ITERATIVE ALGORITHMS FOR FIXED POINT PROBLEMS OF ASYMPTOTICALLY PSEUDOCONTRACTIVE OPERATORS AND THE PROXIMAL SPLIT FEASIBILITY PROBLEMS

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In this paper, we investigate iterative algorithms for solving fixed point problems and the proximal split feasibility problems. With the help of fixed point techniques, we suggest an iterative algorithm for finding an intersection of fixed point problem of an L-Lipschitz asymptotically pseudocontractive operator and the proximal split feasibility problem. Under some mild assumptions, we show that the proposed algorithm has strong convergence.

Keywords: Fixed point, asymptotically pseudocontractive operator, proximal split feasibility problem, proximal operator.

MSC2020: 47H06, 47H09, 49J05, 47J25.

1. Introduction

Let H_1 be a real Hilbert space. For a set $C \subset H_1$, the indicator function of C is defined by

$$\delta_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{if } x \notin C. \end{cases}$$

Given a function $f : H_1 \to \mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$, the domain of f is $dom(f) := \{u \in H_1 : f(u) < +\infty\}$. We say that f is proper if $dom(f) \neq \emptyset$. The class of all proper, convex, and lower semicontinuous functions in H_1 is denoted by $\Gamma_0(H_1)$.

Let H_2 be another real Hilbert space. Let $\psi : H_2 \to \mathbb{R}_{+\infty}$ be a function in $\Gamma_0(H_2)$. For any $\zeta > 0$, the Moreau envelope of ψ of index ζ is the function $\text{env}_\psi^\zeta : H_2 \to \mathbb{R}$ defined by

$$\text{env}_\psi^\zeta(x) = \min_{y \in H_2} \Big\{\psi_(y) + \frac{1}{2\zeta}\|x - y\|^2\Big\}, x \in H_2. \tag{1}$$

The Moreau envelope introduction by Moreau ([13]) (also called Moreau regularization) is ubiquitous in optimization, convex analysis, and variational analysis ([10, 11, 12, 36, 44]). It appears as a natural way to regularize a convex function through an associated optimization problem ([1, 7, 8, 16]).

Note that the minimizer of (1) is attained at a unique point which is used to define the proximal operator:

$$prox_{\zeta\psi}(x) = \arg\min_{y \in H_2} \Big\{\psi(y) + \frac{1}{2\zeta}\|x - y\|^2\Big\},\ x \in H_2. \tag{2}$$

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Note that $prox_{\zeta\psi}$ is characterized by the relation

$$\begin{aligned} y = prox_{\zeta\psi}(x) &\Leftrightarrow 0 \in \partial\Big(\psi(y) + \frac{1}{2\zeta}\|x-y\|^2\Big) = \partial\psi(y) + \frac{1}{\zeta}(y-x) \\ &\Leftrightarrow x \in (I+\zeta\partial)y \\ &\Leftrightarrow y = (I+\zeta\partial\psi)^{-1}x. \end{aligned}$$

Namely,

$$prox_{\zeta\psi}(x) = (I+\zeta\partial\psi)^{-1}x,$$

where $\partial\psi(x)$ is the subdifferential of ψ at x defined by

$$\partial\psi(x) = \{x^* \in H_2 : \psi(u^\dagger) \geq \psi(x) + \langle x^*, u^\dagger - x\rangle, \forall u^\dagger \in H_2\}. \tag{3}$$

It is known that $prox_{\zeta\psi}(x)$ is everywhere defined and firmly nonexpansive and the Moreau envelope $\mathrm{env}_{\psi}^{\zeta}$ is convex and continuously differentiable ([18]). For each $x \in H_2$, the gradient of $\mathrm{env}_{\psi}^{\zeta}$ is given by

$$\nabla\mathrm{env}_{\psi}^{\zeta}(x) = \frac{x - prox_{\zeta\psi}(x)}{\zeta},$$

which is called the Yosida approximation ([41]) of index ζ of the maximal monotone operator $\partial\psi$.

Let $\varphi : H_1 \to \mathbb{R}_\infty$ be a function in $\Gamma_0(H_1)$. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^*. Recall that the proximal split feasibility problem is to solve the following minimization problem

$$\min_{x^\dagger \in H_1} \{\varphi(x^\dagger) + \mathrm{env}_{\psi}^{\zeta}(Ax^\dagger)\}. \tag{4}$$

Use Γ to denote the solution set of (4).

Let $C \subset \mathcal{H}_1$ and $Q \in \mathcal{H}_2$ be two nonempty closed convex sets. Let $proj_Q : H_2 \to Q$ be the orthogonal projection. Taking $\varphi = \delta_C$ and $\psi = \delta_Q$, the proximal split feasibility problem (4) reduces to solve

$$\min_{x^\dagger \in C} \Big\{\frac{1}{2\zeta}\|(I - proj_Q)(Ax^\dagger)\|^2\Big\}, \tag{5}$$

which is equivalent to the following split feasibility problem ([2, 5, 28, 34]) of finding $x^\dagger$ such that

$$x^\dagger \in C \quad \text{and} \quad Ax^\dagger \in Q.$$

Thus, the proximal split feasibility problem (1) includes the split feasibility problem as a special case.

It is well known that the split feasibility problem can be a model for numerous inverse problems where constraints are imposed on the solutions in the domain of a bounded linear operator as well as in its range. The prototype of the split feasibility problem proposed by Censor and Elfving [5] came out of phase retrieval problems and the intensity-modulated radiation therapy. Now, the split feasibility problem has a large number of specific applications in real world such as medical care, image reconstruction and signal processing, see [2, 5, 9, 27, 30, 31, 32] for more details. Since then, the split problems have been studied extensively by many authors, see, for instance, [33, 39, 40, 45].

Fundamental insights into the proximal split feasibility problem come from the study of its Moreau-Yosida regularization and the associated proximal operator. The latter is a fundamental tool in optimization and it was shown that a fixed point iteration on the proximal operator could be used to develop a simple optimization algorithm, namely, the proximal point algorithm ([17, 18]).

Since ψ is subdifferentiable, we have

$$0 \in \partial\psi(x^\dagger) \Leftrightarrow x^\dagger = prox_{\zeta\psi}(x^\dagger). \tag{6}$$

By using (6), we can translate the proximal split feasibility problem (4) into a fixed point problem. As a matter of fact, noting that the Moreau envelope env_ψ^ζ is differentiable, we get

$$\partial\big(\text{env}_\psi^\zeta(Ax^\dagger)\big) = A^*\nabla\text{env}_\psi^\zeta(Ax^\dagger) = A^*\Big(\frac{I - prox_{\zeta\psi}}{\zeta}\Big)(Ax^\dagger).$$

So,

$$\partial\big(\varphi(x^\dagger) + \text{env}_\psi^\zeta(Ax^\dagger)\big) = \partial\varphi(x^\dagger) + A^*\Big(\frac{I - prox_{\zeta\psi}}{\zeta}\Big)(Ax^\dagger). \tag{7}$$

Note that the optimality condition of (4) is $0 \in \partial\big(\varphi(x^\dagger) + \text{env}_\psi^\zeta(Ax^\dagger)\big)$, i.e.,

$$0 \in \zeta\partial\varphi(x^\dagger) + A^*(I - prox_{\zeta\psi})(Ax^\dagger). \tag{8}$$

Based on (6) and (8), we deduce

$$x^\dagger \text{ solves } (4) \Leftrightarrow x^\dagger = prox_{\zeta\varphi}\big(x^\dagger - \varsigma A^*(I - prox_{\zeta\psi})(Ax^\dagger)\big). \tag{9}$$

With the help of the equivalent relations (6) and (9), several iterative algorithms for solving the proximal split feasibility problem (4) have been proposed, see [14, 15, 22].

In the meantime, we focus on iterative approximation of fixed point problems ([20, 21, 29, 42, 48]). It is well known that fixed point theory acts as an important tool for many branches of mathematical analysis and its applications. Especially, iterative algorithms by using fixed point techniques come to be useful in numerous mathematical formulations and theorems ([3, 4, 19, 24, 43]). Often, approximations and solutions to iterative guess strategies utilized in dynamic engineering problems are sought using this method. Recently, fixed point algorithms have attracted so much attention, see [6, 23, 25, 35, 37, 38, 46].

The main purpose of this paper is to investigate iterative algorithms for solving fixed point problems and the proximal split feasibility problem (4). We suggest an iterative algorithm for finding an intersection of fixed point problem of an L-Lipschitz asymptotically pseudocontractive operator and the proximal split feasibility problem (4). We show that the proposed algorithm converges strongly to a common point of the investigated problems.

2. **Preliminaries**

Throughout this paper, H_1 and H_2 are two real Hilbert space endowed with the standard inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$. Weak and strong convergence are denoted by $\rightharpoonup$ and $\to$, respectively. Let $\{z_n\}$ be a given sequence in H_1. We use $\omega_w(z_n)$ to denote the set of all weak cluster points of $\{z_n\}$, i.e.,

$$\omega_w(z_n) = \{z^\ddagger : \exists\{z_{n_k}\} \subset \{z_n\} \text{ such that } z_{n_k} \rightharpoonup z^\ddagger \text{ as } k \to \infty\}.$$

For any $z, z^\dagger \in H_1$ and constant $c \in \mathbb{R}$, there hold

$$\|cz + (1-c)z^\dagger\|^2 = c\|z\|^2 + (1-c)\|z^\dagger\|^2 - c(1-c)\|z - z^\dagger\|^2, \tag{10}$$

and

$$\|z + z^\dagger\|^2 \le \|z\|^2 + 2\langle z^\dagger, z + z^\dagger\rangle. \tag{11}$$

Let $T : H_1 \to H_1$ be an operator. Use $Fix(T)$ to denote the fixed point set of T. Recall that T is said to be

(i) asymptotically pseudocontractive if there exists a real number sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\langle T^n z - T^n z^\dagger, z - z^\dagger\rangle \le k_n\|z - z^\dagger\|^2, \tag{12}$$

for all $n \ge 1$ and for all $z, z^\dagger \in H_1$.

(ii) uniformly L-Lipschitzian if there exists a positive constant L such that

$$\|T^n z - T^n z^\dagger\| \le L\|z - z^\dagger\|,$$

for all $n \ge 1$ and for all $z, z^\dagger \in H_1$.

(iii) demiclosed, if for any given sequence $\{x_n\} \subset H_1$, we have

$$\left.\begin{array}{r}\lambda_n \rightharpoonup \tilde{u} \in H_1 \\ T\lambda_n \to u^\dagger\end{array}\right\} \Rightarrow T\tilde{u} = u^\dagger.$$

(iv) firmly nonexpansive if

$$\|Tz - Tz^\dagger\|^2 \le \langle Tz - Tz^\dagger, z - z^\dagger\rangle, \forall z, z^\dagger \in H_1.$$

It is obviously that $I - T$ is also firmly nonexpansive.

Remark 2.1. *(i) It is easily seen that* (12) *is equivalent to*

$$\|T^n z - T^n z^\dagger\|^2 \le (2k_n - 1)\|z - z^\dagger\|^2 + \|(I - T^n)z - (I - T^n)z^\dagger\|^2. \tag{13}$$

(ii) The proximal operators $prox_{\zeta\psi}$ and $prox_{\zeta\varphi}$ are firmly nonexpansive and $I - prox_{\zeta\psi}$ and $I - prox_{\zeta\varphi}$ are also firmly nonexpansive.

Let C be a nonempty closed convex subset of H_1. For any $x \in H_1$, there exists a unique nearest point $proj_C(x)$ in C satisfying

$$\|x - proj_C(x)\| \le \|x - y\|, \forall y \in C.$$

It is well known that $proj_C$ is firmly nonexpansive and has the following characterization

$$\langle x - proj_C(x), y - proj_C(x)\rangle \le 0 \tag{14}$$

for all $x \in H_1$ and $y \in C$.

Lemma 2.1 ([47]). *Let H_1 be a real Hilbert space. Let $T\colon H_1 \to H_1$ be a uniformly L-Lipschtzian and asymptotically pseudocontractive operator. Then, $I - T$ is demiclosed at zero.*

Lemma 2.2 ([26]). *Let $\{\sigma_n\} \subset \mathbb{R}^+$, $\{\alpha_n\} \subset (0, 1)$ and $\{\tau_n\} \subset \mathbb{R}$ be three real number sequences. Suppose that*

(i) $\sigma_{n+1} \le (1 - \alpha_n)\sigma_n + \tau_n, \forall n \ge 0$;

(ii) $\sum_{n=1}^\infty \alpha_n = \infty$;

(iii) $\limsup_{n\to\infty} \frac{\tau_n}{\alpha_n} \le 0$ *or* $\sum_{n=1}^\infty |\tau_n| < \infty$.

Then, $\lim_{n\to\infty} \sigma_n = 0$.

3. Main results

In this section, we present our main results.

Let H_1 and H_2 be two real Hilbert spaces. Let $\varphi : H_1 \to \mathbb{R}_\infty$ be a function in $\Gamma_0(H_1)$ and $\psi : H_2 \to \mathbb{R}_\infty$ be a function in $\Gamma_0(H_2)$. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^*. Let $T : H_1 \to H_1$ be an L-Lipschitz asymptotically pseudocontractive operator with $L > 1$ and $\{k_n\}$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\eta_n\}$ be three real number sequences in $(0, 1)$. Let $\{\varsigma_n\}$ be a real number sequence in $(0, +\infty)$.

Next, we first introduce an algorithm for solving fixed point problem of asymptotically pseudocontractive operator T and the proximal split feasibility problem (4).

Algorithm 3.1. *Let $u \in H_1$ be a fixed point. Let $x_0 \in H_1$ be an initial guess. Let $n = 0$.*

Step 1. For given x_n, compute

$$y_n = (1 - \beta_n)x_n + \beta_n T^n[(1 - \eta_n)x_n + \eta_n T^n x_n]. \tag{15}$$

Step 2. Compute

$$u_n = A^*(I - prox_{\zeta\psi})Ay_n + (I - prox_{\zeta\varphi})y_n. \tag{16}$$

Criterion: If $u_n = 0$, then set $z_n = y_n$ and go to Step 3. Otherwise, compute

$$z_n = y_n - \frac{\varsigma_n(\lambda_n + \theta_n)}{\|u_n\|^2} u_n, \tag{17}$$

where $\lambda_n = \frac{1}{2}\|(I - prox_{\zeta\psi})Ay_n\|^2$ and $\theta_n = \frac{1}{2}\|(I - prox_{\zeta\varphi})y_n\|^2$.

Step 3. Compute

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) z_n. \tag{18}$$

Step 4. Set $n := n + 1$ and return to Step 1.

Remark 3.1. *If $u_n = 0$, then $y_n \in Fix(prox_{\zeta\varphi})$ and $Ay_n \in Fix(prox_{\zeta\psi})$, i.e., $y_n \in \Gamma$, see* [13].

Theorem 3.1. *Suppose that $Fix(T) \cap \Gamma \neq \emptyset$. Suppose that the following conditions are satisfied*

(C1): $0 < c_1 < \beta_n < c_2 < \eta_n < c_3 < \frac{1}{\sqrt{k_n^2 + L^2} + k_n} (\forall n \geq 0)$;

(C2): $\sum_{n=1}^{\infty}(k_n - 1) < +\infty$ *and* $\lim_{n\to\infty} \frac{k_n - 1}{\alpha_n} = 0$;

(C3): $0 < b_1 < \varsigma_n < b_2 < 4 (\forall n \geq 0)$;

(C4): $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = +\infty$.

Then the sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to $proj_{Fix(T)\cap\Gamma}(u)$.

Proof. Let $x^* \in Fix(T) \cap \Gamma$. Hence, $x^* = prox_{\zeta\varphi}(x^*)$, $Ax^* = prox_{\zeta\psi}(Ax^*)$ and $x^* = Tx^* = T^n x^* (\forall n \geq 1)$. Since T is asymptotically pseudocontractive, we have from (13) that

$$\begin{aligned} \|T^n[(1-\eta_n)x_n + \eta_n T^n x_n] - x^*\|^2 &\leq (2k_n - 1)\|(1-\eta_n)(x_n - x^*) + \eta_n(T^n x_n - x^*)\|^2 \\ &\quad + \|(1-\eta_n)x_n + \eta_n T^n x_n - T^n[(1-\eta_n)x_n + \eta_n T^n x_n]\|^2, \end{aligned} \tag{19}$$

and

$$\|T^n x_n - x^*\|^2 \leq (2k_n - 1)\|x_n - x^*\|^2 + \|T^n x_n - x_n\|^2. \tag{20}$$

Noting that T is uniformly L-Lipschitzian, we have

$$\|T^n x_n - T^n[(1-\eta_n)x_n + \eta_n T^n x_n]\| \leq \eta_n L \|x_n - T^n x_n\|. \tag{21}$$

Using (10) and (20), we have

$$\begin{aligned} &\|(1-\eta_n)(x_n - x^*) + \eta_n(T^n x_n - x^*)\|^2 \\ &\quad = (1-\eta_n)\|x_n - x^*\|^2 + \eta_n\|T^n x_n - x^*\|^2 - \eta_n(1-\eta_n)\|x_n - T^n x_n\|^2 \\ &\quad \leq (1-\eta_n)\|x_n - x^*\|^2 + \eta_n((2k_n - 1)\|x_n - x^*\|^2 + \|T^n x_n - x_n\|^2) \\ &\qquad - \eta_n(1-\eta_n)\|x_n - T^n x_n\|^2 \\ &\quad = [1 + 2(k_n - 1)\eta_n]\|x_n - x^*\|^2 + \eta_n^2\|T^n x_n - x_n\|^2. \end{aligned} \tag{22}$$

In view of (10) and (21), we get

$$\begin{aligned} &\|(1-\eta_n)x_n + \eta_n T^n x_n - T^n[(1-\eta_n)x_n + \eta_n T^n x_n]\|^2 \\ &\quad = \|(1-\eta_n)\big(x_n - T^n[(1-\eta_n)x_n + \eta T^n x_n]\big) \\ &\qquad + \eta_n\big(T^n x_n - T^n[(1-\eta_n)x_n + \eta_n T^n x_n]\big)\|^2 \\ &\quad = (1-\eta_n)\|x_n - T^n[(1-\eta)x_n + \eta_n T^n x_n]\|^2 \\ &\qquad + \eta_n\|T^n x_n - T^n[(1-\eta)x_n + \eta_n T^n x_n]\|^2 \\ &\qquad - \eta_n(1-\eta_n)\|x_n - T^n x_n\|^2 \\ &\quad \leq (1-\eta_n)\|x_n - T^n[(1-\eta_n)x_n + \eta_n T^n x_n]\|^2 \\ &\qquad - \eta_n(1 - \eta_n - L^2\eta_n^2)\|x_n - T^n x_n\|^2. \end{aligned} \tag{23}$$

By (19), (22) and (23), we obtain

$$\begin{aligned}
&\|T^n[(1-\eta_n)x_n+\eta_nT^nx_n]-x^*\|^2 \\
&\qquad \le (2k_n-1)[1+2(k_n-1)\eta_n]\|x_n-x^*\|^2+(2k_n-1)\eta_n^2\|x_n-T^nx_n\|^2 \\
&\qquad\quad +(1-\eta_n)\|x_n-T^n[(1-\eta)x_n+\eta_nT^nx_n]\|^2 \\
&\qquad\quad -\eta_n(1-\eta_n-L^2\eta_n^2)\|x_n-T^nx_n\|^2 \\
&\qquad = (2k_n-1)[1+2(k_n-1)\eta_n]\|x_n-x^*\|^2 \\
&\qquad\quad +(1-\eta_n)\|x_n-T^n[(1-\eta_n)x_n+\eta_nT^nx_n]\|^2 \\
&\qquad\quad -\eta_n(1-2k_n\eta_n-L^2\eta_n^2)\|x_n-T^nx_n\|^2.
\end{aligned} \tag{24}$$

Since $\eta_n < \frac{1}{\sqrt{k_n^2+L^2+k_n}}$, we deduce that $1-2k_n\eta_n-\eta_n^2L^2>0$. According to (24), we obtain

$$\begin{aligned}
\|T^n[(1-\eta)x_n+\eta_nT^nx_n]-x^*\|^2 &\le (2k_n-1)[1+2(k_n-1)\eta_n]\|x_n-x^*\|^2 \\
&\quad +(1-\eta_n)\|x_n-T^n[(1-\eta_n)x_n+\eta_nT^nx_n]\|^2.
\end{aligned} \tag{25}$$

Combine (10) and (25) to get

$$\begin{aligned}
\|y_n-x^*\|^2 &= \|(1-\beta_n)x_n+\beta_nT^n[(1-\eta_n)x_n+\eta_nT^nx_n]-x^*\|^2 \\
&= \|(1-\beta_n)(x_n-x^*)+\beta_n\big(T^n[(1-\eta_n)x_n+\eta_nT^nx_n]-x^*\big)\|^2 \\
&= (1-\beta_n)\|x_n-x^*\|^2+\beta_n\|T^n[(1-\eta_n)x_n+\eta_nT^nx_n]-x^*\|^2 \\
&\quad -\beta_n(1-\beta_n)\|T^n[(1-\eta_n)x_n+\eta_nT^nx_n]-x_n\|^2 \\
&\le \beta(2k_n-1)[1+2(k_n-1)\eta_n]\|x_n-x^*\|^2+(1-\beta_n)\|x_n-x^*\|^2 \\
&\quad +\beta_n(1-\eta_n)\|x_n-T^n[(1-\eta_n)x_n+\eta_nT^nx_n]\|^2 \\
&\quad -\beta_n(1-\beta_n)\|T^n[(1-\eta_n)x_n+\eta_nT^nx_n]-x_n\|^2 \\
&= [1+2(k_n-1)\beta_n+2(k_n-1)(2k_n-1)\eta_n\beta_n]\|x_n-x^*\|^2 \\
&\quad +\beta_n(\beta_n-\eta_n)\|T^n[(1-\eta_n)x_n+\eta_nT^nx_n]-x_n\|^2.
\end{aligned}$$

Note that $\lim_{n\to\infty}\frac{k_n-1}{\alpha_n}=0$. We assume, without loss of generality, that $\frac{k_n-1}{\alpha_n}\le 1$ for all $n\ge 0$. This together with $0<\beta_n<\eta_n<1$ implies that

$$\begin{aligned}
\|y_n-x^*\|^2 &\le [1+8(k_n-1)]\|x_n-x^*\|^2 \\
&\quad +\beta_n(\beta_n-\eta_n)\|T^n[(1-\eta_n)x_n+\eta_nT^nx_n]-x_n\|^2 \\
&\le [1+8(k_n-1)]\|x_n-x^*\|^2.
\end{aligned} \tag{26}$$

By (17), we have

$$\begin{aligned}
\|z_n-x^*\|^2 &= \|y_n-x^*-\frac{\varsigma_n(\lambda_n+\theta_n)}{\|u_n\|^2}u_n\| \\
&= \|y_n-x^*\|^2-2\frac{\varsigma_n(\lambda_n+\theta_n)}{\|u_n\|^2}\langle u_n, y_n-x^*\rangle+\frac{\varsigma_n^2(\lambda_n+\theta_n)^2}{\|u_n\|^2}.
\end{aligned} \tag{27}$$

Since $I-prox_{\zeta\psi}$ and $I-prox_{\zeta\varphi}$ are firmly-nonexpansive, we have

$$\begin{aligned}
\langle (I-prox_{\zeta\varphi})y_n, y_n-x^*\rangle &= \langle (I-prox_{\zeta\varphi})y_n-(I-prox_{\zeta\varphi})x^*, y_n-x^*\rangle \\
&\ge \|(I-prox_{\zeta\varphi})y_n\|^2,
\end{aligned} \tag{28}$$

and

$$\begin{aligned}
\langle (I-prox_{\zeta\psi})Ay_n, Ay_n-Ax^*\rangle &= \langle (I-prox_{\zeta\psi})Ay_n-(I-prox_{\zeta\psi})Ax^*, Ay_n-Ax^*\rangle \\
&\ge \|(I-prox_{\zeta\psi})Ay_n\|^2.
\end{aligned} \tag{29}$$

From (16), we have

$$\begin{aligned}\langle u_n, y_n - x^*\rangle &= \langle A^*(I - prox_{\zeta\psi})Ay_n + (I - prox_{\zeta\varphi})y_n, y_n - x^*\rangle \\ &= \langle A^*(I - prox_{\zeta\psi})Ay_n, y_n - x^*\rangle + \langle (I - prox_{\zeta\varphi})y_n, y_n - x^*\rangle \\ &= \langle (I - prox_{\zeta\psi})Ay_n, Ay_n - Ax^*\rangle + \langle (I - prox_{\zeta\varphi})y_n, y_n - x^*\rangle.\end{aligned} \tag{30}$$

It follows from (28)-(30) that

$$\|(I - prox_{\zeta\varphi})y_n\|^2 + \|(I - prox_{\zeta\psi})Ay_n\|^2 \le \langle u_n, y_n - x^*\rangle,$$

which implies that

$$2(\lambda_n + \theta_n) \le \langle u_n, y_n - x^*\rangle. \tag{31}$$

Combining (27) and (31) to get

$$\begin{aligned}\|z_n - x^*\|^2 &\le \|y_n - x^*\|^2 - \frac{4\varsigma_n(\lambda_n + \theta_n)^2}{\|u_n\|^2} + \frac{\varsigma_n^2(\lambda_n + \theta_n)^2}{\|u_n\|^2} \\ &= \|y_n - x^*\|^2 - \varsigma_n(4 - \varsigma_n)\frac{(\lambda_n + \theta_n)^2}{\|u_n\|^2} \\ &\le \|y_n - x^*\|^2.\end{aligned} \tag{32}$$

Thus, from (18), (26) and (32), we obtain

$$\begin{aligned}\|x_{n+1} - x^*\| &= \|\alpha_n(u - x^*) + (1 - \alpha_n)(z_n - x^*)\| \\ &\le \alpha_n\|u - x^*\| + (1 - \alpha_n)\|y_n - x^*\| \\ &\le \alpha_n\|u - x^*\| + (1 - \alpha_n)[1 + 4(k_n - 1)]\|x_n - x^*\| \\ &\le [1 + 4(k_n - 1)]\max\{\|u - x^*\|, \|x_n - x^*\|\} \\ &\le \prod_{i=1}^{n}[1 + 4(k_i - 1)]\max\{\|x_0 - x^*\|, \|u - x^*\|\}.\end{aligned}$$

This implies that the sequence $\{x_n\}$ is bounded because of $\sum_{n=1}^{\infty}(k_n - 1) < +\infty$. Subsequently, $\{y_n\}$, $\{z_n\}$, $\{Ay_n\}$ and $\{u_n\}$ are all bounded.

According to (26) and (32), we have

$$\begin{aligned}\|z_n - x^*\|^2 \le [1 + 8(k_n - 1)]\|x_n - x^*\|^2 - \varsigma_n(4 - \varsigma_n)\frac{(\lambda_n + \theta_n)^2}{\|u_n\|^2} \\ + \beta_n(\beta_n - \eta_n)\|T^n[(1 - \eta_n)x_n + \eta_n T^n x_n] - x_n\|^2.\end{aligned} \tag{33}$$

From (11) and (18), we have

$$\begin{aligned}\|x_{n+1} - x^*\| &= \|\alpha_n(u - x^*) + (1 - \alpha_n)(z_n - x^*)\|^2 \\ &\le (1 - \alpha_n)\|z_n - x^*\|^2 + 2\alpha_n\langle u - x^*, x_{n+1} - x^*\rangle\end{aligned} \tag{34}$$

On account of (33) and (34), we obtain

$$
\begin{aligned}
\|x_{n+1}-x^*\|^2 \le{}& (1-\alpha_n)[1+8(k_n-1)]\|x_n-x^*\|^2\\
&+(1-\alpha_n)\beta_n(\beta_n-\eta_n)\|T^n[(1-\eta_n)x_n+\eta_nT^nx_n]-x_n\|^2\\
&-(1-\alpha_n)\varsigma_n(4-\varsigma_n)\frac{(\lambda_n+\theta_n)^2}{\|u_n\|^2}+2\alpha_n\langle u-x^*,x_{n+1}-x^*\rangle\\
\le{}& (1-\alpha_n)\|x_n-x^*\|^2+\alpha_n\Big\{\frac{M(k_n-1)}{\alpha_n}+(1-\alpha_n)\beta_n(\beta_n-\eta_n)\\
&\times\frac{\|T^n[(1-\eta_n)x_n+\eta_nT^nx_n]-x_n\|^2}{\alpha_n}\\
&-(1-\alpha_n)\varsigma_n(4-\varsigma_n)\frac{(\lambda_n+\theta_n)^2}{\|u_n\|^2\alpha_n}+2\langle u-x^*,x_{n+1}-x^*\rangle\Big\},
\end{aligned}
\tag{35}
$$

where M is a constant such that $M\ge\sup_n 8(1-\alpha_n)\|x_n-x^*\|^2$.

Set $\sigma_n=\|x_n-x^*\|^2$ and

$$
\begin{aligned}
\tau_n={}&\frac{M(k_n-1)}{\alpha_n}+(1-\alpha_n)\beta_n(\beta_n-\eta_n)\frac{\|T^n[(1-\eta_n)x_n+\eta_nT^nx_n]-x_n\|^2}{\alpha_n}\\
&-(1-\alpha_n)\varsigma_n(4-\varsigma_n)\frac{(\lambda_n+\theta_n)^2}{\|u_n\|^2\alpha_n}+2\langle u-x^*,x_{n+1}-x^*\rangle,
\end{aligned}
\tag{36}
$$

for all $n\ge1$.

By virtue of (35) and (36), we obtain

$$\sigma_{n+1}\le(1-\alpha_n)\sigma_n+\alpha_n\tau_n, n\ge1. \tag{37}$$

Taking into account (36), we get

$$\tau_n\le M+2\|u-x^*\|\|x_{n+1}-x^*\|.$$

By the boundedness of $\{x_n\}$ and the last inequality, we deduce that $\limsup_{n\to\infty}\tau_n<+\infty$. Next we prove $\limsup_{n\to\infty}\tau_n\ge-1$. Assume that $\limsup_{n\to\infty}\tau_n<-1$. There exists a positive integer N_0 such that $\tau_n\le-1$ when $n\ge N_0$. Based on (37), we get

$$\sigma_{n+1}\le\sigma_n-\alpha_n, \forall n\ge N_0.$$

It results in that

$$\sigma_{n+1}\le\sigma_{N_0}-\sum_{k=N_0}^{n}\alpha_k. \tag{38}$$

Taking the superior limit in (38), we have

$$\limsup_{n\to\infty}\sigma_{n+1}\le\sigma_{N_0}-\lim_{n\to\infty}\sum_{k=N_0}^{n}\alpha_k=-\infty,$$

which yields a contradiction. Then,

$$-1\le\limsup_{n\to\infty}\tau_n<+\infty.$$

Hence, $\limsup_{n\to\infty} \tau_n$ exists. Meanwhile, noting that $\{x_{n+1}\}$ is bounded, so there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $x_{n_i+1} \rightharpoonup z^\dagger (i \to \infty)$ and

$$\begin{aligned}\limsup_{n\to\infty} \tau_n &= \lim_{i\to\infty} \tau_{n_i} \\ &= \lim_{i\to\infty} \Bigg[\frac{M(k_{n_i}-1)}{\alpha_{n_i}} + (1-\alpha_{n_i})\beta_{n_i}(\beta_{n_i}-\eta_{n_i}) \frac{\|T^{n_i}[(1-\eta_{n_i})x_{n_i} + \eta_{n_i}T^{n_i}x_{n_i}] - x_{n_i}\|^2}{\alpha_{n_i}} \\ &\quad - (1-\alpha_{n_i})\varsigma_{n_i}(4-\varsigma_{n_i}) \frac{(\lambda_{n_i}+\theta_{n_i})^2}{\|u_{n_i}\|^2 \alpha_{n_i}} + 2\langle u - x^*, x_{n_i+1} - x^* \rangle \Bigg] \\ &= \lim_{i\to\infty} \Bigg[\beta_{n_i}(\beta_{n_i}-\eta_{n_i}) \frac{\|T^{n_i}[(1-\eta_{n_i})x_{n_i} + \eta_{n_i}T^{n_i}x_{n_i}] - x_{n_i}\|^2}{\alpha_{n_i}} \\ &\quad - \varsigma_{n_i}(4-\varsigma_{n_i}) \frac{(\lambda_{n_i}+\theta_{n_i})^2}{\|u_{n_i}\|^2 \alpha_{n_i}} + 2\langle u - x^*, z^\dagger - x^* \rangle \Bigg],\end{aligned} \tag{39}$$

which implies that

$$\lim_{i\to\infty} \beta_{n_i}(\beta_{n_i}-\eta_{n_i}) \frac{\|T^{n_i}[(1-\eta_{n_i})x_{n_i} + \eta_{n_i}T^{n_i}x_{n_i}] - x_{n_i}\|^2}{\alpha_{n_i}} \text{ exists} \tag{40}$$

and

$$\lim_{i\to\infty} \frac{(\lambda_{n_i}+\theta_{n_i})^2}{\|u_{n_i}\|^2 \alpha_{n_i}} \text{ exists.} \tag{41}$$

By conditions (C1) and (C3), from (40) and (41), we have

$$\lim_{i\to\infty} \|T^{n_i}[(1-\eta_{n_i})x_{n_i} + \eta_{n_i}T^{n_i}x_{n_i}] - x_{n_i}\| = 0 \tag{42}$$

and

$$\lim_{i\to\infty} \frac{\lambda_{n_i}+\theta_{n_i}}{\|u_{n_i}\|} = 0. \tag{43}$$

Since u_{n_i} is bounded, by (43), we obtain $\lim_{i\to\infty}(\lambda_{n_i}+\theta_{n_i}) = 0$. Therefore,

$$\lim_{i\to\infty} \|(I - prox_{\zeta\psi})Ay_{n_i}\| = \lim_{i\to\infty} \|(I - prox_{\zeta\varphi})y_{n_i}\| = 0. \tag{44}$$

According to (15) and (42), we deduce

$$\lim_{i\to\infty} \|y_{n_i} - x_{n_i}\| = 0. \tag{45}$$

By (17) and (43), we deduce

$$\lim_{i\to\infty} \|y_{n_i} - z_{n_i}\| = 0. \tag{46}$$

It follows from (18), (45) and (46) that

$$\lim_{i\to\infty} \|x_{n_i+1} - x_{n_i}\| = 0, \tag{47}$$

which together with $x_{n_i+1} \rightharpoonup z^\dagger (i \to \infty)$ imply that $x_{n_i} \rightharpoonup z^\dagger (i \to \infty)$ and $y_{n_i} \rightharpoonup z^\dagger (i \to \infty)$. The weak lower semicontinuity of the norm gives

$$0 \le \|(I - prox_{\zeta\varphi})z^\dagger\| \le \liminf_{i\to\infty} \|(I - prox_{\zeta\varphi})y_{n_i}\| = 0,$$

and

$$0 \le \|(I - prox_{\zeta\psi})Az^\dagger\| \le \liminf_{i\to\infty} \|(I - prox_{\zeta\psi})Ay_{n_i}\| = 0.$$

Thus, we conclude that $z^\dagger \in Fix(prox_{\zeta\varphi})$ and $Az^\dagger \in Fix(prox_{\zeta\psi})$, i.e., $z^\dagger \in \Gamma$.

Since T is uniformly L-Lipschitzian, we derive

$$\begin{aligned}\|T^{n_i}x_{n_i} - x_{n_i}\| &\le \|T^{n_i}x_{n_i} - T^{n_i}[(1-\eta_{n_i})x_{n_i} + \eta_{n_i}T^{n_i}x_{n_i}]\| \\ &\quad + \|T^{n_i}[(1-\eta_{n_i})x_{n_i} + \eta_{n_i}T^{n_i}x_{n_i}] - x_{n_i}\| \\ &\le L\eta_{n_i}\|T^{n_i}x_{n_i} - x_{n_i}\| + \|T^{n_i}[(1-\eta_{n_i})x_{n_i} + \eta_{n_i}T^{n_i}x_{n_i}] - x_{n_i}\|\end{aligned}$$

which leads to

$$\|T^{n_i}x_{n_i} - x_{n_i}\| \leq \frac{1}{1 - L\eta_{n_i}} \|T^{n_i}[(1-\eta_{n_i})x_{n_i} + \eta_{n_i}T^{n_i}x_{n_i}] - x_{n_i}\|.$$

This together with (42) implies

$$\lim_{i\to\infty} \|T^{n_i}x_{n_i} - x_{n_i}\| = 0. \tag{48}$$

Again, by using uniformly L-Lipschitzian continuity of T, we have

$$\begin{aligned} \|x_{n_i+1} - Tx_{n_i+1}\| &\leq \|x_{n_i+1} - T^{n_i+1}x_{n_i+1}\| + \|T^{n_i+1}x_{n_i+1} - T^{n_i+1}x_{n_i}\| \\ &\quad + \|T^{n_i+1}x_{n_i} - Tx_{n_i+1}\| \\ &\leq \|x_{n_i+1} - T^{n_i+1}x_{n_i+1}\| + L\|x_{n_i+1} - x_{n_i}\| + L\|T^{n_i}x_{n_i} - x_{n_i+1}\| \\ &\leq \|x_{n_i+1} - T^{n_i+1}x_{n_i+1}\| + 2L\|x_{n_i+1} - x_{n_i}\| + L\|T^{n_i}x_{n_i} - x_{n_i}\|. \end{aligned} \tag{49}$$

By (47)-(49), we have immediately that

$$\lim_{i\to\infty} \|x_{n_i} - Tx_{n_i}\| = 0. \tag{50}$$

By Lemma 2.1 and (50), we deduce that $z^\dagger \in Fix(T)$. Therefore, $z^\dagger \in Fix(T) \cap \Gamma$. So, $\omega_w(x_n) \subset Fix(T) \cap \Gamma$.

With the help of (39), we have

$$\limsup_{n\to\infty} \tau_n = \lim_{i\to\infty} \tau_{n_i} \leq 2\langle u - proj_{Fix(T)\cap\Gamma}(u), z^\dagger - proj_{Fix(T)\cap\Gamma}(u)\rangle \leq 0. \tag{51}$$

From (35), we obtain

$$\begin{aligned} \|x_{n+1} - proj_{Fix(T)\cap\Gamma}(u)\|^2 &\leq (1-\alpha_n)\|x_n - proj_{Fix(T)\cap\Gamma}(u)\|^2 \\ + \alpha_n &\left\{ \frac{M(k_n - 1)}{\alpha_n} + 2\langle u - proj_{Fix(T)\cap\Gamma}(u), x_{n+1} - proj_{Fix(T)\cap\Gamma}(u)\rangle \right\}. \end{aligned} \tag{52}$$

By Lemma 2.2, (51) and (52), we conclude that $x_n \to proj_{Fix(T)\cap\Gamma}(u)$. This completes the proof. □

Algorithm 3.2. *Let $u \in H_1$ be a fixed point. Let $x_0 \in H_1$ be an initial guess. Let $n = 0$.*

Step 1. For given x_n, compute

$$u_n = A^*(I - prox_{\zeta\psi})Ax_n + (I - prox_{\zeta\varphi})x_n.$$

Criterion: If $u_n = 0$, then set $z_n = x_n$ and go to Step 2. Otherwise, compute

$$z_n = x_n - \frac{\varsigma_n(\lambda_n + \theta_n)}{\|u_n\|^2} u_n,$$

where $\lambda_n = \frac{1}{2}\|(I - prox_{\zeta\psi})Ax_n\|^2$ and $\theta_n = \frac{1}{2}\|(I - prox_{\zeta\varphi})x_n\|^2$.

Step 2. Compute

$$x_{n+1} = \alpha_n u + (1-\alpha_n)z_n.$$

Step 3. Set $n := n+1$ and return to Step 1.

Corollary 3.1. *Suppose that $\Gamma \neq \emptyset$. Suppose that the following conditions are satisfied*
(C3): $0 < b_1 < \varsigma_n < b_2 < 4 (\forall n \geq 0)$;
(C4): $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = +\infty$.
Then the sequence $\{x_n\}$ generated by Algorithm 3.2 converges strongly to $proj_\Gamma(u)$.

4. **Concluding remarks**

An iterative algorithm (Algorithm 3.1) has been introduced for finding an intersection of fixed point problem of an L-Lipschitz asymptotically pseudocontractive operator T and the proximal split feasibility problem (4). The basic iteration used in this paper is to apply the Moreau regularization method and the fixed point method with a self-adaptive technique. The strong convergence of the iterates has been obtained under some mild conditions.

Acknowledgments

Zhangsong Yao was partially supported by the Grant 19KJD100003. Ching-Feng Wen was supported by the Grant of MOST 109-2115-M-037-001.

REFERENCES

[1] F. Bernard and L. Thibault, *Prox-regular functions in Hilbert spaces*, J. Math. Anal. Appl., **303** (2005), 1–14.
[2] C. Byrne, *A unified treatment of some iterative algorithms in signal processing and image reconstruction*, Inverse Probl., **20** (2004), 103–120.
[3] L.C. Ceng, A. Petrusel, J.C. Yao and Y. Yao, *Hybrid viscosity extragradient method for systems of variational inequalities, fixed Points of nonexpansive mappings, zero points of accretive operators in Banach spaces*, Fixed Point Theory, **19(2)** (2018), 487–502.
[4] L.C. Ceng, A. Petrusel, J.C. Yao and Y. Yao, *Systems of variational inequalities with hierarchical variational inequality constraints for Lipschitzian pseudocontractions*, Fixed Point Theory, **20(1)** (2019), 113–133.
[5] Y. Censor and T. Elfving, *A multiprojection algorithm using Bregman projections in a product space*, Numer. Algori., **8** (1994), 221–239.
[6] S.S. Chang, J.C. Yao, C.F. Wen and L.C. Zhao, *On the split equality fixed point problem of quasi-pseudo-contractive mappings without a priori knowledge of operator norms with applications*, J. Optim. Theory Appl., **185** (2020), 343–360.
[7] C. Chaux, P.L. Combettes, J.C. Pesquet and V.R. Wajs, *A variational formulation for frame-based inverse problems*, Inverse Probl., **23** (2007), 1495–1518.
[8] Z.F.H. Clarke, Y.S. Ledyaev, R.J. Stern and P.R. Wolenski, Nonsmooth Analysis and Control Theory, Springer, New York, 1998.
[9] Q.L. Dong, Y. Peng and Y. Yao, *Alternated inertial projection methods for the split equality problem*, J. Nonlinear Convex Anal., **22** (2021), 53–67.
[10] A. Hantoute, M.A. Lopez and C. Zalinescu, *Subdifferential calculus rules in convex analysis: a unifying approach via pointwise supremum functions*, SIAM J. Optim., **19** (2008), 863–882.
[11] A. Jourani, L. Thibault and D. Zagrodny, *Differential properties of the Moreau envelope*, J. Funct. Anal., **266** (2014), 1185–1237.
[12] A. Jourani and E. Vilches, *Moreau-Yosida regularization of state-dependent sweeping processes with nonregular sets*, J. Optim. Theory Appl., **173** (2017), 91–116.
[13] J.J. Moreau, *Proximite et dualite dans un espace hilbertien*, Bull. Soc. Math. Fr., **93** (1965), 273–299.
[14] A. Moudafi, *A proximal iterative approach to a nonconvex optimization problem*, Nonlinear Anal., **72** (2010), 704–709.
[15] A. Moudafi and B.S. Thakur, *Solving proximal split feasibility problems without prior knowledge of operator norms*, Optim. Letts., **8** (2014), 2099–2110.
[16] R.A. Poliquin and R.T. Rockafellar, *Prox-regular functions in variational analysis*, Trans. Am. Math. Soc., **348** (1996), 1805–1838.
[17] R.T. Rockafellar, *Monotone operators and the proximal point algorithm*, SIAM J. Control Optim., **14** (1976),877–898.
[18] R.T. Rockafellar and R. Wets, Variational Analysis, Springer, Berlin, 1988.
[19] D.R. Sahu, A. Pitea and M. Verma, *A new iteration technique for nonlinear operators as concerns convex programming and feasibility problems*, Numer. Algor., **83** (2020), 421–449.
[20] D.R. Sahu, H.K. Xu and J.C. Yao, *Asymptotically strict pseudocontractive mappings in the intermediate sense*, Nonlinear Anal., **70(10)** (2009), 3502–3511.
[21] J. Schu, *Iterative construction of fixed points of asymptotically nonexpansive mappings*, J. Math. Anal. Appl., **158** (1991), 407–413.
[22] Y. Shehu and O.S. Iyiola, *Strong convergence result for proximal split feasibility problem in Hilbert spaces*, Optim., **66** (2017), 2275–2290.

[23] W. Takahashi, C.F. Wen and J.C. Yao, *Iterative methods for the split common fixed point problem with families of demimetric mappings in Banach spaces*, J. Nonlinear Convex Anal., **19(1)** (2018), 1–18.

[24] G.I. Usurelu, *Split feasibility handled by a single-projection three-step iteration with comparative analysis*, J. Nonlinear Convex Anal., **22** (2021), 544–558.

[25] X. Wu, Y.C. Liou, Z. Wu and P.X. Yang, *Viscosity methods of asymptotically pseudocontractive and asymptotically nonexpansive mappings for variational inequalities*, Abstr. Appl. Anal., **2012** (2012), Article ID 453452, 14 pages.

[26] H.K. Xu, *Iterative algorithms for nonlinear operators*, J. London Math. Soc., **66** (2002), 240–256.

[27] H.K. Xu, *Iterative methods for the split feasibility problem in infinite-dimensional Hilbert spaces*, Inverse Probl., **26** (2010), Article Number 105018.

[28] Y. Yao, R.P. Agarwal, M. Postolache and Y.C. Liou, *Algorithms with strong convergence for the split common solution of the feasibility problem and fixed point problem*, Fixed Point Theory Appl., **2014** (2014), Article Number 183.

[29] Y. Yao, L. Leng, M. Postolache and X. Zheng, *Mann-type iteration method for solving the split common fixed point problem*, J. Nonlinear Convex Anal., **18** (2017), 875–882.

[30] Y. Yao, H. Li and M. Postolache, *Iterative algorithms for split equilibrium problems of monotone operators and fixed point problems of pseudo-contractions*, Optim., DOI: 10.1080/02331934.2020.1857757.

[31] Y. Yao, Y.C. Liou and M. Postolache, *Self-adaptive algorithms for the split problem of the demicontractive operators*, Optim., **67** (2018), 1309–1319.

[32] Y. Yao, Y.C. Liou and J.C. Yao, *Split common fixed point problem for two quasi-pseudocontractive operators and its algorithm construction*, Fixed Point Theory Appl., **2015** (2015), Article Number 127.

[33] Y. Yao, Y.C. Liou and J.C. Yao, *Iterative algorithms for the split variational inequality and fixed point problems under nonlinear transformations*, J. Nonlinear Sci. Appl., **10** (2017), 843–854.

[34] Y. Yao, M. Postolache and Y.C. Liou, *Strong convergence of a self-adaptive method for the split feasibility problem*, Fixed Point Theory Appl., **2013** (2013), Article Number 201.

[35] Y. Yao, M. Postolache and J.C. Yao, *An iterative algorithm for solving the generalized variational inequalities and fixed points problems*, Math., **7** (2019), Article Number 61.

[36] Y. Yao, M. Postolache and J.C. Yao, *Iterative algorithms for generalized variational inequalities*, U.P.B. Sci. Bull., Series A, **81** (2019), 3–16.

[37] Y. Yao, M. Postolache and J.C. Yao, *Strong convergence of an extragradient algorithm for variational inequality and fixed point problems*, U.P.B. Sci. Bull., Series A, **82(1)** (2020),3–12.

[38] Y. Yao, X. Qin and J.C. Yao, *Projection methods for firmly type nonexpansive operators*, J. Nonlinear Convex Anal., **19** (2018), 407–415.

[39] Y. Yao, M. Postolache and Z. Zhu, *Gradient methods with selection technique for the multiple-sets split feasibility problem*, Optim., **69** (2020), 269–281.

[40] Y. Yao, J.C. Yao, Y.C. Liou and M. Postolache, *Iterative algorithms for split common fixed points of demicontractive operators without priori knowledge of operator norms*, Carpathian J. Math., **34** (2018), 459–466.

[41] K. Yosida, Functional Analysis, Springer, Berlin, 1964.

[42] H. Zegeye, M. Robdera and B. Choudhary, *Convergence theorems for asymptotically pseudocontractive mappings in the intermediate sense*, Comput. Math. Appl., **62** (2011), 326–332.

[43] H. Zegeye, N. Shahzad and Y. Yao, *Minimum-norm solution of variational inequality and fixed point problem in Banach spaces*, Optim., **64** (2015), 453–471.

[44] X. Zhao, M.A. Kobis, Y. Yao and J.C. Yao, *A projected subgradient method for nondifferentiable quasiconvex multiobjective optimization problems*, J. Optim. Theory Appl., **190** (2021), 82–107.

[45] X.P. Zhao, J.C. Yao and Y. Yao, *A proximal algorithm for solving split monotone variational inclusions*, U.P.B. Sci. Bull., Series A, **82(3)** (2020), 43–52.

[46] X.P. Zhao and Y. Yao, *Modified extragradient algorithms for solving monotone variational inequalities and fixed point problems*, Optim., **69(9)** (2020), 1987–2002.

[47] H. Zhou, *Demiclosedness principle with applications for asymptotically pseudo-contractions in Hilbert spaces*, Nonlinear Anal., **70** (2009), 3140–3145.

[48] L.J. Zhu, Y. Yao and M. Postolache, *Projection methods with linesearch technique for pseudomonotone equilibrium problems and fixed point problems*, U.P.B. Sci. Bull., Series A, **83** (2021), 3–14.