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LIGHTLIKE HYPERSURFACES OF AN INDEFINITE NEARLY KAEHLER MANIFOLD

Chul Woo Lee[1], Dae Ho Jin[2], Jae Won Lee[3]

We study the geometry of lightlike hypersurfaces M of an indefinite nearly Kaehler manifold $\bar{M}$ and an indefinite nearly complex space form $\bar{M}(c)$. The purpose of this paper is to prove several characterization theorems for such lightlike hypersurfacs M which have recurrent, nearly recurrent or Lie recurrent induced structure tensor fields F.

Keywords: recurrent, nearly recurrent, Lie recurrent, lightlike hypersurface, indefinite nearly Kaehler manifold

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1. Introduction

A hypersurfaces M of an almost complex manifold $(\bar{M}, J, \bar{g})$ has an almost contact metric structure (F, u, U) induced from the almost complex structure J of $\bar{M}$, where F is a tensor field of type (1, 1), U is a vector field and u is a 1-form associated with U. Then there exist three types of hypersurfaces of an almost complex manifold. First, the structure vector field U is called *principal* if $AU = \alpha U$, where A is the shape operator of M and α is a smooth function on M. A real hypersurface of an almost complex manifold $\bar{M}$ is said to be a *Hopf hypersurface* if its structure vector field U is principal. Next, the structure tensor field F is called *recurrent* (resp. *Lie recurrent*) if there exists a 1-form ω (resp. a 1-form θ) on M such that

$$(\nabla_X F)Y = \omega(X)FY \qquad (\text{resp. } (\mathcal{L}_X F)Y = \theta(X)FY\,),$$

for any vector fields X and Y on M, where the symbols ∇ and $\mathcal{L}$ denote the covariant and Lie derivative on M, respectively. A real hypersurface is said to be a *recurrent* (resp. *Lie recurrent*) *hypersurface* if its structure tensor field F is recurrent (resp. Lie recurrent) ([1], [7]∼[10]).

The theory of lightlike hypersurfaces is an important topic of research in differential geometry and mathematical physics. The study of such notion was initiated by Duggal-Bejancu [2] and later studied by many authors [3, 4]. The objective of this paper is to study on the differential geometry of lightlike hypersurfaces M of an indefinite nearly Kaehler manifold $\bar{M}$ and an indefinite nearly complex space form $\bar{M}(c)$. The main results are characterization theorems for such lightlike hypersufaces M which have recurrent, nearly recurrent or Lie recurrent induced structure tensor fields F.

[1]Researcher, Kyungpook National University, 41566 Daegu, Republic of Korea, e-mail: mathisu@knu.ac.kr

[2]Professor, Dongguk University, 30866 Kyongju, Republic of Korea, e-mail:jindh@dongguk.ac.kr

[3]Professor, Gyeongsang National University and RINS, 52828 Jinju, Republic of Korea, e-mail:leejaew@gnu.ac.kr

2. **Lightlike hypersurfaces**

An even dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called an *indefinite almost Hermitian manifold* if there exists a set $(J, \bar{g})$, where J is a $(1,1)$-type tensor field and $\bar{g}$ is the semi-Riemannian metric such that

$$J^2 = -I, \qquad \bar{g}(J\bar{X}, J\bar{Y}) = \bar{g}(\bar{X}, \bar{Y}). \tag{1}$$

In this paper, we denote by $\bar{X}$, $\bar{Y}$ and $\bar{Z}$ the smooth vector fields on $\bar{M}$.

A hypersurface M of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called a *lightlike hypersurface* if the normal bundle $TM^{\perp}$ of M is a subbundle of the tangent bundle TM of M, and coincides with the radical distribution $Rad(TM)$. Then there exists a non-degenerate complementary vector bundle $S(TM)$ of $TM^{\perp}$ in TM, called a *screen distribution* of M, such that

$$TM = TM^{\perp} \oplus_{orth} S(TM),$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over M. Also, denote by $(1)_i$ the i-th equation of (1). We use same notations for any others. It is well-known [2] that, for any null section ξ of $TM^{\perp}$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique null section N of a unique lightlike vector bundle $tr(TM)$ of rank 1 in the orthogonal complement $S(TM)^{\perp}$ of $S(TM)$ in $\bar{M}$ satisfying

$$\bar{g}(\xi, N) = 1, \quad \bar{g}(N, N) = \bar{g}(N, X) = 0, \ \forall\, X \in \Gamma(S(TM)).$$

In this case, the tangent bundle $T\bar{M}$ of $\bar{M}$ is decomposed as follows:

$$T\bar{M} = TM \oplus tr(TM) = \{TM^{\perp} \oplus tr(TM)\} \oplus_{orth} S(TM).$$

We call $tr(TM)$ and N the *transversal vector bundle* and the *null transversal vector field* with respect to the screen distribution $S(TM)$, respectively.

From now and in the sequel, we denote by X, Y and Z the smooth vector fields on M, unless otherwise specified. Let $\bar{\nabla}$ be the Levi-Civita connection of $\bar{M}$ and P the projection morphism of TM on $S(TM)$. Then the local Gauss and Weingarten formulae of M and $S(TM)$ are given by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \tag{2}$$
$$\bar{\nabla}_X N = -A_N X + \tau(X)N; \tag{3}$$
$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi, \tag{4}$$
$$\nabla_X \xi = -A^*_{\xi} X - \tau(X)\xi, \tag{5}$$

respectively, where ∇ and ∇^* are the liner connections on TM and $S(TM)$, respectively, B and C are the local second fundamental forms on TM and $S(TM)$, respectively, A_N and A^*_{ξ} are the shape operators and τ is a 1-form.

The induced connection ∇ on M is not metric and satisfies

$$(\nabla_X g)(Y, Z) = B(X, Y)\,\eta(Z) + B(X, Z)\,\eta(Y), \tag{6}$$

where η is a 1-form such that $\eta(X) = \bar{g}(X, N)$. But the connection ∇^* on $S(TM)$ is metric. Since $\bar{\nabla}$ is torsion-free, ∇ is also torsion-free and B is symmetric. From the fact that $B(X, Y) = \bar{g}(\bar{\nabla}_X Y, \xi)$, we show that B is independent of the choice of the screen distribution $S(TM)$ and satisfies

$$B(X, \xi) = 0. \tag{7}$$

The above two local second fundamental forms B and C of M and $S(TM)$ respectively are related to their shape operators by

$$B(X,Y) = g(A^*_\xi X, Y), \qquad \bar{g}(A^*_\xi X, N) = 0, \tag{8}$$

$$C(X,PY) = g(A_N X, PY), \quad \bar{g}(A_N X, N) = 0. \tag{9}$$

From (8), the operator A^*_ξ is $S(TM)$-valued self-adjoint and $A^*_\xi \xi = 0$.

3. Indefinite almost Hermitian manifolds

It is well known [2, 5] that, for any lightlike hypersurface M of an indefinite almost Hermitian manifold $\bar{M}$, $J(TM^\perp)$ and $J(tr(TM))$ are subbundles of $S(TM)$, of rank 1, and $J(TM^\perp) \cap J(tr(TM)) = \{0\}$. Consequently $J(TM^\perp) \oplus J(tr(TM))$ is a subbundle of $S(TM)$ of rank 2. Thus there exist two non-degenerate almost complex distributions D_o and D with respect to the structure tensor J, *i.e.*, $J(D_o) = D_o$ and $J(D) = D$, such that

$$S(TM) = \{J(TM^\perp) \oplus J(tr(TM))\} \oplus_{orth} D_o,$$
$$D = TM^\perp \oplus_{orth} J(TM^\perp) \oplus_{orth} D_o.$$

Hence, the decomposition form of TM is reformed as follow:

$$TM = D \oplus J(tr(TM)).$$

Consider two lightlike vector fields U and V and their 1-forms such that

$$U = -JN, \quad V = -J\xi, \quad u(X) = g(X,V), \quad v(X) = g(X,U). \tag{10}$$

Denote by S the projection morphism of TM on D. Any vector field X of M is expressed as $X = SX + u(X)U$. Applying J to this form, we have

$$JX = FX + u(X)N, \tag{11}$$

where F is a tensor field of type $(1,1)$ globally defined on M by $FX = JSX$. Applying J to (11) and using $(1)_1$ and $(10)_1$, we have

$$F^2 X = -X + u(X)U. \tag{12}$$

Applying ∇_X to $F\xi = -V$ and $FV = \xi$ by turns and using (5), we get

$$(\nabla_X F)\xi = -\nabla_X V + F(A^*_\xi X) - \tau(X)V, \tag{13}$$

$$(\nabla_X F)V = -F\nabla_X V - A^*_\xi X - \tau(X)\xi. \tag{14}$$

Definition 3.1. *The structure tensor field F of M is said to be recurrent* [6] *if there exists a* 1*-form ω on M such that*

$$(\nabla_X F)Y = \omega(X)FY. \tag{15}$$

Theorem 3.1. *Let M be a lightlike hypersurface of an indefinite almost Hermitian manifold $\bar{M}$. If the structure tensor field F is recurrent, then F is parallel with respect to the connection ∇ of M.*

Proof. Taking $Y = V$ to (15) and using the fact that $FV = \xi$, we get

$$(\nabla_X F)V = \omega(X)\xi.$$

Comparing this equation with (14), we obtain

$$F\nabla_X V + A^*_\xi X + \{\omega(X) + \tau(X)\}\xi = 0. \tag{16}$$

On the other hand, replacing Y by ξ to (15), we have

$$(\nabla_X F)\xi = -\omega(X)V.$$

Comparing this equation with (13), we obtain

$$\nabla_X V - F(A^*_\xi X) - \{\omega(X) - \tau(X)\}V = 0. \tag{17}$$

Taking the scalar product with V to (17), we have

$$u(\nabla_X V) = 0. \tag{18}$$

Applying F to (16) and using (12) and (18) and then, comparing this result with (17), we get $\omega = 0$. Thus F is parallel with respect to ∇. □

Definition 3.2. *The structure tensor field F of M is said to be Lie recurrent* [6] *if there exists a 1-form θ on M such that*

$$(\mathcal{L}_X F)Y = \sigma(X)FY, \tag{19}$$

where $\mathcal{L}_X$ denotes the Lie derivative on M with respect to X, that is,

$$(\mathcal{L}_X F)Y = [X, FY] - F[X, Y]. \tag{20}$$

The structure tensor field F is called Lie parallel if $\mathcal{L}_X F = 0$.

Theorem 3.2. *Let M be a lightlike hypersurface of an indefinite almost Hermitian manifold $\bar{M}$. If F is Lie recurrent, then F is Lie parallel.*

Proof. As ∇ is torsion-free, from (19) and (20) we have

$$(\nabla_X F)Y = \nabla_{FY} X - F\nabla_Y X + \sigma(X)FY. \tag{21}$$

Replacing Y by V to (21) and using the fact that $FV = \xi$, we have

$$(\nabla_X F)V = \nabla_\xi X - F\nabla_V X + \sigma(X)\xi.$$

Comparing this equation with (14), we obtain

$$\nabla_\xi X = -F(\nabla_X V - \nabla_V X) - A^*_\xi X - \{\sigma(X) + \tau(X)\}\xi. \tag{22}$$

On the other hand, replacing Y by ξ to (21), we have

$$(\nabla_X F)\xi = -\nabla_V X - F\nabla_\xi X - \sigma(X)V.$$

Comparing this equation with (13), we obtain

$$F\nabla_\xi X = \nabla_X V - \nabla_V X - F(A^*_\xi X) - \{\sigma(X) - \tau(X)\}V. \tag{23}$$

Taking the scalar product with V to (23), we obtain

$$u(\nabla_X V - \nabla_V X) = 0. \tag{24}$$

Applying F to (22) and using (12) and (24) and then, comparing this result with (23), we have $\sigma = 0$. Thus F is Lie parallel. □

4. **Indefinite nearly Kaehler manifolds**

Definition 4.1. *An indefinite almost Hermitian manifold $\bar{M}$ is called an indefinite nearly Kaehler manifold if the structure tensor field J of $\bar{M}$ satisfies*

$$(\bar{\nabla}_{\bar{X}}J)\bar{Y} + (\bar{\nabla}_{\bar{Y}}J)\bar{X} = 0. \tag{25}$$

By using (2), (3), (11) and (25), we obtain

$$(\nabla_X F)Y + (\nabla_Y F)X = u(X)A_N Y + u(Y)A_N X - 2B(X,Y)U. \tag{26}$$

Definition 4.2. *The structure tensor field F of M is said to be nearly recurrent if there exists a 1-form ω on M such that*

$$(\nabla_X F)Y + (\nabla_Y F)X = \omega(X)FY + \omega(Y)FX. \tag{27}$$

Theorem 4.1. *Let M be a lightlike hypersurface of an indefinite nearly Kaehler manifold $\bar{M}$ such that ζ is tangent to M. If F is nearly recurrent, then the 1-form ω given by (27) satisfies $\omega = 0$.*

Proof. If F is nearly recurrent, then, from (26) and (27), we obtain

$$\begin{aligned}&\omega(Y)FX + \omega(X)FY \\ &= u(X)A_N Y + u(Y)A_N X - 2B(X,Y)U.\end{aligned}$$

Taking the scalar product with N to this and using $(9)_2$, we have

$$\omega(Y)v(X) + \omega(X)v(Y) = 0.$$

Replacing Y by V to this, we get $\omega(X) = -\omega(V)v(X)$. Taking $X = V$ to this result, we have $\omega(V) = 0$. Thus $\omega(X) = 0$ for all $X \in \Gamma(TM)$. □

Applying ∇_X to $FU = 0$, we obtain

$$(\nabla_X F)U = -F\nabla_X U.$$

From this equation with $X = V$ and (14) with $X = U$, we see that

$$(\nabla_U F)V + (\nabla_V F)U = -F(\nabla_U V + \nabla_V U) - A^*_\xi U - \tau(U)\xi.$$

Comparing this result with (26) such that $X = U$ and $Y = V$, we have

$$F(\nabla_U V + \nabla_V U) + A^*_\xi U + \tau(U)\xi = -A_N V + 2B(U,V)U.$$

Taking the scalar product with V, U and N to (14) by turn, we obtain

$$\begin{gathered}B(U,V) = C(V,V), \qquad B(U,U) = C(U,V), \\ v(\nabla_U V + \nabla_V U) = -\tau(U).\end{gathered} \tag{28}$$

Theorem 4.2. *Let M be a lightlike hypersurface of an indefinite nearly Kaehler manifold $\bar{M}$. If one of the following conditions is satisfied,*

(1) $(\nabla_X F)Y + (\nabla_Y F)X = 0$,
(2) *F is parallel with respect to the connection ∇,*
(3) *F is recurrent,*
(4) *F is nearly recurrent,*

then the shape operators A^*_ξ and A_N satisfy

$$A^*_\xi V = 0, \quad A_N V = 0, \quad A_N \xi = 0, \quad A_N X = C(X,V)U. \tag{29}$$

Proof. (1) Assume that $(\nabla_X F)Y + (\nabla_Y F)X = 0$. Taking the scalar product with V to (26), we have

$$2B(X,Y) = u(Y)u(A_N X) + u(X)u(A_N Y). \tag{30}$$

Taking $Y = V$ to this equation and using $(9)_1$, we obtain

$$2B(X,V) = u(X)C(V,V).$$

Replacing X by U to this equation, we have $2B(U,V) = C(V,V)$. Comparing this result with $(28)_1$, we have $C(V,V) = 0$. Thus we obtain

$$C(V,V) = B(U,V) = 0, \qquad B(X,V) = 0. \tag{31}$$

From (8), $(31)_2$ and the facts that B is symmetric and $S(TM)$ is non-degenerate, we have $A^*_\xi V = 0$. Taking $X = U$ and $Y = V$ to (26) and using $(31)_1$, we get $A_N V = 0$. Also, taking $X = U$ and $Y = \xi$ to (26) and using (7), we have $A_N \xi = 0$. Taking $Y = U$ to (30), we obtain

$$2B(X,U) = u(A_N X) + u(X)u(A_N U).$$

Replacing Y by U to (26) and using the last equation, we get

$$A_N X - u(A_N X)U + u(X)\{A_N U - u(A_N U)U\} = 0.$$

Taking $X = U$ to this, we have $A_N U = u(A_N U)U$. Thus we have

$$A_N X = u(A_N X)U.$$

(2) If F is parallel with respect to the induced connection ∇ of M, then we have $(\nabla_X F)Y + (\nabla_Y F)X = 0$. Thus we have (29) by (1).

(3) If F is recurrent, then F is parallel with respect to the induced connection ∇ of M by Theorem 3.1. Thus we have (29) by (2).

(4) If F is nearly recurrent, then we have $(\nabla_X F)Y + (\nabla_Y F)X = 0$ by Theorem 4.1. Thus we have (29) by (1). □

5. Indefinite nearly complex space forms

Denote by $\bar{R}$, R and R^* the curvature tensors of the Levi-Civita connection $\bar{\nabla}$ of $\bar{M}$ and the induced connection ∇ and ∇^* on M and $S(TM)$, respectively. Using the Gauss-Weingarten formulae for M and $S(TM)$, we obtain the following two Gauss equations for M and $S(TM)$ such that

$$\begin{aligned}\bar{R}(X,Y)Z = R(X,Y)Z &+ B(X,Z)A_N Y - B(Y,Z)A_N X \\ &+ \{(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) \\ &+ \tau(X)B(Y,Z) - \tau(Y)B(X,Z)\}N,\end{aligned} \tag{32}$$

$$\begin{aligned}R(X,Y)PZ = R^*(X,Y)PZ &+ C(X,PZ)A^*_\xi Y - C(Y,PZ)A^*_\xi X \\ &+ \{(\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) \\ &- \tau(X)C(Y,PZ) + \tau(Y)C(X,PZ)\tau(Y)\}\xi.\end{aligned} \tag{33}$$

Definition 5.1. *Given an indefinite nearly Kaehler manifold $(\bar{M}, J, \bar{g})$ is called an indefinite nearly complex space form, denoted by $\bar{M}(c)$, if there exists a constant holomorphic sectional curvature c such that*

$$\begin{aligned}\bar{R}(\bar{X},\bar{Y})\bar{Z} = \frac{c}{4}\{&\bar{g}(\bar{Y},\bar{Z})\bar{X} - \bar{g}(\bar{X},\bar{Z})\bar{Y} + \bar{g}(\bar{X},J\bar{Z})J\bar{Y} \\ &- \bar{g}(\bar{Y},J\bar{Z})J\bar{X} + 2\bar{g}(\bar{X},J\bar{Y})J\bar{Z}\}.\end{aligned} \tag{34}$$

Taking the scalar product with ξ and N to (34) by turns and using $(9)_2$, (11), (32) and (33), we obtain

$$\begin{aligned}(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z) \\ = \frac{c}{4}\{\bar{g}(X,JZ)u(Y) - \bar{g}(Y,JZ)u(X) + 2\bar{g}(X,JY)u(Z)\},\end{aligned} \tag{35}$$

$$\begin{aligned}(\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) - \tau(X)C(Y,PZ) + \tau(Y)C(X,PZ) \\ = \frac{c}{4}\{g(Y,PZ)\eta(X) - g(X,PZ)\eta(Y) + \bar{g}(X,JPZ)v(Y) \\ - \bar{g}(Y,JPZ)v(X) + 2\bar{g}(X,JY)v(PZ)\}.\end{aligned} \tag{36}$$

Theorem 5.1. *Let M be a lightlike hypersurface of an indefinite nearly complex space form $\bar{M}(c)$. If one of the following conditions is satisfied,*

(1) $(\nabla_X F)Y + (\nabla_Y F)X = 0$,
(2) *F is parallel with respect to the connection ∇,*
(3) *F is recurrent,*
(4) *F is nearly recurrent,*

then $c = 0$ and $\bar{M}(c)$ is flat.

Proof. If one of the conditions (1) ∼ (4) is satisfied, then we have (29). Taking the scalar product with U to $(29)_4$ and using $(9)_1$, we have

$$C(X,U) = 0.$$

Applying ∇_X to $C(Y,U) = 0$ and using the last equation, we have

$$(\nabla_X C)(Y,U) = -C(Y,\nabla_X U).$$

Substituting the last two equations into (36) with $PZ = U$, we obtain

$$C(X,\nabla_Y U) - C(Y,\nabla_X U) = \frac{c}{2}\{v(Y)\eta(X) - v(X)\eta(Y)\}.$$

Taking $Y = V$ and $X = \xi$ to this equation and using $(9)_1$ and $(29)_{2,3}$, we obtain $c = 0$. Thus we have our theorem. □

Definition 5.2. *A lightlike hypersurface M is said to be a Hopf lightlike hypersurface if the structure vector field U is an eigenvector of A^*_ξ.*

Theorem 5.2. *Let M be a lightlike hypersurface of an indefinite nearly complex space form $\bar{M}(c)$ such that F is Lie recurrent. Then*

$$c = -1 \iff g(A^*_\xi U, A^*_\xi U) = 0.$$

If M is a Hopf lightlike hypersurface of $\bar{M}(c)$, then $c = -1$.

Proof. Replacing Y by U to (21), we have $(\nabla_X F)U = -F\nabla_U X$. Applying ∇_X to $FU = 0$, we get $(\nabla_X F)U = -F\nabla_X U$. Therefore we have

$$F(\nabla_X U - \nabla_U X) = 0. \tag{37}$$

Taking the scalar product with N to this equation, we obtain

$$v(\nabla_X U - \nabla_U X) = 0. \tag{38}$$

Applying ∇_X to $g(U,U) = 0$ and using (6), $(28)_3$ and (38), we get

$$v(\nabla_X U) = 0, \qquad v(\nabla_U X) = 0, \qquad \tau(U) = 0. \tag{39}$$

Taking $X = U$ to (22) and using (37) and $(39)_3$, we obtain

$$\nabla_\xi U = -A^*_\xi U. \tag{40}$$

Taking the scalar product with U to (40) and using $(39)_1$, we have

$$B(U, U) = 0. \tag{41}$$

Applying ∇_ξ to (41): $B(U, U) = 0$ and using (40), we see that

$$(\nabla_\xi B)(U, U) = 2g(A^*_\xi U, A^*_\xi U).$$

Also, applying ∇_U to $B(\xi, U) = 0$ and using (5) and (7), we see that

$$(\nabla_U B)(\xi, U) = g(A^*_\xi U, A^*_\xi U).$$

Taking $X = \xi$, $Y = U$ and $Z = U$ to (35) and using (7), $(39)_3$, (41) and the last two equations, we obtain

$$g(A^*_\xi U, A^*_\xi U) = \frac{3}{4}c.$$

Thus we see that $c = 0 \iff g(A^*_\xi U, A^*_\xi U) = 0$.

If M is a Hopf lightlike hypersurface of $\bar{M}(c)$, that is, $A^*_\xi U = \alpha U$ for some smooth function α, then $g(A^*_\xi U, A^*_\xi U) = 0$. Thus we have $c = 0$. □

Definition 5.3. *A lightlike hypersurface M is said to be screen conformal* [3] *if there exist a non-vanishing smooth function φ on any coordinate neighborhood $\mathcal{U}$ in M such that $A_N = \varphi A^*_\xi$, or equivalently,*

$$C(X, PY) = \varphi B(X, PY). \tag{42}$$

Theorem 5.3. *Let M be a screen conformal lightlike hypersurface of an indefinite nearly complex space form $\bar{M}(c)$. Then $c = 0$ and $\bar{M}(c)$ is flat.*

Proof. Let μ be the vector field on $S(TM)$ given by $\mu = U - \varphi V$. We get

$$g(V, \mu) = 1, \qquad g(U, \mu) = -\varphi, \qquad g(\mu, \mu) = -2\varphi. \tag{43}$$

And, from $(28)_{1,2}$ and (42), we obtain

$$B(V, \mu) = 0, \qquad\qquad B(U, \mu) = 0. \tag{44}$$

Applying ∇_X to $C(Y, PZ) = \varphi B(Y, PZ)$, we have

$$(\nabla_X C)(Y, PZ) = (X\varphi)B(Y, PZ) + \varphi(\nabla_X B)(Y, PZ).$$

Substituting this equation and (42) into (36), we have

$$\begin{aligned} &\{X\varphi - 2\varphi\tau(X)\}B(Y, PZ) - \{Y\varphi - 2\varphi\tau(Y)\}B(X, PZ) \\ &= \frac{c}{4}\{g(Y, PZ)\eta(X) - g(X, PZ)\eta(Y) + \bar{g}(X, JPZ)g(Y, \mu) \\ &\quad - \bar{g}(Y, JPZ)g(X, \mu) + 2\bar{g}(X, JY)g(PZ, \mu)\}. \end{aligned}$$

Taking $X = \xi$ to this equation and using (7), we have

$$\begin{aligned} &\{\xi\varphi - 2\varphi\tau(\xi)\}B(Y, PZ) \\ &\quad = \frac{c}{4}\{g(Y, PZ) + u(PZ)g(Y, \mu) + 2u(Y)g(PZ, \mu)\}. \end{aligned} \tag{45}$$

Taking $Y = V$ and $PZ = \mu$ to (45) and using (43) and $(44)_1$, we get $c = 0$. Thus we have our theorem. □

Definition 5.4. *A lightlike hypersurface M of $(\bar{M}, \bar{g})$ is said to be*

(1) *totally umbilical* [2] *if there is a smooth function* β *on any coordinate neighborhood* $\mathcal{U}$ *in* M *such that* $A^*_\xi X = \beta PX$*, or equivalently,*

$$B(X,Y) = \beta g(X,Y). \tag{46}$$

(2) *screen totally umbilical* [2] *if there exist a smooth function* γ *on* $\mathcal{U}$ *such that* $A_N X = \gamma PX$*, or equivalently,*

$$C(X,PY) = \gamma g(X,PY). \tag{47}$$

In case $\gamma = 0$ *on* $\mathcal{U}$*, we say that* M *is screen totally geodesic.*

Theorem 5.4. *Let* M *be a totally umbilical lightlike hypersurface of an indefinite nearly complex space form* $\bar{M}(c)$*. Then* $c = 0$ *and* β *satisfies*

$$\xi\beta + \beta\tau(\xi) - \beta^2 = 0.$$

Proof. Substituting (46) into (35) and using (6), we have

$$\begin{aligned}&\{X\beta + \beta\tau(X) - \beta^2\eta(X)\}g(Y,Z)\\ &\qquad - \{Y\beta + \beta\tau(Y) - \beta^2\eta(Y)\}g(X,Z)\\ =\ &\frac{c}{4}\{\bar{g}(X,JZ)u(Y) - \bar{g}(\bar{Y},JZ)u(X) + 2\bar{g}(X,JY)u(Z)\}.\end{aligned}$$

Replacing X by ξ to this equation, we have

$$\{\xi\beta + \beta\tau(\xi) - \beta^2\}g(Y,Z) = \frac{3}{4}c\,u(Y)u(Z).$$

Taking $Y = Z = U$ to this equation, we get $c = 0$. And, taking $Y = U$ and $Z = V$ to the last equation, we have $\xi\beta + \beta\tau(\xi) - \beta^2 = 0$. □

Theorem 5.5. *Let* M *be a screen totally umbilical lightlike hypersurface of an indefinite nearly complex space form* $\bar{M}(c)$*. Then* $c = 0$ *and* $\gamma = 0$*, i.e.,* M *is screen totally geodesic.*

Proof. From $(28)_{1,2}$ and (47), we have

$$B(U,V) = 0, \qquad B(U,U) = \gamma. \tag{48}$$

Substituting (47) into (36) and using (6), we have

$$\begin{aligned}&\{X\gamma - \gamma\tau(X)\}g(Y,PZ) - \{Y\gamma - \gamma\tau(Y)\}g(X,PZ)\\ &\quad + \gamma\{B(X,PZ)\eta(Y) - B(Y,PZ)\eta(X)\}\\ =\ &\frac{c}{4}\{g(Y,PZ)\eta(X) - g(X,PZ)\eta(Y) + \bar{g}(X,JPZ)v(Y)\\ &\qquad - \bar{g}(Y,JPZ)v(X) + 2\bar{g}(X,JY)v(PZ)\}.\end{aligned}$$

Replacing X by ξ to this equation, we have

$$\begin{aligned}&\{\xi\gamma - \gamma\tau(\xi)\}g(Y,PZ) - \gamma B(Y,PZ)\\ =\ &\frac{c}{4}\{g(Y,PZ) + v(Y)u(PZ) + 2u(Y)v(PZ)\}.\end{aligned}$$

Taking $Y = PZ = U$ to this equation and using $(48)_2$, we get $\gamma = 0$. Therefore, M is screen totally geodesic. And, taking $Y = U$ and $PZ = V$ to the last equation, we have $c = 0$. □

6. **Conclusions**

We studied the geometry of lightlike hypersurfaces on various indefinite ambient spaces, like almost Hermitian manifolds, nearly Kaehler manifolds and nearly complex space forms. In this case, depending on the structure vector field(parallel, recurrent and nearly recurrent), we derived the flatness of an entire manifold.

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