}\left[\frac{\partial^{\eta_\varepsilon}f}{\partial t^{\eta_\varepsilon}}(0_\varepsilon^+;k)\right]\end{aligned}$$

where

$$E_{\gamma'}^1=\{\varepsilon\in E_\gamma\mid\{1\}\not\subset\varepsilon\},\quad E_{\gamma'}^2=\{\varepsilon\in E_\gamma\mid\{1\}\subset\varepsilon\}.$$

Therefore in $\sum^{*}$, $\{1\}\subset\overline{\varepsilon}$, $\gamma'_\varepsilon=\gamma_\varepsilon$ and $s_1s_1^{\gamma_1}=s_1^{\gamma'_1}$; in $\sum^{**}$, $\gamma'_{\overline{\varepsilon}}=\gamma_{\overline{\varepsilon}}$ and the term with s_0^1, i.e. with $\eta_1=\gamma'_1-1=\gamma_1$ lacks. In the same manner, we denote $\hat{\gamma}=(\gamma_2,\ldots,\gamma_p)$ and by a modified version of (2.11ii) we obtain

$$-\mathcal{L}_{n,m}^{\bar{1},\langle m\rangle}\left[\frac{\partial^{\gamma} f}{\partial t^{\gamma}}(0_1^+;k)\right]=-\mathcal{L}_{n,m}^{\bar{1},\langle m\rangle}\left[\frac{\partial^{\hat{\gamma}}}{\partial t^{\hat{\gamma}}}\left(\frac{\partial^{\gamma_1} f}{\partial t_1^{\gamma_1}}(0_1^+;k)\right)\right]=-s^{\hat{\gamma}}\mathcal{L}_{n,m}^{\bar{1},\langle m\rangle}\left[\frac{\partial^{\gamma_1} f}{\partial t_1^{\gamma_1}}(0_1^+;k)\right]-$$

$$-\sum_{\varepsilon\in E_{\hat{\gamma}}}(-1)^{|\varepsilon|}s_{\bar{\varepsilon}}^{\hat{\gamma}_{\bar{\varepsilon}}}\sum_{\eta_\varepsilon\le\hat{\gamma}_\varepsilon-1}s_\varepsilon^{\hat{\gamma}_\varepsilon-\eta_\varepsilon-1}\mathcal{L}_{n,m}^{\overline{\varepsilon\cup\{1\}},\langle m\rangle}\left[\frac{\partial^{\gamma_1+\eta_\varepsilon} f}{\partial t_1^{\gamma_1}\partial t_\varepsilon^{\eta_\varepsilon}}(0_{\varepsilon\cup\{1\}}^+;k)\right].$$

If we denote $\varepsilon\cup\{1\}$ by ε_1 we have $|\varepsilon|+1=|\varepsilon_1|$, $-s^{\hat{\gamma}}=(-1)^{|\varepsilon_1|}s_{\bar{1}}^{\gamma'_1}s_1^0$ and the previous expression becomes

$$\sum_{\varepsilon_1\in E_{\gamma'}}(-1)^{|\varepsilon_1|}s_{\bar{\varepsilon}_1}^{\hat{\gamma}_{\bar{\varepsilon}_1}}\sum_{\eta_\varepsilon\le\gamma'_\varepsilon-1}^{***}s_{\varepsilon_1}^{\gamma'_{\varepsilon_1}-\eta_{\varepsilon_1}-1}\mathcal{L}_{n,m}^{\bar{\varepsilon}_1,\langle m\rangle}\left[\frac{\partial^{\eta_{\varepsilon_1}} f}{\partial t^{\eta_{\varepsilon_1}}}(0_{\varepsilon_1}^+;k)\right]$$

where in $\sum^{***}$, $\eta_1=\gamma_1=\gamma'_1-1$. By combining $\sum^{*},\sum^{**},\sum^{***}$, formula (2.13) becomes (2.12ii) with $\gamma'=(\gamma_1+1,\gamma_2,\dots,\gamma_p)$ instead of $\gamma=(\gamma_1,\gamma_2,\dots,\gamma_p)$.

Now, for $\gamma''=(\gamma_1,\dots,\gamma_p,\gamma_{p+1})$ with $\gamma_{p+1}=1$, we have, by (2.12i) and (2.12ii):

$$\mathcal{L}_{n,m}\left[\frac{\partial^{\gamma''} f}{\partial t^{\gamma''}}(t;k)\right]=\mathcal{L}_{n,m}\left[\frac{\partial}{\partial t_{p+1}}\left(\frac{\partial^{\gamma} f}{\partial t^{\gamma}}(t;k)\right)\right]=$$

$$=s_{p+1}\mathcal{L}_{n,m}\left[\frac{\partial^{\gamma} f}{\partial t^{\gamma}}(t;k)\right]-\mathcal{L}_{n,m}^{\overline{p+1},\langle m\rangle}\left[\frac{\partial^{\gamma} f}{\partial t^{\gamma}}(0_{p+1}^+;k)\right]=s_{p+1}s^{\gamma}F(s;z)+$$

$$+\sum_{\varepsilon\in E_\gamma}^{1}(-1)^{|\varepsilon|}s_{p+1}s_{\bar{\varepsilon}}^{\gamma_{\bar{\varepsilon}}}\sum_{\eta_\varepsilon\le\gamma_\varepsilon-1}s_\varepsilon^{\gamma_\varepsilon-\eta_\varepsilon-1}\mathcal{L}_{n,m}^{\bar{\varepsilon},\langle m\rangle}\left[\frac{\partial^{\eta_\varepsilon} f}{\partial t^{\eta_\varepsilon}}(0_\varepsilon^+;k)\right]-s^{\gamma}\mathcal{L}_{n,m}^{\overline{p+1},\langle m\rangle}[f(0_{p+1}^+;k)]-$$

$$-\sum_{\varepsilon\in E_\gamma}^{2}(-1)^{|\varepsilon|}s_{\bar{\varepsilon}}^{\gamma_{\bar{\varepsilon}}}\sum_{\eta_\varepsilon\le\gamma_\varepsilon-1}s_\varepsilon^{\gamma_\varepsilon-\eta_\varepsilon-1}\mathcal{L}_{n,m}^{\overline{\varepsilon\cup\{p+1\}},\langle m\rangle}\left[\frac{\partial^{\eta_\varepsilon} f}{\partial t^{\eta_\varepsilon}}(0_\varepsilon^+,0_{p+1}^+;k)\right].$$

In $\overset{1}{\Sigma}$ the subsets $\varepsilon\in E_\gamma$ are the subsets $\varepsilon\in E_{\gamma''}$ which does not include $p+1$, while in $\overset{2}{\Sigma}$ we can replace ε by $\varepsilon_1=\varepsilon\cup\{p+1\}\in E_{\gamma''}$, hence $-(-1)|\varepsilon|=(-1)^{|\varepsilon_1|}$, $\sum_{\eta_\varepsilon\le\gamma_\varepsilon-1}s_\varepsilon^{\gamma_\varepsilon-\eta_\varepsilon-1}=\sum_{\eta''_{\varepsilon_1}\le\gamma''_{\varepsilon_1}-1}s_{\varepsilon_1}^{\gamma''_{\varepsilon_1}-\eta''_{\varepsilon_1}-1}$, where $\gamma''_{p+1}=1$ and $\eta''_{p+1}=0$, hence finally we obtain (2.12ii) for γ'', which completes the proof of (2.12ii) by induction.

□

Theorem 2.18 (Differentiation and delay). *For any* $i\in\langle n\rangle$, $j\in\langle m\rangle$,

$$\mathbf{L}_{n,m}\left[\frac{\partial f}{\partial t_i}(t_1,\dots,t_n;k_1,\dots,k_{j-1},k_j+1,k_{j+1},\dots,k_m)\right]=s_iz_jF(s,z)- \\ -s_iz_j\mathbf{L}_{n,m}^{\langle n\rangle,\bar{j}}[f(t;0_j)]-z_j\mathbf{L}_{n,m}^{\bar{i},\langle m\rangle}f[0_i^+;k)]+z_jf(0_i^+;0_j). \tag{2.14i}$$

For any $\gamma=(\gamma_{i_1},\dots,\gamma_{i_p})\in\mathbf{N}^p$, $\beta=\{j_1,\dots,j_q\}\subset\langle m\rangle$ $\quad b=(b_{j_1},\dots,b_{j_q})\in\mathrm{N}^q$,

$$\mathcal{L}_{n,m}\left[\frac{\partial^\gamma f}{\partial t^\gamma}(t;k+b)\right]=s^\gamma z^bF(s,z)+z^b\sum_{\varepsilon\in E_\gamma}\sum_{\delta\in E_\beta}(-1)^{|\varepsilon|+|\delta|}s_{\bar{\varepsilon}}^{\gamma_{\bar{\varepsilon}}} \\ \cdot\sum_{\eta_\varepsilon\le\gamma_\varepsilon-1}s_\varepsilon^{\gamma_\varepsilon-\eta_\varepsilon-1}\sum_{D'_{b,\delta}}\mathcal{L}_{n,m}^{\bar{\varepsilon},\bar{\delta}}\left[\frac{\partial^{\eta_\varepsilon}f}{\partial t^{\eta_\varepsilon}}(0_\varepsilon^+;0_\delta)\right]\left(\prod_{j\in\delta}z_j^{-k_j}\right). \tag{2.14ii}$$

Proof. We can combine the proofs of Theorems 2.15 and 2.17.

□

Definition 2.19. For $j\in\langle m\rangle$, the *j-first difference* (*$(j,1)$-difference*) of $f\in O_{n,m}$ is the function

$$\Delta_jf(t;k)=\begin{cases}0 & \text{if}\quad t_i<0\quad\text{or}\quad k_j<0\quad\text{for some}\quad i\in\langle m\rangle\ \text{or}\ j\in\langle m\rangle\\ f(t;k+e_j)-f(t;k) & \text{otherwise}\end{cases}$$

where $e_j=(\underbrace{0,\dots,0}_{j-1},1,0,\dots,0\in\mathbf{Z}^m$.

For $j\in\langle m\rangle$ and $\gamma\in\mathbf{N}$, $\gamma\ge 2$, the *(j,γ)-difference* of f is defined by induction by $\Delta_j^\gamma f(t;k)=\Delta_j(\Delta_j^{\gamma-1}f(t;k))$.

For $\beta=\{j_1,\dots,j_q\}\subset\langle m\rangle$ and $\gamma=\{\gamma_1,\dots,\gamma_q\}\in(\mathbf{N}^*)^q$ the *$(\beta,1)$-difference* and the *(β,γ)-difference* of f are the functions $\Delta_\beta f(t;k)=\Delta_{j_1}\cdots\Delta_{j_q}f(t;k)$ and $\Delta_\beta^\gamma f(t;k)=\Delta_{j_1}^{\gamma_1}\cdots\Delta_{j_q}^{\gamma_q}f(t;k)$.

We can prove by induction the following result, where C_n^m represents combinations :

Proposition 2.20. *If* $\beta=\{j_1,\dots,j_q\}\subset\langle m\rangle$ *and* $\gamma=\{\gamma_1,\dots,\gamma_q\}\in(\mathbf{N}^*)^q$, *then*

$$\Delta_\beta^\gamma f(t;k) = \sum_{l_1=0}^{\gamma_1} \dots \sum_{l_q=0}^{\gamma_q} (-1)^{\sum_{j=1}^{q}(\gamma_j - l_j)} \left(\prod_{j=1}^{q} C_{\gamma_j}^{l_j} \right) f(t;k+l). \tag{2.15}$$

We denote by $(z-1)^\beta$ the product $\prod_{j\in\beta} (z_j - 1)$.

Theorem 2.21 (Differentiation and difference of the original). *For any* $j \in \langle m \rangle$, $\beta = \{j_1, \dots, j_q\} \subset \langle m \rangle$ *and* $\gamma = \{\gamma_{i_1}, \dots, \gamma_{i_p}\} \in \mathbf{N}^p$, $p \in \langle n \rangle$ *we have*

$$\mathcal{L}_{n,m}[\Delta_j f(t;k)] = (z_j - 1)F(s;z) - \mathcal{L}_{n,m}^{\langle n \rangle, \bar{j}}[f(t;0_j)] \tag{2.16i}$$

$$\mathcal{L}_{n,m}[\Delta_\beta f(t;k)] = (z-1)^\beta F(s;z) + \sum_{\substack{\delta\in E_\beta \\ \delta \neq \varnothing}} (-1)^{|\delta|} z^\delta (z-1)^{\beta-\delta} \mathcal{L}_{n,m}^{\langle n \rangle, \bar{\delta}}[f(t;0_j)] \tag{2.16ii}$$

$$\begin{aligned} \mathcal{L}_{n,m}\left[\frac{\partial^\gamma}{\partial t^\gamma} \Delta_\beta f(t;k)\right] &= s^\gamma (z-1) F(s;z) + \sum_{\varepsilon\in E_\gamma} \sum_{\delta\in E_\beta} (-1)^{|\varepsilon|+|\delta|} \cdot \\ &\cdot s_{\bar{\varepsilon}}^{\gamma_{\bar{\varepsilon}}} z^\delta (z-1)^{\beta-\delta} \sum_{\eta_\varepsilon \le \gamma_\varepsilon - 1} s_\varepsilon^{\gamma_\varepsilon - \eta_\varepsilon - 1} \mathcal{L}_{n,m}^{\bar{\varepsilon},\bar{\delta}}\left[\frac{\partial^{\eta_\varepsilon} f}{\partial t^{\eta_\varepsilon}}(0_\varepsilon^+;0_\delta)\right]. \end{aligned} \tag{2.16iii}$$

Proof. By linearity and by Theorem 2.15 we get

$$\mathcal{L}_{n,m}[\Delta_j f(t;k)] = \mathcal{L}_{n,m}[f(t;k+e_j)] - \mathcal{L}_{n,m}[f(t;k)] = z_j[F(s;z) - \mathcal{L}_{n,m}^{\langle n \rangle, \bar{j}}[f(t;0_j)] - F(s;z)$$

which gives (2.16i).

We shall prove (2.16ii) for $\beta = (j_1, j_2)$ and the general case can be obtained similarly. Again by linearity and Theorem 2.15

$$\begin{aligned} &\mathcal{L}_{n,m}[\Delta_{j_1}\Delta_{j_2} f(t;k)] = \mathcal{L}_{n,m}[f(t;k+e_{j_1}+e_{j_2})] - \mathcal{L}_{n,m}[f(t;k+e_{j_1})] - \\ &- \mathcal{L}_{n,m}[f(t;k+e_{j_2})] + \mathcal{L}_{n,m}[f(t;k)] = z_{j_1} z_{j_2} (F(s;z) - \mathcal{L}_{n,m}^{\langle n \rangle, j_1}[f(t;0_{j_2})] - \\ &- \mathcal{L}_{n,m}^{\langle n \rangle, j_2}[f(t;0_{j_1})] + \mathcal{L}_{n,m}^{\langle n \rangle, \varnothing}[f(t;0_\beta)] - z_{j_2}(F(s;z) - \mathcal{L}_{n,m}^{\langle n \rangle, j_1}[f(t;0_{j_2})]) - \\ &- z_{j_1}(F(s;z) - \mathcal{L}_{n,m}^{\langle n \rangle, j_2}[f(t;0_{j_1})]) + F(s;z) = \\ &= (z_{j_1} - 1)(z_{j_2} - 1)F(s;z) - z_{j_2}(z_{j_1} - 1)\mathcal{L}_{n,m}^{\langle n \rangle, j_1}[f(t;0_{j_2})] - \\ &- z_{j_1}(z_{j_2} - 1)\mathcal{L}_{n,m}^{\langle n \rangle, j_2}[f(t;0_{j_1})] + z_{j_1} z_{j_2} \mathcal{L}_{n,m}^{\langle n \rangle, \varnothing}[f(t;0_\beta)]. \end{aligned}$$

Formula (2.16iii) can be obtained by combining (2.16ii) and (2.12ii).

□

Theorem 2.22 (Differentiation of the image). *For any* $\gamma=\{\gamma_{i_1},\ldots,\gamma_{i_p}\}\in(\mathbf{N}^*)^p$, $\alpha=\{i_1,\ldots,i_p\}\subset\langle n\rangle$ and $b=(b_{j_1},\ldots,b_{j_q})\in(N^*)^q$, $\beta=\{j_1,\ldots,j_q\}\subset\langle m\rangle$,

$$\mathcal{L}_{n,m}\left[(-1)^{[\gamma]+[b]}\left(\prod_{i\in\alpha}t_i^{\gamma_i}\right)\left(\prod_{j\in\beta}k_j(k_j+1)\cdots(k_j+b_j-1)\right)f(t;k)\right]= \\ =\left(\prod_{j\in\beta}z_j^{b_j}\right)\frac{\partial^{\gamma_{i_1}+\cdots+\gamma_{i_p}+b_{j_1}+\cdots+b_{j_q}}}{\partial s_{i_1}^{\gamma_{i_1}}\cdots\partial s_{i_p}^{\gamma_{i_p}}\partial z_{j_1}^{b_{j_1}}\cdots\partial z_{j_q}^{b_{j_q}}}F(s;z) \tag{2.17}$$

where $[\gamma]=\sum_{i\in I}\gamma_i$ and $[b]=\sum_{j\in J}b_j$.

Proof. We derive the image $F(s;z)$ given by (2.3) as in (2.17) and we obtain that the right hand member of (2.17) equals

$$\prod_{j\in\beta}z_j^{b_j}\int_0^\infty\cdots\int_0^\infty\sum_{k_1=0}^\infty\cdots\sum_{k_m=0}^\infty f(t;k)\left(\prod_{i\in\alpha}(-t_i)^{\gamma_i}\right)\exp\left(-\sum_{i=1}^n s_i t_i\right)\cdot \\ \cdot\left[\prod_{j\in\beta}(-k_j)(-b_j-1)\cdots(-k_j-b_j+1)z_j^{-k_j-b_j}\right]dt_1\ldots dt_n$$

which is the left hand member of (2.17).

□

3. Conclusion

In this paper a complete theory of a multiple (n,m)-Hybrid Laplace transformation has been developed. Other properties, such as integration and sum of the original, integration of the image, convolution, product of originals, initial and final values, inversion formulas and applications of this transformation will be presented in two subsequent papers, including differential-difference and integral equations, as well as the frequency-domain representation of multidimensional hybrid control systems.

REFERENCES

[1] *M. Dymkov, F. Gaishun, K. Galkovski, E. Rogers, D.H. Owens*, Exponential stability of discrete linear repetitive processes, Int. J. Control 75, 12(2002), 861-869.

[2] *K. Galkovski, E. Rogers and D.H. Owens*, New 2D models and a transition matrix for discrete linear repetitive processes, Int. J. Control, 72, 15 (1999), 1365-1380.

[3] K. Galkovski, State-space Realizations of Linear 2-D Systems with Extensions to the General nD $(n > 2)$ Case, Lecture Notes in Control and Information Sciences, 263, Springer Verlag London, 2001.
[4] *T. Kaczorek*, Controllability and minimum energy control of 2D continuous-discrete linear systems, Appl. Math. and Comp. Sci., 5, 1 (1995), 5-21.
[5] *T. Kaczorek*, Singular two-dimensional continuous-discrete linear systems, Dynamics of Continuous, Discrete and Impulsive Systems, 2 (1996), 193-204.
[6] J. *Kurek and M.B. Zaremba*, Iterative learning control synthesis on 2D system theory, IEEE Trans. Aut. Control, AC-38, 1 (1993), 121-125.
[7] *V. Prepeliță*, Criteria of reachability for 2D continuous-discrete separable systems, Rev. Roumaine Math. Pures Appl., 48, 1 (2003), 81-93.
[8] *V. Prepeliță*, Generalized Ho-Kalman algorithm for 2D continuous-discrete linear systems, Analysis and Optimization of Differential Systems, V. Barbu, I. Lasiecka, D. Tiba, C. Varsan eds., Kluwer Academic Publishers, Boston/Dordrecht/ London (2003), 321-332.
[9] *V. Prepeliță*, 2D Continuous-Discrete Laplace Transformation and Applications to 2D Systems, Rev. Roum. Math. Pures Appl., 49 (2004), 4, 355-376.
[10] *E. Rogers, D.H. Owens*, Stability Analysis for Linear Repetitive Processes, Lecture Notes in Control and Information Sciences, 175, Ed. Thoma H, Wyner W., Springer Verlag Berlin, 1999.