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SUB–RIEMANNIAN GEODESICS ASSOCIATED TO CR-GEOMETRY ON HEISENBERG GROUP $\mathbb{H}_3$.

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We introduce a class of curves corresponding to the class of metrics of sub-Riemannian geometry on the Heisenberg group $\mathbb{H}_3$. Any curve from the class belongs to a set of geodesics corresponding to a sub-Riemannian geometry on the Heisenberg group $\mathbb{H}_3$. We use the symmetries of CR-geometry as an additional geometric structure on $\mathbb{H}_3$ to construct these curves. We demonstrate how these classes of geodesics can be used for the corresponding control systems.

Keywords: Geometric control theory, Heisenberg group, Symmetries, Sub–Riemannian geodesics, CR–structure

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1. Introduction

We discuss the non-trivial sub-Rimannian geometry based on Heisenberg group $\mathbb{H}_3$. The control problem on $\mathbb{H}_3$ is well known [3, 5]. We discuss the geodesics of sub-Rimeannian geometry with respect to additional geometric structures, [1, 2] and [4, 9]. Our solutions are optimal trajectories with respect to a special class of metrics on distribution, [6]. On top of that, we do not choose arbitrary symmetries for it, but to control the optimality we choose an additional geometric structure on the distribution. The choice of the additional geometric structure can be motivated for example by an application.

We introduce an algorithm, which embeds the Lie algebra of left invariant vector fields into the algebra of its infinitesimal automorphisms. This algorithm is called Tanaka's prolongation. The algorithm has two steps. At first, it constructs the Lie algebra of automorphisms - this is called algebraic Tanaka's prolongation - next, this algebra is seen as an algebra of infinitesimal automorphisms. The second step is called geometric Tanaka's prolongation, [10, 11].

2. Heisenberg group $\mathbb{H}_3$

We will start with the classical definition of Heisenberg group $\mathbb{H}_3$ as a group of matrices. The subset of 3 by 3 matrices over real numbers $\mathcal{M}_3(\mathbb{R})$ given by the following set

$$\mathbb{H}_3 = \left\{ \begin{pmatrix} 1 & x_1 & t \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix}; x_1, x_2, t \in \mathbb{R} \right\}$$

defines a noncommutative group with the usual matrix multiplication. Consider the matrices $A, B \in \mathbb{H}_3$.

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Then we can compute a product of matrices A and B as

$$AB = \begin{pmatrix} 1 & x_1 + \bar{x}_1 & t + \bar{t} + x_1\bar{x}_2 \\ 0 & 1 & x_2 + \bar{x}_2 \\ 0 & 0 & 1 \end{pmatrix}$$

and inspect the inverse matrices to A and B

$$A^{-1} = \begin{pmatrix} 1 & -x_1 & x_1x_2 - t \\ 0 & 1 & -x_2 \\ 0 & 0 & 1 \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} 1 & -\bar{x}_1 & \bar{x}_1\bar{x}_2 - \bar{t} \\ 0 & 1 & -\bar{x}_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then we can inspect the group commutator $[A, B]$ as

$$[A, B] = ABA^{-1}B^{-1} = \begin{pmatrix} 1 & 0 & x_1\bar{x}_2 - \bar{x}_1x_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

hence the commutator subgroup is

$$\Gamma_1(\mathbb{H}_3) = [\mathbb{H}_3, \mathbb{H}_3] = \langle [A, B]; A, B \in \mathbb{H}_3 \rangle = \left\{ \begin{pmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; k \in \mathbb{R} \right\}.$$

Now, assume $C \in \Gamma_1(\mathbb{H}_3)$ be an element of the commutator subgroup and we compute the product of a matrix A with its commutator C

$$AC = \begin{pmatrix} 1 & x_1 & t \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x_1 & t + k \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} = CA,$$

and therefore $[A, C] = AC(AC)^{-1} = I_3$. Hence $\Gamma_2(\mathbb{H}_3) = [\Gamma_1(\mathbb{H}_3), \mathbb{H}_3] = I_3 = e$ and the group $\mathbb{H}_3$ is nilpotent of class 2. Lie group $\mathbb{H}_3$ is called the Heisenberg group with 3 parameters. The nilpotence class measures the noncommutativity of the group. The Lie algebra of the Heisenberg group over the real numbers is known as Heisenberg algebra $\mathfrak{h}_3$, [5, 8]. It is represented using the space of 3 by 3 matrices of the form

$$\mathfrak{h}_3 = \left\{ \begin{pmatrix} 0 & x_1 & t \\ 0 & 0 & x_2 \\ 0 & 0 & 0 \end{pmatrix}; a, b, c \in \mathbb{R} \right\}$$

where $a, b, c \in \mathbb{R}$. The basis elements of the algebra

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

satisfy the commutation relations

$$[X, Y] = Z, \quad [X, Z] = 0, \quad [Y, Z] = 0.$$

In the following, we shall associate with this group a noncommutative geometry of step 2. This geometry will have the Heisenberg principle built in. The bijection $\Phi : \mathbb{R}^3 \to \mathcal{M}_3(\mathbb{R})$,

$$\Phi(x_1, x_2, t) = \begin{pmatrix} 1 & x_1 & t \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix}$$

induces a noncommutative group law structure on $\mathbb{R}^3$

$$(x_1, x_2, t) \circ (x_1', x_2', t') = (x_1 + x_1', x_2 + x_2', t + t' + x_1x_2'). \tag{1}$$

The zero element is $e = (0, 0, 0)$ and the inverse of (x_1, x_2, t) is $(-x_1, -x_2, x_1x_2 - t)$. The set $\mathbb{R}^3$ together with the group law (1) will be called the nonsymmetric 3-dimensional Heisenberg group. This group can be viewed as a Lie group. The left translation $L_a : \mathbb{H}_3 \to \mathbb{H}_3, L_a g = ag, \forall g \in \mathbb{H}_3$ is an analytic diffeomorphism with inverse $L_a^{-1} = L_{a^{-1}}$. A vector field x on $\mathbb{H}_3$ is called left invariant if

$$(L_a)_*(X_g) = X_{ag}, \quad \forall a, g \in \mathbb{H}_3.$$

The set of all left invariant vector fields form the Lie algebra of $\mathbb{H}_3$, dentoed by $L(\mathbb{H}_3)$. The Lie algebra of $\mathbb{H}_3$ has the same dimension as $\mathbb{H}_3$ and it is isomorphic to the tangent space $T_e\mathbb{H}_3$.

It is obvious that vector fields

$$n_1 = \partial_{x_1}, \quad n_2 = \partial_{x_2} + x_1\partial_t, \quad n_3 = \partial_t \tag{2}$$

are left invariant with respect to the Lie group law (1).

On the Heisenberg group, an important role is played by the distribution $\mathcal{H}$ generated by linearly independent vector fields n_1 and n_2 :

$$x \to \mathcal{H}_x = \text{span}_x\{n_1, n_2\},$$

called the horizontal distribution. As $[n_1, n_2] = \partial_t \notin \mathcal{H}$, the horizontal distribution $\mathcal{H}$ is not involutive, and hence, by Frobenius theorem, it is not integrable, i.e. there is no surface locally tangent to $\mathcal{H}$. A vector field V on $\mathbb{R}^3$ is called horizontal if and only if $V_x \in \mathcal{H}_x, \forall x$. A curve $q : [0, 1] \to \mathbb{R}^3$ is called horizontal if the velocity vector $\dot{q}(s)$ is a horizontal vector field along $q(s)$. Horizontality is a constraint on velocities and it is called a non-holonomic constraint. It is easy to show that any two points in $\mathbb{H}_3$ can be joined by a piece-wise horizontal curve, i.e. a curve tangent to the horizontal distribution. The choice of vector fields n_1 and n_2 as an orthonormal basis on the distribution $\mathcal{H}$ introduces a sub-Riemannian structure on the Heisenberg group $\mathbb{H}_3$.

We say that the pair $(\hat{u}(t), \hat{q}(t))$ is an optimal pair if $\hat{q}(t)$ is a length minimizer curve with respect to sub-Riemannian structure on $\mathcal{H}$ and satisfies $\dot{q} = u_1n_1(q) + u_2n_2(q), q \in M$ with the control function $u = \hat{u}(t)$. The Hamiltonian of the maximum principle is a family of smooth functions parameterized by controls $u = (u_1, u_2) \in \mathbb{R}^2$ and a real number, $\nu \leq 0$ given by

$$H(\nu, f) = \langle u_1n_1 + u_2n_2, f\rangle + \frac{\nu}{2}(u_1^2 + u_2^2) = u_1h_1 + u_2h_2 + \frac{\nu}{2}(u_1^2 + u_2^2),$$

where $\langle\cdot,\cdot\rangle$ denotes evaluation. If $\nu = 0$, we speak about an abnormal Hamiltonian. Otherwise, we speak about a normal Hamiltonian and we normalize H by $\nu = -1$. The curve $q(t)$ is called an extremal, and it is a normal (abnormal) if it corresponds to a normal (abnormal) Hamiltonian of Pontryagin's maximum principle. The projection of each abnormal to M coincides with a projection of a normal for 1-step filtrations, so for Heisenberg group $\mathbb{H}_3$. Thus we focus only on normal extremals.

Now, let us construct the adjoint representation of left invariant vector fields n_1, n_2.

$$\text{ad } n_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \text{ad } n_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

and left-invariance leads to the following system which takes the form

$$\dot{h}_i = \langle(\text{ad } \text{d}\mathcal{H})e_i, \xi\rangle = \langle e_i, (\text{ad } \text{d}\mathcal{H})^*\xi\rangle, \tag{3}$$

where e_i is the basis of $\mathfrak{h}_3$ corresponding to n_i and $\mathcal{H} : \mathfrak{h}_3^* \to \mathbb{R}$ corresponds to H and $\xi \in T_0^*\mathbb{H}_3$.

The left-invariant Hamiltonian H satisfies $\text{d}H = h_1\text{d}h_1 + h_2\text{d}h_2$ and thus $\mathcal{H} = h_1e_1 + h_2e_2$ in formula (3). Direct computation gives the adjoint action $\text{ad}(h_1e_1 + h_2e_2)$ viewed as a linear endomorphism represented in the basis e_i as

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -h_2 & h_1 & 0 \end{pmatrix} \tag{4}$$

According to (3) we read off directly the vertical system from the action of the matrix (4) and the horizontal system follows from the form of generators of the distribution (2). These considerations leads to the formulation of the following proposition.

Proposition 1. *Normal extremals of the approximation are solutions of the system*

$$\dot{h}_1 = -h_3 h_2, \quad \dot{h}_2 = h_3 h_1, \quad \dot{h}_3 = 0 \tag{5}$$

$$\dot{x} = h_2, \quad \dot{y} = \phi h_2, \quad \dot{\phi} = h_1 \tag{6}$$

where (5) *is the vertical system and* (6) *is the horizontal system.*

Let us start with the solution of the vertical system (5), which is independent of the horizontal part (6).

Proposition 2. *The system* (5) *has the solution of the form*

$$\begin{aligned} h_1 &= C_2 \sin(C_1 t) + C_3 \cos(C_1 t) \\ h_2 &= C_3 \sin(C_1 t) - C_2 \cos(C_1 t) \\ h_3 &= C_1 \end{aligned}$$

in the generic case $h_3 \neq 0$ for constants C_1, C_2, C_3. In the case $h_3 = 0$ we get $h_2 = C_2, h_1 = C_3$ for constants C_1, C_2, C_3.

Proof. The equation $\dot{h}_3 = 0$ implies $h_3 = C_1$ for some constant C_1. If $C_1 = 0$ then $h_2 = C_2$ and $h_1 = C_3$ for constants C_2, C_3. If $C_1 \neq 0$, then we get the system

$$\begin{pmatrix} \dot{h}_1 \\ \dot{h}_2 \end{pmatrix} = \begin{pmatrix} 0 & -C_1 \\ C_1 & 0 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

with constant coefficients and we get the solution by the discussion of eigenvalues and eigenvectors of the corresponding matrix. □

The horizontal system can be solved by the direct integration. Moreover, it is sufficient to consider the initial condition $x(0) = y(0) = \phi(0) = 0$, because we get solutions with different initial conditions using the group structure of $\mathbb{H}_3$.

Proposition 3. *In the case $h_3 \neq 0$, the horizontal system* (6) *has solutions satisfying $x(0) = y(0) = \phi(0) = 0$*

$$\begin{aligned} x_1 &= \frac{1}{C_1}(C_3 - C_2 \sin(C_1 t) - C_3 \cos(C_1 t)), \\ x_2 &= \frac{1}{4C_1^2}(2C_1(C_2^2 + C_3^2)t - 4C_2C_3 \cos(C_1 t) + 2C_2C_3 \cos(2C_1 t) \\ &\quad - 4C_2^2 \sin(C_1 t) + (C_2^2 - C_3^2) \sin(2C_1 t) + 2C_2C_3), \\ t &= \frac{1}{C_1}(C_2 - C_2 \cos(C_1 t) + C_3 \sin(C_1 t)) \end{aligned}$$

for constants C_1, C_2, C_3 from proposition 2. In the degenerate case $h_3 = 0$ we get $x = C_2 t, y = 12C_2C_3t^2, \phi = C_3 t$ for C_2, C_3 from proposition 2.

3. Symmetries of CR-structure on $\mathbb{H}_3$

There exist two well-known nondegenerated geometric structures on top of $\mathbb{H}_3$. These are Lagrange contact structure and CR-structure. From now on we have chosen CR-structure for further investigations. We will compute both algebraic and geometric Tanaka's prolongation in the case of algebra $\mathbb{H}_3$. Formally a CR (Cauchy-Riemann) structure is a smooth manifold N equipped with a distribution $\mathcal{D}$ and an almost complex structure on $\mathcal{D}$ on this distribution (i.e. an affinor J, such that $J^2 = -\mathrm{Id}_{\mathcal{D}}$). Now, let us introduce it in a greater detail.

Definition 1. Let M be a smooth manifold. Let $J : TM \to TM$ be a smooth tensor field such that $J^2 = -Id_M$. We call tensor field J *the almost complex structure* on M.

Definition 2. Let M be a smooth manifold. Let $\mathcal{D}$ be a distribution on M and $J : \mathcal{D} \to \mathcal{D}$ is an almost complex structure on $\mathcal{D}$. We call $(M, \mathcal{D}, J)$ the CR-structure structure on M.

Now we will define an additional CR-structure on distribution $\mathcal{H} = \langle n_1, n_2\rangle$ on $\mathbb{H}_3$. An almost complex structure on distribution $\mathcal{H}$, is defined on basis elements as

$$J(n_1) = -n_2, J(n_2) = n_1. \tag{7}$$

In particular Lie algebra

$$\mathfrak{g}_0 \cong \{A \in \mathrm{Hom}(\mathcal{H}, \mathcal{H}), AJ = JA\}$$

preserve CR-structure. So $A \in \mathfrak{g}_0$ if and only if $A = \mathrm{id}$ or $A = J$ because of another automorphism

$$A_1(n_1) = n_2, \quad A_1(n_2) = n_1, \quad A_2(n_1) = -n_1, \quad A_2(n_2) = -n_1,$$

do not belong to $\mathfrak{g}_0$ because $A_1J \neq JA_1$ and $A_2J \neq JA_2$. This geometric structure is also called Cauchy–Riemannian structure and is frequently studied, [7]. To embed the control algebra to the algebra of infinitesimal automorphisms we use Tanaka's prolongation. In the rest of the chapter, we will explicitly show how to calculate algebraic and geometric extensions of the Lie algebra $\mathfrak{g}_0$.

3.1. Algebraic Tanaka's prolongation.

Algebraic Tanaka's prolongation is the standard algebraic method, which can be found in [10, 11].

Theorem 3.2. *Let $\mathbb{H}_3$ be the Heisenberg group equipped with distribution generated by vector fields $X_1 = n_1$ and $X_2 = n_2$ from* (2) *and with CR–structure* (7)*. The algebra of infinitesimal authomorphisms of CR–structure* (7) *is generated by eight basis elements it forms a Lie algebra with multiplication Table 1.*

Proof. Now, let us proceed in the construction of algebra of infinitesimal automorphisms of $\mathfrak{h}$. Let

$$\mathfrak{g}^0 = \mathrm{span}\{\Lambda^0_1, \Lambda^0_2\},$$

where we choose automorphisms as follows

$$\Lambda^0_1(X_1) = X_1, \quad \Lambda^0_1(X_2) = X_2, \quad \Lambda^0_2(X_1) = X_2, \quad \Lambda^0_2(X_2) = -X_1.$$

Because Λ^0_i are derivations, then we can compute

$$\Lambda^0_1(X_3) = \Lambda^0_1([X_1, X_2]) = [\Lambda^0_1(X_1), X_2] + [X_1, \Lambda^0_1(X_2)] = [X_1, X_2] + [X_1, X_2] = 2X_3,$$
$$\Lambda^0_2(X_3) = \Lambda^0_2([X_1, X_2]) = [\Lambda^0_2(X_1), X_2] + [X_1, \Lambda^0_2(X_2)] = [X_2, X_2] + [X_1, -X_1] = 0.$$

Because $\delta^1 \in \mathrm{Hom}\ (\mathfrak{g}^{-1}, \mathfrak{g}^0) \bigoplus \mathrm{Hom}\ (\mathfrak{g}^{-2}, \mathfrak{g}^{-1})$ then

$$\delta^1(X_1) = \alpha_{11}\Lambda^0_1 + \alpha_{12}\Lambda^0_2, \quad \delta^1(X_2) = \alpha_{21}\Lambda^0_1 + \alpha_{22}\Lambda^0_2,$$

$[\cdot,\cdot]$	X_1	X_2	X_3	Λ^0_1	Λ^0_2	Λ^1_1	Λ^1_2	Λ^2
X_1	0	X_3	0	$-X_1$	$-X_2$	$-\Lambda^0_1$	$3\Lambda^0_2$	Λ^1_2
X_2	$-X_3$	0	0	$-X_2$	X_1	$-3\Lambda^0_2$	$-\Lambda^0_1$	$-\Lambda^1_1$
X_3	0	0	0	$-2X_3$	0	$2X_2$	$-2X_1$	0
Λ^0_1	X_1	X_2	$2X_3$	0	0	$-\Lambda^1_1$	$-\Lambda^1_2$	$-2\Lambda^2$
Λ^0_2	X_2	$-X_1$	0	0	0	Λ^1_2	$-\Lambda^1_1$	0
Λ^1_1	Λ^0_1	$3\Lambda^0_2$	$-2X_2$	Λ^1_1	$-\Lambda^1_2$	0	$-2\Lambda^2$	0
Λ^1_2	$-3\Lambda^0_2$	Λ^0_1	$2X_1$	Λ^1_2	Λ^1_1	$2\Lambda^2$	0	0
Λ^2	$-\Lambda^1_2$	Λ^1_1	0	$2\Lambda^2$	0	0	0	0

TABLE 1. Multiplication table of Tanaka's prolongation for Heisenberg group $\mathbb{H}_3$

for some $\alpha_{ij}, 1 \le i,j \le 2$. if $\delta^1 \in \mathfrak{g}^1$ then

$$\begin{aligned}\delta^1(X_3) &= [\delta^1(X_1), X_2] + [X_1, \delta^1(X_2)] = [\alpha_{11}\Lambda^0_1 + \alpha_{12}\Lambda^0_2, X_2] + [X_1, \alpha_{21}\Lambda^0_1 + \alpha_{22}\Lambda^0_2]\\ &= \alpha_{11}\Lambda^0_1(X_2) + \alpha_{12}\Lambda^0_2(X_2) - \alpha_{21}\Lambda^0_1(X_1) - \alpha_{22}\Lambda^0_2(X_1)\\ &= \alpha_{11}X_2 - \alpha_{12}X_1 - \alpha_{21}X_1 - \alpha_{22}X_2 = (\alpha_{11} - \alpha_{22})X_2 + (-\alpha_{12} - \alpha_{21})X_1.\end{aligned}$$

Similarly we can compute

$$\begin{aligned}\delta^1([X_1, X_3]) &= [\delta^1(X_1), X_3] + [X_1, \delta^1(X_3)]\\ &= [\alpha_{11}\Lambda^0_1 + \alpha_{12}\Lambda^0_2, X_3] + [X_1, (-\alpha_{12} - \alpha_{21})X_1 + (\alpha_{11} - \alpha_{22})X_2]\\ &= \alpha_{11}\Lambda^0_1(X_3) + \alpha_{12}\Lambda^0_2(X_3) + (\alpha_{12} + \alpha_{21})[X_1, X_1] + (\alpha_{11} - \alpha_{22})[X_1, X_2]\\ &= 2\alpha_{11}X_3 + (\alpha_{11} - \alpha_{22})X_3 = 0,\end{aligned}$$

which obviously implies $3\alpha_{11} = \alpha_{22}$. Furthermore, we compute

$$\begin{aligned}\delta^1([X_2, X_3]) &= [\delta^1(X_2), X_3] + [X_2, \delta^1(X_3)]\\ &= [\alpha_{21}\Lambda^0_1 + \alpha_{22}\Lambda^0_2, X_3] + [X_2, (-\alpha_{12} - \alpha_{21})X_1 + (\alpha_{11} - \alpha_{22})X_2]\\ &= \alpha_{21}\Lambda^0_1(X_3) + \alpha_{22}\Lambda^0_2(X_3) + (\alpha_{12} + \alpha_{21})[X_1, X_2] + (\alpha_{11} - \alpha_{22})[X_2, X_2]\\ &= 2\alpha_{21}X_3 + (\alpha_{12} + \alpha_{21})X_3 = 0\end{aligned}$$

which similarly implies $3\alpha_{21} = -\alpha_{12}$ thus we can evaluate δ^1 as

$$\delta^1(X_1) = \alpha_{11}\Lambda^0_1 - 3\alpha_{21}\Lambda^0_2, \quad \delta^1(X_2) = \alpha_{21}\Lambda^0_1 + 3\alpha_{11}\Lambda^0_2.$$

To find $\mathfrak{g}^2$ let us choose parameters α_{11}, α_{21} as $(\alpha_{11}, \alpha_{21}) = (1, 0)$ respectively $(\alpha_{11}, \alpha_{21}) = (0, 1)$, so we get

$$\Lambda^1_1(X_1) = \Lambda^0_1 \quad \Lambda^1_1(X_2) = 3\Lambda^0_2, \quad \Lambda^1_2(X_1) = -3\Lambda^0_2, \quad \Lambda^1_2(X_2) = \Lambda^0_1.$$

Now, let us evaluate X_3 in Λ^1_i.

$$\begin{aligned}\Lambda^1_1(X_3) &= [\Lambda^1_1(X_1), X_2] + [X_1, \Lambda^1_1(X_2)] = [\Lambda^0_1, X_2] - [3\Lambda^0_2, X_1] = X_2 - 3X_2 = -2X_2\\ \Lambda^1_2(X_3) &= [\Lambda^1_2(X_1), X_2] + [X_1, \Lambda^1_2(X_2)] = [-3\Lambda^0_2, X_2] - [\Lambda^0_1, X_1] = 3X_1 - X_1 = 2X_1\end{aligned}$$

Assume, that $\delta^2 \in \mathrm{Hom}\ (\mathfrak{g}^{-1}, \mathfrak{g}^1) \bigoplus \mathrm{Hom}\ (\mathfrak{g}^{-2}, \mathfrak{g}^0)$, then

$$\delta^2(X_1) = \beta_{11}\Lambda^1_1 + \beta_{12}\Lambda^1_2, \quad \delta^2(X_2) = \beta_{21}\Lambda^1_1 + \beta_{22}\Lambda^1_2,$$

for some $\beta_{ij}, 1 \le i,j \le 2$. Let us compute $X_3, [X_1, X_3], [X_2, X_3]$ for $\delta^2 \in \mathfrak{g}^2$.

$$\begin{aligned}\delta^2(X_3) &= [\delta^2(X_1), X_2] + [X_1, \delta^2(X_2)] = [\beta_{11}\Lambda^1_1 + \beta_{12}\Lambda^1_2, X_2] - [\beta_{21}\Lambda^1_1 + \beta_{22}\Lambda^1_2, X_1]\\ &= 3\beta_{11}\Lambda^0_2 + \beta_{12}\Lambda^0_1 - \beta_{21}\Lambda^0_1 + 3\beta_{22}\Lambda^0_2 = (\beta_{12} - \beta_{21})\Lambda^0_1 + (3\beta_{11} + 3\beta_{22})\Lambda^0_2\end{aligned}$$

From calculation of $[X_1, X_3], [X_2, X_3]$ in δ^2 we are able to compute values of coefficients β_{ij}.

$$\begin{aligned}\delta^2([X_1, X_3]) &= [\beta_{11}\Lambda_1^1 + \beta_{12}\Lambda_2^1, X_3] - [(\beta_{12} - \beta_{21})\Lambda_1^0 + (3\beta_{11} + 3\beta_{22})\Lambda_2^0, X_1]\\ &= -2\beta_{11}X_2 + 2\beta_{12}X_1 + (\beta_{21} - \beta_{12})X_1 - (3\beta_{11} + 3\beta_{22})X_2\\ &= (\beta_{12} + \beta_{21})X_1 + (-5\beta_{11} - 3\beta_{22})X_2 = 0\end{aligned}$$

$$\begin{aligned}\delta^2([X_2, X_3]) &= [\beta_{21}\Lambda_1^1 + \beta_{22}\Lambda_2^1, X_3] - [(\beta_{12} - \beta_{21})\Lambda_1^0 + (3\beta_{11} + 3\beta_{22})\Lambda_2^0, X_2]\\ &= -2\beta_{21}X_2 + 2\beta_{22}X_1 + (\beta_{21} - \beta_{12})X_2 + (3\beta_{11} + 3\beta_{22})X_1\\ &= (3\beta_{11} + 5\beta_{22})X_1 + (-\beta_{21} - \beta_{12})X_2 = 0\end{aligned}$$

These two equations imply following system of equations

$$\begin{aligned}\beta_{12} + \beta_{21} = 0 \quad & -5\beta_{11} - 3\beta_{22} = 0\\ 3\beta_{11} + 5\beta_{22} = 0 \quad & -\beta_{21} - \beta_{12} = 0\end{aligned}$$

whose solution is $\beta_{11} = \beta_{22} = 0, \beta_{12} = -\beta_{21}$, so

$$\delta^2(X_1) = -\beta_{21}\Lambda_2^1, \quad \delta^2(X_2) = \beta_{21}\Lambda_1^1.$$

Let us try to compute $\mathfrak{g}^3$. Chose $\beta_{21} = 1$ and then $\beta_{21} = 0$. It is obvious that such mapping degenerates with such choice.

$$\Lambda_1^2(X_1) = -\Lambda_2^1, \quad \Lambda_1^2(X_2) = \Lambda_1^1, \quad \Lambda_2^2(X_1) = 0, \quad \Lambda_2^2(X_2) = 0$$

Compute X_3 in Λ_i^2.

$$\begin{aligned}\Lambda_1^2(X_3) &= [\Lambda_1^2(X_1), X_2] + [X_1, \Lambda_1^2(X_2)] = [-\Lambda_2^1, X_2] - [\Lambda_1^1, X_1] = \Lambda_1^0 - \Lambda_1^0 = 0\\ \Lambda_2^2(X_3) &= [\Lambda_2^2(X_1), X_2] + [X_1, \Lambda_2^2(X_2)] = [0, X_2] - [0, X_1] = 0\end{aligned}$$

Now assume, that $\delta^3 \in \text{Hom}\ (\mathfrak{g}^{-1}, \mathfrak{g}^2)$, then

$$\delta^3(X_1) = \gamma_1\Lambda_1^2, \quad \delta^3(X_2) = \gamma_2\Lambda_1^2,$$

for some $\gamma_i, i = 1, 2$. Now we will compute $X_3, [X_1, X_3], [X_2, X_3]$ in $\delta^3 \in \mathfrak{g}^3$.

$$\begin{aligned}\delta^3(X_3) &= [\delta^3(X_1), X_2] + [X_1, \delta^3(X_2)] = [\gamma_1\Lambda_1^2, X_2] - [\gamma_2\Lambda_1^2, X_1] = \gamma_1\Lambda_1^1 + \gamma_2\Lambda_2^1\\ \delta^3([X_1, X_3]) &= [\gamma_1\Lambda_1^2, X_3] - [\gamma_1\Lambda_1^1 + \gamma_2\Lambda_2^1, X_1] = 0 - \gamma_1\Lambda_1^0 - 3\gamma_2\Lambda_2^0\\ \delta^3([X_2, X_3]) &= [\gamma_2\Lambda_1^2, X_3] - [\gamma_1\Lambda_1^1 + \gamma_2\Lambda_2^1, X_2] = 0 - 3\gamma_1\Lambda_2^0 - \gamma_2\Lambda_1^0\end{aligned}$$

Obviously, the system has a single solution $\gamma_1 = \gamma_2 = 0$. Compute the rest of the brackets $[\Lambda_i^j, \Lambda_k^l]$ and construct multiplication table of this Lie algebra.

$$\begin{aligned}[\Lambda_1^0, \Lambda_2^0](X_1) &= [X_1, \Lambda_2^0] + [\Lambda_1^0, X_2] = -X_2 + X_2 = 0\\ [\Lambda_1^0, \Lambda_2^0](X_2) &= [X_2, \Lambda_2^0] + [\Lambda_1^0, -X_1] = -(-X_1) - X_1 = 0\\ [\Lambda_1^1, \Lambda_1^0](X_1) &= [\Lambda_1^0, \Lambda_1^0] + [\Lambda_1^1, X_1] = \Lambda_1^0\\ [\Lambda_1^1, \Lambda_1^0](X_2) &= [3\Lambda_2^0, \Lambda_1^0] + [\Lambda_1^1, X_2] = 3\Lambda_2^0\\ [\Lambda_1^1, \Lambda_2^0](X_1) &= [\Lambda_1^0, \Lambda_2^0] + [\Lambda_1^1, X_2] = 3\Lambda_2^0\\ [\Lambda_1^1, \Lambda_2^0](X_2) &= [3\Lambda_2^0, \Lambda_2^0] + [\Lambda_1^1, -X_1] = -\Lambda_1^0\\ [\Lambda_2^1, \Lambda_1^0](X_1) &= [-3\Lambda_2^0, \Lambda_1^0] + [\Lambda_2^1, X_1] = -3\Lambda_2^0\\ [\Lambda_2^1, \Lambda_1^0](X_2) &= [\Lambda_1^0, \Lambda_1^0] + [\Lambda_2^1, X_2] = \Lambda_1^0\\ [\Lambda_2^1, \Lambda_2^0](X_1) &= [-3\Lambda_2^0, \Lambda_2^0] + [\Lambda_2^1, X_2] = \Lambda_1^0\\ [\Lambda_2^1, \Lambda_2^0](X_2) &= [\Lambda_1^0, \Lambda_2^0] + [\Lambda_2^1, -X_1] = 3\Lambda_2^0\\ [\Lambda_1^1, \Lambda_2^1](X_1) &= [\Lambda_1^0, \Lambda_2^1] + [\Lambda_1^1, -3\Lambda_2^0] = 2\Lambda_2^1\\ [\Lambda_1^1, \Lambda_2^1](X_2) &= [3\Lambda_2^0, \Lambda_2^1] + [\Lambda_1^1, \Lambda_1^0] = -2\Lambda_1^1\end{aligned}$$

$$[\Lambda^2,\Lambda^0_1](X_1)=[-\Lambda^1_2,\Lambda^0_1]+[\Lambda^2,X_1]=-2\Lambda^1_2$$
$$[\Lambda^2,\Lambda^0_1](X_2)=[\Lambda^1_1,\Lambda^0_1]+[\Lambda^2,X_2]=2\Lambda^1_1$$
$$[\Lambda^2,\Lambda^0_2](X_1)=[-\Lambda^1_2,\Lambda^0_2]+[\Lambda^2,X_2]=0$$
$$[\Lambda^2,\Lambda^0_2](X_2)=[\Lambda^1_1,\Lambda^0_2]+[\Lambda^2,-X_1]=0$$
$$[\Lambda^2,\Lambda^1_1](X_1)=[-\Lambda^1_2,\Lambda^1_1]+[\Lambda^2,\Lambda^0_1]=0$$
$$[\Lambda^2,\Lambda^1_1](X_2)=[\Lambda^1_1,\Lambda^1_1]+[\Lambda^2,3\Lambda^0_2]=0$$
$$[\Lambda^2,\Lambda^1_2](X_1)=[-\Lambda^1_2,\Lambda^1_2]+[\Lambda^2,-3\Lambda^0_2]=0$$
$$[\Lambda^2,\Lambda^1_2](X_2)=[\Lambda^1_1,\Lambda^1_2]+[\Lambda^2,\Lambda^0_1]=0$$

which completes the proof □

3.3. Geometric Tanaka's prolongation.

We got the 8-dimensional Lie algebra with the multiplication table (1). For simplicity denote elements of Tanaka's prolongation $(\mathfrak{m}_0)^\infty$ as e_i, so we got the following multiplication table.

$[\cdot,\cdot]$	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
e_1	0	e_3	0	$-e_1$	$-e_2$	$-e_4$	$3e_5$	e_7
e_2	$-e_3$	0	0	$-e_2$	e_1	$-3e_5$	$-e_4$	$-e_6$
e_3	0	0	0	$-2e_3$	0	$2e_2$	$-2e_1$	0
e_4	e_1	e_2	$2e_3$	0	0	$-e_6$	$-e_7$	$-2e_8$
e_5	e_2	$-e_1$	0	0	0	e_7	$-e_6$	0
e_6	e_4	$3e_5$	$-2e_2$	e_6	$-e_7$	0	$-2e_8$	0
e_7	$-3e_5$	e_4	$2e_1$	e_7	e_6	$2e_8$	0	0
e_8	$-e_7$	e_6	0	$2e_8$	0	0	0	0

TABLE 2. Relabeled multiplication table

The next computation is based on the algorithm, which can be found in [11]. To find infinitesimal automorphisms, we will introduce so-called Maurer-Cartan form, precisely the mapping $\omega : TM \to \mathfrak{g}$ such that $\omega = \omega_{x_1}\mathrm{d}x_1 + \omega_{x_2}\mathrm{d}x_2 + \omega_t \mathrm{d}t$ holds $\omega(X_i) = e_i$.

Lemma 1. *Let $\mathbb{H}_3$ be a Heisenberg group equipped with distribution generated by vector fields n_1 and n_2 from* (2) *and $\mathfrak{g}$ is a Lie algebra generated by elements e_i, $i = 1,\dots,3$ such that $[e_1,e_2]=e_3$. The Maurer-Cartan form is a mapping $\omega : T\mathbb{H}_3 \to \mathfrak{g}$ such that*

$$\omega = (e_2 - te_3)dx_1 + e_3 dx_2 + e_1 dx_1 = dte_1 + dx_1e_2 + (dx_2 - t\,dx_1)e_3$$

and denote its parts as

$$\omega^{-1} = dte_1 + dx_1e_2,$$
$$\omega^{-2} = (dx_2 - t\,dx_1)e_3.$$

Proof. It yields

$$e_1 = (\omega_{x_1}\mathrm{d}x_1 + \omega_{x_2}\mathrm{d}x_2 + \omega_t\mathrm{d}t)(\partial_t) = \omega_t,$$
$$e_2 = (\omega_{x_1}\mathrm{d}x_1 + \omega_{x_2}\mathrm{d}x_2 + \omega_t\mathrm{d}t)(\partial_{x_1} + t\partial_{x_2}) = \omega_{x_1} + t\omega_{x_2},$$
$$e_3 = (\omega_{x_1}\mathrm{d}x_1 + \omega_{x_2}\mathrm{d}x_2 + \omega_t\mathrm{d}t)(\partial_{x_2}) = \omega_{x_2}.$$

and substituting to the Maurer-Cartan form we complete the proof. □

Theorem 1. *The Lie algebra of infinitesimal automorphisms $\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is generated by following vector fields:*

$$
\begin{aligned}
Y_{e_1} &= \partial_t + x_1\partial_{x_2}\\
Y_{e_2} &= \partial_{x_1}\\
Y_{e_3} &= \partial_{x_2}\\
Y_{e_4} &= x_1\partial_{x_1} + 2x_2\partial_{x_2} + t\partial_t\\
Y_{e_5} &= t\partial_{x_1} - \frac{1}{2}({x_1}^2 - t^2)\partial_{x_2} - x_1\partial_t\\
Y_{e_6} &= \left(\frac{1}{2}t^2 - 3{x_1}^2\right)\partial_t + (3x_1t - 2x_2)\partial_{x_1} + \left(\frac{3}{2}t^2x_1 - 2x_2t + 2x_2 - \frac{1}{2}{x_1}^3\right)\partial_{x_2}\\
Y_{e_7} &= (2x_2 + tx_1)\,\partial_t + \frac{1}{2}({x_1}^2 - 3t^2)\partial_{x_1} + \left(2x_1x_2 - t^3\right)\partial_{x_2}\\
Y_{e_8} &= -\frac{1}{2}[Y_{e_6}, Y_{e_7}]
\end{aligned}
$$

Proof. Let $e_3 \in \mathfrak{g}^{-2}$, then desired Y_{e_3} holding

$$\omega(Y_{e_3}) = u^{-2},$$

where $u^{-2} = e_3$, is exactly $Y_{e_3} = \partial_{x_2}$. Now let $e_2 \in \mathfrak{g}^{-1}$ such that we look for Y_{e_2} holding $\omega(Y_{e_2}) = u^{-1} + u^{-2}$, where

$$u^{-1} = e_2, \qquad \mathrm{d}u^{-2} \quad = [u^{-1}, \omega^{-1}].$$

Evaluation of these equations

$$
\begin{aligned}
u^{-1} &= e_2\\
\mathrm{d}u^{-2} &= [e_2, \mathrm{d}te_1 + \mathrm{d}x_1e_2] = -\mathrm{d}te_3
\end{aligned}
$$

yielding $u^{-1} = e_2, u^{-2} = -te_3$, is solution Y_{e_2} of equation $\omega(Y_{e_2}) = e_2 - te_3$.

$$Y_{e_2} = \partial_{x_1} + t\partial_{x_2} - t\partial_{x_2} = \partial_{x_1}$$

Furthermore let $e_1 \in \mathfrak{g}^{-1}$ such that we look for Y_{e_1} holding $\omega(Y_{e_1}) = u^{-1} + u^{-2}$, where

$$u^{-1} = e_1, \qquad \mathrm{d}u^{-2} \quad = [u^{-1}, \omega^{-1}].$$

Evaluation of these equations

$$
\begin{aligned}
u^{-1} &= e_1\\
\mathrm{d}u^{-2} &= [e_1, \mathrm{d}te_1 + \mathrm{d}x_1e_2] = \mathrm{d}x_1e_3
\end{aligned}
$$

yielding $u^{-1} = e_1, u^{-2} = x_1e_3$, is solution Y_{e_1} of equation $\omega(Y_{e_1}) = e_1 + x_1e_3$.

$$Y_{e_1} = \partial_t + x_1\partial_{x_2}$$

It yields algebra $\langle Y_{e_1}, Y_{e_2}, Y_{e_3}\rangle$ isomorphic with algebra $\langle X_1, X_2, X_3\rangle$. Now, let $e_4 \in \mathfrak{g}^0$ such that we look for Y_{e_4} holding $\omega(Y_{e_4}) = u^0 + u^{-1} + u^{-2}$, where

$$
\begin{aligned}
u^0 &= e_4,\\
\mathrm{d}u^{-1} &= [u^0, \omega^{-1}],\\
\mathrm{d}u^{-2} &= [u^{-1}, \omega^{-1}] + [u^0, \omega^{-2}].
\end{aligned}
$$

Evaluation of these equations

$$
\begin{aligned}
u^0 &= e_4\\
\mathrm{d}u^{-1} &= [e_4, \mathrm{d}te_1 + \mathrm{d}x_1e_2] = \mathrm{d}te_1 + \mathrm{d}x_1e_2
\end{aligned}
$$

yielding $u^0 = e_4, u^{-1} = te_1 + x_1e_2$ and moreover

$$\mathrm{d}u^{-2} = [te_1 + x_1e_2, \mathrm{d}te_1 + \mathrm{d}x_1e_2] + [e_4, (\mathrm{d}x_2 - t\mathrm{d}x_1)e_3] = (t\mathrm{d}x_1 - x_1\mathrm{d}t)e_3 + 2(\mathrm{d}x_2 - tx_1)e_3$$

yielding $u^{-2} = (2x_2 - x_1t)e_3$, we get solution Y_{e_4} of equation $\omega(Y_{e_4}) = e_4 + te_1 + x_1e_2 + (2x_2 - x_1t)e_3$.

$$Y_{e_4} = x_1\partial_{x_1} + 2x_2\partial_{x_2} + t\partial_t$$

Next let $e_5 \in \mathfrak{g}^0$ such that we look for Y_{e_5} holding $\omega(Y_{e_5}) = u^0 + u^{-1} + u^{-2}$, where

$$\begin{aligned} u^0 &= e_5, \\ \mathrm{d}u^{-1} &= [u^0, \omega^{-1}], \\ \mathrm{d}u^{-2} &= [u^{-1}, \omega^{-1}] + [u^0, \omega^{-2}]. \end{aligned}$$

Evaluation of these equations

$$\begin{aligned} u^0 &= e_5 \\ \mathrm{d}u^{-1} &= [e_5, \mathrm{d}te_1 + \mathrm{d}x_1e_2] = \mathrm{d}te_2 - \mathrm{d}x_1e_1 \end{aligned}$$

yielding $u^0 = e_4, u^{-1} = te_2 - x_1e_1$ and moreover

$$\mathrm{d}u^{-2} = [te_2 - x_1e_1, \mathrm{d}te_1 + \mathrm{d}x_1e_2] + [e_5, (\mathrm{d}x_2 - t\mathrm{d}x_1)e_3] = (-t\mathrm{d}t - x_1\mathrm{d}x_1)e_3$$

yielding $u^{-2} = -\frac{1}{2}(t^2 + {x_1}^2)e_3$, we get solution Y_{e_5} of equation $\omega(Y_{e_5}) = e_5 + te_2 - x_1e_1 - \frac{1}{2}(t^2 + {x_1}^2)e_3$.

$$Y_{e_5} = t\partial_{x_1} - \frac{1}{2}({x_1}^2 - t^2)\partial_{x_2} - x_1\partial_t$$

Now let $e_6 \in \mathfrak{g}^1$ such that we look for Y_{e_6} holding $\omega(Y_{e_6}) = u^1 + u^0 + u^{-1} + u^{-2}$, where

$$\begin{aligned} u^1 &= e_6, \\ \mathrm{d}u^0 &= [u^1, \omega^{-1}], \\ \mathrm{d}u^{-1} &= [u^1, \omega^{-2}] + [u^0, \omega^{-1}], \\ \mathrm{d}u^{-2} &= [u^{-1}, \omega^{-1}] + [u^0, \omega^{-2}]. \end{aligned}$$

Evaluation of these equations

$$\begin{aligned} u^1 &= e_6 \\ \mathrm{d}u^0 &= [e_6, \mathrm{d}te_1 + \mathrm{d}x_1e_2] = \mathrm{d}te_4 + 3\mathrm{d}x_1e_5 \end{aligned}$$

yielding $u^1 = e_6, u^0 = te_4 + 3x_1e_5$ and moreover

$$\begin{aligned} \mathrm{d}u^{-1} &= [te_4 + 3x_1e_5, \mathrm{d}te_1 + \mathrm{d}x_1e_2] + [e_6, (\mathrm{d}x_2 - t\mathrm{d}x_1)e_3] \\ &= (3x_1\mathrm{d}t + 3t\mathrm{d}x_1 - 2\mathrm{d}x_2)e_2 + (t\mathrm{d}t - 3x_1\mathrm{d}x_1)e_1, \end{aligned}$$

yielding $u^{-1} = \frac{1}{2}(t^2 - 3{x_1}^2)e_1 + (3x_1t - 2x_2)e_2$ and additionally

$$\begin{aligned} \mathrm{d}u^{-2} &= \left[\frac{1}{2}(t^2 - 3{x_1}^2)e_1 + (3x_1t - 2x_2)e_2, \mathrm{d}te_1 + \mathrm{d}x_1e_2\right] + [te_4 + 3x_1e_5, (\mathrm{d}x_2 - t\mathrm{d}x_1)e_3] \\ &= \left(2t\mathrm{d}x_2 - 2t^2\mathrm{d}x_1 - 3tx_1\mathrm{d}t + 2x_2\mathrm{d}t + \frac{1}{2}t^2\mathrm{d}x_1 - \frac{3}{2}{x_1}^2\mathrm{d}x_1\right)e_3 \end{aligned}$$

we get $u^{-2} = \left(2x_2t - \frac{3}{2}t^2x_1 - 3t{x_1}^3\right)e_3$.
From those equations we get solution Y_{e_6} of equation $\omega(Y_{e_6}) = e_6 + te_4 + 3x_1e_5 + \frac{1}{2}\left(t^2 - 3{x_1}^2\right)e_1 + (3x_1t - 2x_2)e_2 + \left(2x_2 - \frac{3}{2}t^2x_1 - \frac{1}{2}{x_1}^3\right)e_3$.

$$Y_{e_6} = \left(\frac{1}{2}t^2 - 3{x_1}^2\right)\partial_t + (3x_1t - 2x_2)\partial_{x_1} + \left(\frac{3}{2}t^2x_1 - 2x_2t + 2x_2 - \frac{1}{2}{x_1}^3\right)\partial_{x_2}$$

Finally let $e_7 \in \mathfrak{g}^1$ such that we look for Y_{e_7} holding $\omega(Y_{e_7}) = u^1 + u^0 + u^{-1} + u^{-2}$, where

$$\begin{aligned} u^1 &= e_7, \\ \mathrm{d}u^0 &= [u^1, \omega^{-1}], \\ \mathrm{d}u^{-1} &= [u^1, \omega^{-2}] + [u^0, \omega^{-1}], \\ \mathrm{d}u^{-2} &= [u^{-1}, \omega^{-1}] + [u^0, \omega^{-2}]. \end{aligned}$$

Evaluation of these equations

$$\begin{aligned} u^1 &= e_7 \\ \mathrm{d}u^0 &= [e_7, \mathrm{d}te_1 + \mathrm{d}x_1e_2] = \mathrm{d}x_1e_4 - 3\mathrm{d}te_5 \end{aligned}$$

yielding $u^1 = e_7, u^0 = x_1e_4 - 3te_5$ and moreover

$$\begin{aligned} \mathrm{d}u^{-1} &= [x_1e_4 - 3te_5, \mathrm{d}te_1 + \mathrm{d}x_1e_2] + [e_7, (\mathrm{d}x_2 - t\mathrm{d}x_1)e_3] \\ &= (x_1\mathrm{d}x_1 - 3t\mathrm{d}t - 2\mathrm{d}x_2)e_2 + (t\mathrm{d}x_1 + x_1\mathrm{d}t + 2\mathrm{d}x_2)e_1, \end{aligned}$$

yielding $u^{-1} = (tx_1 + 2x_2)e_1 + \frac{1}{2}({x_1}^2 - 3t^2)e_2$ and additionally from equation

$$\begin{aligned} \mathrm{d}u^{-2} &= \left[(tx_1 + 2x_2)e_1 + \frac{1}{2}({x_1}^2 - 3t^2)e_2, \mathrm{d}te_1 + \mathrm{d}x_1e_2\right] + [x_1e_4 - 3te_5, (\mathrm{d}x_2 - t\mathrm{d}x_1)e_3] \\ &= \left(2x_1\mathrm{d}x_2 + 2x_2\mathrm{d}x_1 - tx_1\mathrm{d}x_1 - \frac{1}{2}{x_1}^2\mathrm{d}t + \frac{3}{2}t^2\mathrm{d}t\right)e_3 \end{aligned}$$

we get $u^{-2} = \left(2x_1x_2 - \frac{1}{2}t{x_1}^2 + \frac{1}{2}t^2\right)e_3$.
Accordingly we get solution Y_{e_7} of equation $\omega(Y_{e_7}) = e_7 + x_1e_4 - 3te_5 + (tx_1 + 2x_2)e_1 + \frac{1}{2}({x_1}^2 - 3t^2)e_2 + \left(2x_1x_2 - \frac{1}{2}t{x_1}^2 + \frac{1}{2}t^2\right)e_3$.

$$Y_{e_7} = (2x_2 + tx_1)\,\partial_t + \frac{1}{2}({x_1}^2 - 3t^2)\partial_{x_1} + \left(2x_1x_2 - t^3\right)\partial_{x_2}$$

We computed seven infinitesimal automorphisms, which preserves the horizontal distribution of our mechanism. The Lie bracket of vector fields Y_{e_6} and Y_{e_7} is defined by Lie algebra multiplicative table $\frac{1}{2}[Y_{e_6}, Y_{e_7}] = -2Y_{e_8}$ which completes the proof. □

4. Notes on almost optimal control

We will use flows of vector fields from Theorem 1 given from the Tanaka's prolongation to design geodesics of Sub-Riemannian geometry. For every vector field Y from Theorem 1, we can find its flow and use this flow to map a class of geodesics in sub-Riemannian geometry 3, which was defined by the choice of n_1 and n_2 from (2) as the orthonormal basis. The left-invariance of the Hamiltonian guarantees that the mapped class of curves will form geodesics of the pullback of the original metric. Therefore, we consider the vector fields $[Y, n_1]$ and $[Y, n_2]$ orthonormal. We continue with a direct computation that we demonstrate on two vector fields. Here, as an example, we will display flows of vector fields of $Y_{e_4}, Y_{e_5} \in \mathfrak{g}^0$.

Lemma 4.1. *Let Y_{e_4} and Y_{e_5} are vector fields from Theorem 1 then their flows are the following maps*

$$Flow_t^{e_4} = \begin{pmatrix} x_1e^t \\ x_2e^{2t} \\ te^t \end{pmatrix}$$

$$Flow_t^{e_5} = \begin{pmatrix} t\sin(t) + x_1\cos(t) \\ \frac{t^2\sin(t)\cos(t)}{2} - \frac{{x_1}^2\sin(t)\cos(t)}{2} + \cos^2(t)tx_1 - tx_1 + x_2 \\ t\cos(t) - x_1\sin(t) \end{pmatrix}$$

Proof. Direct computation completes the proof. □

At this moment, we can construct classes of geodesics corresponding to different metrics on $\mathcal{H}$. The procedure is straightforward, so we demonstrate it with one example. For example in the simple case of the flow $\mathrm{Flow}_t^{e_4}$, the curves

$$\begin{aligned} x_1 &= \frac{e^s}{C_1}(C_3 - C_2\sin(C_1 t) - C_3\cos(C_1 t)),\\ x_2 &= \frac{e^{2s}}{4C_1^2}(2C_1(C_2^2 + C_3^2)t - 4C_2C_3\cos(C_1 t) + 2C_2C_3\cos(2C_1 t)\\ &\quad - 4C_2^2\sin(C_1 t) + (C_2^2 - C_3^2)\sin(2C_1 t) + 2C_2C_3),\\ t &= \frac{e^s}{C_1}(C_2 - C_2\cos(C_1 t) + C_3\sin(C_1 t)) \end{aligned}$$

are geodesics of Sub-Riemannian structure based on vector fields

$$\begin{aligned} \bar{n}_1 &= [x_1\partial_{x_1} + 2x_2\partial_{x_2} + t\partial_t, \partial_{x_1} - x_2\partial_t] = -2x_2\partial_t + 2x_2\partial_t - \partial_{x_1} = -\partial_{x_1}\\ \bar{n}_2 &= [x_1\partial_{x_1} + 2x_2\partial_{x_2} + t\partial_t, \partial_{x_2}] = -2\partial_{x_2} \end{aligned}$$

With this procedure, we are able to construct classes of geodesics corresponding to the class of metrics on the Heisenberg group $\mathbb{H}_3$. That means we have proposed a method, how to transform a geodesic to an almost optimal curve with respect to a class of metrics. In the sequel, we could use an elementary set of initial geodesics which would be extended to any almost optimal curve, under some assumptions, to find solution, as studied in [6]. Further analysis could, for example, propose how to classify these geodesics or how to control a dynamic system based on $\mathbb{H}_3$.

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