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TOTAL EDGE IRREGULARITY STRENGTH OF STRONG PRODUCT OF CYCLES AND PATHS*

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As an edge variant of the well-known irregularity strength of a graph $G=(V,E)$ *we investigate edge irregular total* k*-labeling* $\varphi: V\cup E\to\{1,2,\ldots,k\}$ *such that* $\varphi(u)+\varphi(uv)+\varphi(v)\neq\varphi(u')+\varphi(u'v')+\varphi(v')$ *for every pair of different edges* $uv,u'v'\in E$. *The smallest possible* k *is the total edge irregularity strength of* G. *We determine the exact value of the total edge irregularity strength of the strong product of graphs over cycles and paths.*

Keywords *:* irregular assignments, total edge irregularity strength, edge irregular total labeling, strong product of cycle and path.
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1. Introduction

An irregular assignment is a k-labeling of the edges $f: E\to\{1,2,\ldots,k\}$ such that the vertex weights (label sums of edges incident with the vertex) are different for all vertices of G. The smallest k for which there is an irregular assignment is the *irregularity strength*. The notion of irregularity strength was introduced by Chartrand et al. [8] and studied by numerous authors, see [6,10,11,14,17,19,21].

Motivated by these papers, Bača et al. [5] started to investigate a total edge irregularity strength of graphs. For a graph G they define a labeling $\varphi: V\cup E\to\{1,2,\ldots,k\}$ to be an *edge irregular total* k*-labeling* of the graph G if for every two different edges uv and $u'v'$ of G one has $w_\varphi(uv)=\varphi(u)+\varphi(uv)+\varphi(v)\neq\varphi(u')+\varphi(u'v')+\varphi(v')=w_\varphi(u'v')$. The minimum k for which the graph G has an edge irregular total k-labeling is called the *total edge irregularity strength* of G, $tes(G)$.

In [5] is given a lower bound on the total edge irregularity strength of a graph, $tes(G)\geq\max\left\{\left\lceil\frac{|E(G)|+2}{3}\right\rceil,\left\lceil\frac{\Delta(G)+1}{2}\right\rceil\right\}$, where $\Delta(G)$ is the maximum degree of

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G. The authors of [5] present also a few families of graphs for which they found the exact values of the total edge irregularity strength. Recently Ivančo and Jendrol̆ [13] determined the total edge irregularity strength for any tree and posed a conjecture that for every graph G with size $E(G)$ and maximum degree $\Delta(G)$, that is different from K_5, $tes(G)$ is equal to its lower bound.

The conjecture has been verified for complete graphs and complete bipartite graphs in [15, 16], for the Cartesian product of two paths [18], for corona product of a path with certain graphs [20], for large dense graphs with $\frac{|E(G)|+2}{3} \leq \frac{\Delta(G)+1}{2}$ [7], for the categorical product of a cycle and a path [1], for toroidal grid [9] and for categorical product of two paths [2], for strong product of two path [3], for hexagonal grid graph [4].

In this paper we deal with the strong product of cycles and paths. The strong product $G_1 \boxtimes G_2$ of graphs G_1 and G_2 has as vertices the pairs (u,v) where $u \in V(G_1)$ and $v \in V(G_2)$. Vertices (u_1,v_1) and (u_2,v_2) are adjacent if either u_1u_2 is an edge of G_1 and $v_1 = v_2$ or if $u_1 = u_2$ and v_1v_2 is an edge of G_2 or if u_1u_2 is an edge of G_1 and v_1v_2 is an edge of G_2, see e.g. [12].

For integers a and b let $[a,b]$ be an interval of integers c, $a \leq c \leq b$. If we consider graph G_1 as a cycle C_n with $V(C_n) = \{u_i : i \in [1,n]\}$, $E(C_n) = \{u_iu_{i+1} : i \in [1,n-1]\} \cup \{u_nu_1\}$ and graph G_2 as a path P_m with $V(P_m) = \{v_j : j \in [1,m]\}$, $E(P_m) = \{v_jv_{j+1} : j \in [1,m-1]\}$ then

$V(C_n \boxtimes P_m) = \{(u_i,v_j) : i \in [1,n],\ j \in [1,m]\}$ is the vertex set and

$$E(C_n \boxtimes P_m) = \{(u_i,v_j)(u_{i+1},v_j) : i \in [1,n-1],\ j \in [1,m]\} \cup \{(u_{i+1},v_j)(u_i,v_{j+1}) :$$
$$i \in [1,n-1],\ j \in [1,m-1]\} \cup \{(u_i,v_j)(u_{i+1},v_{j+1}) : i \in [1,n-1],\ j \in [1,m-1]\}$$
$$\cup \{(u_i,v_j)(u_i,v_{j+1}) : i \in [1,n],\ j \in [1,m-1]\} \cup \{(u_1,v_j)(u_n,v_{j+1}) : j \in [1,m-1]\}$$
$$\cup \{(u_n,v_j)(u_1,v_{j+1}) : j \in [1,m-1]\} \cup \{(u_1,v_j)(u_n,v_j) : j = 1,m\}$$

is the edge set of $C_n \boxtimes P_m$. Thus $|V(C_n \boxtimes P_m)| = nm$ and $|E(C_n \boxtimes P_m)| = n(4m-3)$. We show that the strong product $C_n \boxtimes P_m$ has total edge irregularity strength equal to its lower bound.

2. Preliminary Lemmas

The total edge irregularity strength of $C_n \boxtimes P_m$ for small values of n we determine in the following lemmas.

Lemma. 1. 1 *Let* $m \geq 2$. *Then* $tes(C_3 \boxtimes P_m) = \left\lceil \frac{12m-7}{3} \right\rceil$.

Proof. From lower bound of the total edge irregularity strength we have $tes(C_3 \boxtimes P_m) \geq \max\left\{\left\lceil \frac{|E(C_3 \boxtimes P_m)|+2}{3} \right\rceil, \left\lceil \frac{\Delta(C_3 \boxtimes P_m)+1}{2} \right\rceil\right\} = \left\lceil \frac{12m-7}{3} \right\rceil = k$. The converse inequality proves the existence of the optimal labelings.

Let us consider three cases.

Case 1. $m = 2$. Define the total labeling φ_1 as follows:

$$\varphi_1(u_2, v_1) = 2, \varphi_1(u_2, v_2) = 4,$$

$$\varphi_1(u_1, v_1) = \varphi_1(u_1, v_2) = 1, \varphi_1(u_3, v_1) = \varphi_1(u_3, v_2) = 6,$$

$$\varphi_1((u_1, v_1)(u_1, v_2)) = \varphi_1((u_1, v_1)(u_2, v_1)) = \varphi_1((u_1, v_2)(u_2, v_2)) = 1,$$

$$\varphi_1((u_2, v_1)(u_1, v_2)) = \varphi_1((u_2, v_1)(u_2, v_2)) = \varphi_1((u_1, v_1)(u_2, v_2)) = 2,$$

$$\varphi_1((u_2, v_1)(u_3, v_2)) = \varphi_1((u_3, v_1)(u_3, v_2)) = \varphi_1((u_3, v_1)(u_2, v_2)) = 5,$$

$$\varphi_1((u_2, v_1)(u_3, v_1)) = \varphi_1((u_2, v_2)(u_3, v_2)) = 6, \varphi_1((u_1, v_2)(u_3, v_2)) = 5,$$

$$\varphi_1((u_3, v_1)(u_1, v_2)) = 3, \varphi_1((u_1, v_1)(u_3, v_1)) = 4, \varphi_1((u_1, v_1)(u_3, v_2)) = 2.$$

Case 2. $m = 3$. Construct the total labeling φ_2 in the following way:

$$\varphi_2(u_1, v_1) = \varphi_2(u_1, v_2) = \varphi_2(u_1, v_3) = 1, \varphi_2(u_2, v_1) = 3,$$

$$\varphi_2(u_3, v_1) = \varphi_2(u_3, v_2) = \varphi_2(u_3, v_3) = 10, \varphi_2(u_2, v_2) = 5, \varphi_2(u_2, v_3) = 8,$$

$$\varphi_2((u_1, v_1)(u_1, v_2)) = \varphi_2((u_1, v_1)(u_2, v_1)) = \varphi_2((u_1, v_2)(u_2, v_2)) = 1,$$

$$\varphi_2((u_1, v_2)(u_1, v_3)) = \varphi_2((u_2, v_1)(u_1, v_2)) = \varphi_2((u_2, v_2)(u_1, v_3)) = 2,$$

$$\varphi_2((u_2, v_1)(u_3, v_1)) = \varphi_2((u_2, v_2)(u_3, v_2)) = \varphi_2((u_2, v_3)(u_3, v_3)) = 9,$$

$$\varphi_2((u_3, v_1)(u_2, v_2)) = \varphi_2((u_3, v_2)(u_2, v_3)) = \varphi_2((u_1, v_3)(u_3, v_3)) = 8,$$

$$\varphi_2((u_1, v_1)(u_3, v_2)) = \varphi_2((u_1, v_2)(u_2, v_3)) = 2, \varphi_2((u_1, v_3)(u_2, v_3)) = 1,$$

$$\varphi_2((u_1, v_2)(u_3, v_3)) = \varphi_2((u_1, v_1)(u_2, v_2)) = 3, \varphi_2((u_3, v_2)(u_1, v_3)) = 5,$$

$$\varphi_2((u_2, v_1)(u_2, v_2)) = \varphi_2((u_3, v_1)(u_1, v_2)) = 4, \varphi_2((u_1, v_1)(u_3, v_1)) = 6,$$

$$\varphi_2((u_2, v_1)(u_3, v_2)) = \varphi_2((u_3, v_1)(u_3, v_2)) = 8, \varphi_2((u_3, v_2)(u_3, v_3)) = 9,$$

$$\varphi_2((u_1, v_2)(u_3, v_2)) = \varphi_2((u_2, v_2)(u_2, v_3)) = 7, \varphi_2((u_2, v_2)(u_3, v_3)) = 10.$$

Case 3. $m \geq 4$. Construct the total labeling φ_3 in the following way:

$$\varphi_3(u_2, v_1) = m, \varphi_3((u_1, v_1)(u_2, v_1)) = 1, \varphi_3((u_2, v_1)(u_3, v_1)) = 7m - k - 3,$$

$$\varphi_3((u_2, v_1)(u_1, v_2)) = 2, \varphi_3((u_2, v_1)(u_3, v_2)) = 7m - k - 2,$$

$$\varphi_3((u_2, v_1)(u_2, v_2)) = 2m - 2, \varphi_3((u_2, v_2)(u_2, v_3)) = 2m - 6.$$

If $1 \leq j \leq m$ then we put $\varphi_3(u_1, v_j) = 1$, $\varphi_3(u_3, v_j) = k$ and $\varphi_3((u_1, v_j)(u_3, v_j)) = 6m - k - 2 + j$.

If $2 \le j \le m$ then we put $\varphi_3(u_2, v_j) = m - 4 + 3j$, $\varphi_3((u_1, v_j)(u_2, v_j)) = 2$ and $\varphi_3((u_2, v_j)(u_3, v_j)) = 7m - k - 2$.

If $1 \le j \le m-1$ then we put $\varphi_3((u_1, v_j)(u_1, v_{j+1})) = j$, $\varphi_3((u_1, v_j)(u_2, v_{j+1})) = 1$, $\varphi_3((u_1, v_j)(u_3, v_{j+1})) = 4m - k + j$, $\varphi_3((u_3, v_j)(u_1, v_{j+1})) = 5m - k - 1 + j$, $\varphi_3((u_3, v_j)(u_2, v_{j+1})) = 7m - k - 3$, $\varphi_3((u_3, v_j)(u_3, v_{j+1})) = 7m - k - 4 + j$.

If $2 \le j \le m-1$ then we put $\varphi_3((u_2, v_j)(u_1, v_{j+1})) = 3$, $\varphi_3((u_2, v_j)(u_3, v_{j+1})) = 7m - k - 1$.

If $3 \le j \le m-1$ then we put $\varphi_3((u_2, v_j)(u_2, v_{j+1})) = 5m + 2 - 5j$.

It is easy to verify that in each case all vertex and edge labels are at most k and that weights of the edges are pairwise distinct. It proves that total labelings φ_1, φ_2 and φ_3 are optimal labelings. □

Lemma. 2. 2 *Let* $m \ge 2$. *Then* $tes(C_4 \boxtimes P_m) = \left\lceil \frac{16m-10}{3} \right\rceil$.

Proof. Suppose that $m \ge 2$ and $k = \left\lceil \frac{16m-10}{3} \right\rceil$. From lower bound of the total edge irregularity strength we have that $tes(C_4 \boxtimes P_m) \ge \left\lceil \frac{16m-10}{3} \right\rceil$. It is enough to prove the converse inequality. We construct the function φ_4 as follows:

$$\varphi_4((u_i, v_j)) = \begin{cases} j, & \text{if } i = 1,2 \\ k - m + j, & \text{if } i = 3 \\ k, & \text{if } i = 4 \end{cases}$$

$$\varphi_4((u_i, v_j)(u_{i+1}, v_j)) = \varphi_4((u_{i+1}, v_j)(u_i, v_{j+1})) = \begin{cases} 1, & \text{if } i = 1 \\ 6m - k - 3, & \text{if } i = 2 \\ 12m - 2k - 7 + j, & \text{if } i = 3 \end{cases}$$

$$\varphi_4((u_i, v_j)(u_{i+1}, v_{j+1})) = \begin{cases} 3m - 1 - j, & \text{if } i = 1 \\ 8m - k - 4 - j, & \text{if } i = 2 \\ 15m - 2k - 8, & \text{if } i = 3 \end{cases}$$

$$\varphi_4((u_i, v_j)(u_i, v_{j+1})) = \begin{cases} 2m - j, & \text{if } i = 1 \\ 4m - 2 - j, & \text{if } i = 2 \\ 15m - 2k - 8 - j, & \text{if } i = 3 \\ 15m - 2k - 9 + j, & \text{if } i = 4 \end{cases}$$

$\varphi_4((u_1,v_j)(u_4,v_j)) = 10m-k-6$, $\varphi_4((u_1,v_j)(u_4,v_{j+1})) = 8m-k-4$ and $\varphi_4((u_4,v_j)(u_1,v_{j+1})) = 9m-k-6$.

Observe that under the total labeling φ_4 all vertex and edge labels are at most k and moreover the edge weights create the integer interval $[3,16m-10]$, i.e. are pairwise distinct. Thus, φ_4 is the desired edge irregular total k-labeling. □

Lemma. 3. 3 *Let* $m \geq 2$. *Then* $tes(C_5 \boxtimes P_m) = \left\lceil \frac{20m-13}{3} \right\rceil$.

Proof. Again from lower bound of $tes(G)$ it follows that $tes(C_5 \boxtimes P_m) \geq \left\lceil \frac{20m-13}{3} \right\rceil$. Let $m \geq 2$ and $k = \left\lceil \frac{20m-13}{3} \right\rceil$. Define a vertex labeling $\varphi_5 : V(C_5 \boxtimes P_m) \to \{1,2,\dots,k\}$ in the following way:

$$\varphi_5((u_i,v_j)) = \begin{cases} j, & \text{if } i = 1,2 \\ k+m(i-5)+j, & \text{if } i = 3,4 \\ k, & \text{if } i = 5 \end{cases}$$

To complete the labeling φ_5 to a total one we label the edges of the graph $C_5 \boxtimes P_m$ as follows:

(i) $\varphi_5((u_1,v_j)(u_2,v_j)) = \varphi_5((u_2,v_j)(u_1,v_{j+1})) = 1$ for all j. For edge weights we have $w_{\varphi_5}((u_1,v_j)(u_2,v_j)) = \varphi_5((u_1,v_j)) + \varphi_5((u_1,v_j)(u_2,v_j)) + \varphi_5((u_2,v_j)) = 2j+1$ for $1 \leq j \leq m$ and

$w_{\varphi_5}((u_2,v_j)(u_1,v_{j+1})) = \varphi_5((u_2,v_j)) + \varphi_5((u_2,v_j)(u_1,v_{j+1})) + \varphi_5((u_1,v_{j+1})) = 2j+2$ for $1 \leq j \leq m-1$.

(ii) $\varphi_5((u_1,v_j)(u_1,v_{j+1})) = 2m-j$ for $1 \leq j \leq m-1$. Then $w_{\varphi_5}((u_1,v_j)) + \varphi_5((u_1,v_j)(u_1,v_{j+1})) + \varphi_5((u_1,v_{j+1})) = 2m+j+1$ for $1 \leq j \leq m-1$.

(iii) $\varphi_5((u_1,v_j)(u_2,v_{j+1})) = 3m-2$ and $\varphi_5((u_2,v_j)(u_2,v_{j+1})) = 3m-1$ for $1 \leq j \leq m-1$. For edge weights we have $w_{\varphi_5}((u_1,v_j)(u_2,v_{j+1})) = 3m+2j-1$ and $w_{\varphi_5}((u_2,v_j)(u_2,v_{j+1})) = 3m+2j$ for $1 \leq j \leq m-1$.

(iv) $\varphi_5((u_2,v_j)(u_3,v_j)) = \varphi_5((u_3,v_j)(u_2,v_{j+1})) = 7m-k-3$ for all j. Then $w_{\varphi_5}((u_2,v_j)(u_3,v_j)) = 5m+2j-3$ for $1 \leq j \leq m$ and $w_{\varphi_5}((u_2,v_j)(u_3,v_j)) = 5m+2j-2$ for $1 \leq j \leq m-1$.

(v) $\varphi_5((u_1,v_j)(u_5,v_{j+1})) = 7m-k-3$ and $\varphi_5((u_5,v_j)(u_1,v_{j+1})) = 8m-k-5$ for $1 \le j \le m-1$. For edge weights we have $w_{\varphi_5}((u_1,v_j)(u_5,v_{j+1})) = 7m+j-3$ and $w_{\varphi_5}((u_5,v_j)(u_1,v_{j+1})) = 8m+j-4$ for $1 \le j \le m-1$.

(vi) $\varphi_5((u_1,v_j)(u_5,v_j)) = 9m-k-5$ for $1 \le j \le m$.

Then $w_{\varphi_5}((u_1,v_j)(u_5,v_j)) = 9m+j-5$.

(vii) $\varphi_5((u_2,v_j)(u_3,v_{j+1})) = 12m-k-7$ and $\varphi_5((u_3,v_j)(u_3,v_{j+1})) = 14m-2k-6$ for $1 \le j \le m-1$. Then $w_{\varphi_5}((u_2,v_j)(u_3,v_{j+1})) = 10m+2j-6$ and $w_{\varphi_5}((u_3,v_j)(u_3,v_{j+1})) = 10m+2j-5$ for $1 \le j \le m-1$.

(viii) $\varphi_5((u_3,v_j)(u_4,v_j)) = \varphi_5((u_4,v_j)(u_3,v_{j+1})) = 15m-2k-8$ for all j. For edge weights we have $w_{\varphi_5}((u_3,v_j)(u_4,v_j)) = 12m+2j-8$ for $1 \le j \le m$ and $w_{\varphi_5}((u_4,v_j)(u_3,v_{j+1})) = 12m+2j-7$ for $1 \le j \le m-1$.

(ix) $\varphi_5((u_3,v_j)(u_4,v_{j+1})) = 17m-2k-10$ and $\varphi_5((u_4,v_j)(u_4,v_{j+1})) = 16m-2k-9$ for $1 \le j \le m-1$. Then $w_{\varphi_5}((u_3,v_j)(u_4,v_{j+1})) = 14m+2j-9$ and $w_{\varphi_5}((u_4,v_j)(u_4,v_{j+1})) = 14m+2j-8$ for $1 \le j \le m-1$.

(x) $\varphi_5((u_4,v_j)(u_5,v_j)) = \varphi_5((u_5,v_j)(u_4,v_{j+1})) = 17m-2k-11+j$ for all j. Then $w_{\varphi_5}((u_4,v_j)(u_5,v_j)) = 16m+2j-11$ for $1 \le j \le m$ and $w_{\varphi_5}((u_5,v_j)(u_4,v_{j+1})) = 16m+2j-10$ for $1 \le j \le m-1$.

(xi) $\varphi_5((u_4,v_j)(u_5,v_{j+1})) = 19m-2k-11$,and
$\varphi_5((u_5,v_j)(u_5,v_{j+1})) = 19m-2k-12+j$ for $1 \le j \le m-1$. For edge weights we have $w_{\varphi_5}((u_4,v_j)(u_5,v_{j+1})) = 18m+j-11$ and $w_{\varphi_5}((u_5,v_j)(u_5,v_{j+1})) = 19m+j-12$ for $1 \le j \le m-1$.

It is routine matter to check that under the labeling φ_5 the edge labels are at most k and the weights of the edges successively attain values from 3 up to $20m-13$. Thus the total k-labeling φ_5 has the required property. □

3. Main Result

Theorem. 1. 4 *Let $n \ge 3$, $m \ge 2$ be positive integers and $C_n \boxtimes P_m$ be the strong product of cycle C_n and path P_m. Then*

$$tes(C_n \boxtimes P_m) = \left\lceil \frac{n(4m-3)+2}{3} \right\rceil.$$

Proof. According to the lower bound of the total edge irregularity strength we have that $tes(C_n \boxtimes P_m) \geq \left\lceil \frac{|E(C_n \boxtimes P_m)|+2}{3} \right\rceil = \left\lceil \frac{n(4m-3)+2}{3} \right\rceil$. In order to show the converse inequality, it only remains to describe an edge irregular total $\left\lceil \frac{n(4m-3)+2}{3} \right\rceil$-labeling.

The cases when $m \geq 2$ and $3 \leq n \leq 5$ were discussed in the previous lemmas. Now we suppose that $n \geq 6$, $m \geq 2$ and $k = \left\lceil \frac{n(4m-3)+2}{3} \right\rceil$.

Define the function $\varphi_6 : V(C_n \boxtimes P_m) \to \{1, 2, \ldots, k\}$ as follows:

$$\varphi_6((u_i, v_j)) = \begin{cases} j, & \text{if } i = 1 \\ m(i-2) + j, & \text{if } i \in \left[2, \left\lceil \frac{n}{2} \right\rceil\right] \\ k - m(n-i) + j, & \text{if } i \in \left[\left\lceil \frac{n}{2} \right\rceil + 1, n-1\right] \\ k, & \text{if } i = n. \end{cases}$$

To complete the vertex labeling φ_6 to a total edge irregular one we label the edges of the graph $C_n \boxtimes P_m$ and describe the corresponding edge weights in the following way:

(i) $\varphi_6((u_1, v_j)(u_2, v_j)) = \varphi_6((u_2, v_j)(u_1, v_{j+1})) = 1$ for all j. For edge weights we have $w_{\varphi_6}((u_1, v_j)(u_2, v_j)) = \varphi_6((u_1, v_j)) + \varphi_6((u_1, v_j)(u_2, v_j)) + \varphi_6((u_2, v_j)) = 2j+1$ for $j \in [1, m]$ and $w_{\varphi_6}((u_2, v_j)(u_1, v_{j+1})) = \varphi_6((u_2, v_j)) + \varphi_6((u_2, v_j)(u_1, v_{j+1})) +$ $\varphi_6((u_1, v_{j+1})) = 2j+2$ for $j \in [1, m-1]$.

(ii) $\varphi_6((u_1, v_j)(u_1, v_{j+1})) = 2m - j$ for $j \in [1, m-1]$. Then $w_{\varphi_6}((u_1, v_j)) + \varphi_6((u_1, v_j)(u_1, v_{j+1})) + \varphi_6((u_1, v_{j+1})) = 2m + j + 1$ for $j \in [1, m-1]$.

(iii) $\varphi_6((u_1, v_j)(u_2, v_{j+1})) = 3m-2$ and $\varphi_6((u_2, v_j)(u_2, v_{j+1})) = 3m-1$ for $j \in [1, m-1]$. For edge weights we have $w_{\varphi_5}((u_1, v_j)(u_2, v_{j+1})) = 3m+2j-1$ and $w_{\varphi_5}((u_2, v_j)(u_2, v_{j+1})) = 3m+2j$ for $j \in [1, m-1]$.

(iv) $\varphi_6((u_i, v_j)(u_{i+1}, v_j)) = \varphi_6((u_{i+1}, v_j)(u_i, v_{j+1})) = (2m-3)i+3$ for all j and $i \in [2, \left\lceil \frac{n}{2} \right\rceil - 1]$. Then $w_{\varphi_6}((u_i, v_j)(u_{i+1}, v_j)) = (4m-3)i - 3m + 2j + 3$ for $j \in [1, m]$

and $w_{\varphi_6}((u_{i+1},v_j)(u_i,v_{j+1})) = (4m-3)i-3m+2j+4$ for $j\in[1,m-1]$ and $i\in[2,\left\lceil\frac{n}{2}\right\rceil-1]$.

(v) $\varphi_6((u_i,v_j)(u_{i+1},v_{j+1})) = (2m-3)i+2m+1$ and

$\varphi_6((u_{i+1},v_j)(u_{i+1},v_{j+1})) = (2m-3)i-m+5$ for $j\in[1,m-1]$ and $i\in[2,\left\lceil\frac{n}{2}\right\rceil-1]$. For edge weights we have $w_{\varphi_6}((u_i,v_j)(u_{i+1},v_{j+1})) = (4m-3)i-m+2j+2$ and $w_{\varphi_6}((u_{i+1},v_j)(u_{i+1},v_{j+1})) = (4m-3)i-m+2j+3$ for $j\in[1,m-1]$ and $i\in[2,\left\lceil\frac{n}{2}\right\rceil-1]$.

(vi) $\varphi_6((u_1,v_j)(u_n,v_{j+1})) = (4m-3)\lceil\frac{n}{2}\rceil-3m-k+4$ and

$\varphi_6((u_n,v_j)(u_1,v_{j+1})) = (4m-3)\lceil\frac{n}{2}\rceil-2m-k+2$ for $j\in[1,m-1]$. For edge weights we have $w_{\varphi_6}((u_1,v_j)(u_n,v_{j+1})) = (4m-3)\lceil\frac{n}{2}\rceil-3m+j+4$ and $w_{\varphi_6}((u_n,v_j)(u_1,v_{j+1})) = (4m-3)\lceil\frac{n}{2}\rceil-2m+j+3$ for $j\in[1,m-1]$.

(vii) $\varphi_6((u_1,v_j)(u_n,v_j)) = (4m-3)\lceil\frac{n}{2}\rceil-m-k+2$ for $j\in[1,m]$. Then $w_{\varphi_6}((u_1,v_j)(u_n,v_j)) = (4m-3)\lceil\frac{n}{2}\rceil-m+j+2$.

(viii)

$\varphi_6((u_{\lceil\frac{n}{2}\rceil},v_j)(u_{\lceil\frac{n}{2}\rceil+1},v_j)) = \varphi_6((u_{\lceil\frac{n}{2}\rceil+1},v_j)(u_{\lceil\frac{n}{2}\rceil},v_{j+1})) = (2m-3)\lceil\frac{n}{2}\rceil+m(n+1)+1-k$

for all j. Then $w_{\varphi_6}((u_{\lceil\frac{n}{2}\rceil},v_j)(u_{\lceil\frac{n}{2}\rceil+1},v_j)) = (4m-3)\lceil\frac{n}{2}\rceil+2j+1$ for $j\in[1,m]$ and $w_{\varphi_6}((u_{\lceil\frac{n}{2}\rceil+1},v_j)(u_{\lceil\frac{n}{2}\rceil},v_{j+1})) = (4m-3)\lceil\frac{n}{2}\rceil+2j+2$ for $j\in[1,m-1]$.

(ix) $\varphi_6((u_{\lceil\frac{n}{2}\rceil},v_j)(u_{\lceil\frac{n}{2}\rceil+1},v_{j+1})) = (2m-3)\lceil\frac{n}{2}\rceil+m(n+3)-k-1$ and

$\varphi_6((u_{\lceil\frac{n}{2}\rceil+1},v_j)(u_{\lceil\frac{n}{2}\rceil+1},v_{j+1})) = (2m-3)(\lceil\frac{n}{2}\rceil)+2(mn-k)$ for $j\in[1,m-1]$. Then

$w_{\varphi_6}((u_{\lceil \frac{n}{2} \rceil}, v_j)(u_{\lceil \frac{n}{2} \rceil+1}, v_{j+1})) = (4m-3)\lceil \frac{n}{2} \rceil + 2m + 2j$ and

$w_{\varphi_6}((u_{\lceil \frac{n}{2} \rceil+1}, v_j)(u_{\lceil \frac{n}{2} \rceil+1}, v_{j+1})) = (4m-3)\lceil \frac{n}{2} \rceil + 2m + 2j + 1$ for $j \in [1, m-1]$.

(x) $\varphi_6((u_i, v_j)(u_{i+1}, v_j)) = \varphi_6((u_{i+1}, v_j)(u_i, v_{j+1})) = (2m-3)i + m(2n-1) - 2k + 1$ for all j and $i \in [\left\lceil \frac{n}{2} \right\rceil + 1, n-2]$. Then $w_{\varphi_6}((u_i, v_j)(u_{i+1}, v_j)) = (4m-3)i + 2j + 1$ for $j \in [1, m]$ and $w_{\varphi_6}((u_{i+1}, v_j)(u_i, v_{j+1})) = (4m-3)i + 2j + 2$ for $j \in [1, m-1]$ and $i \in [\left\lceil \frac{n}{2} \right\rceil - 1, n-2]$.

(xi) $\varphi_6((u_i, v_j)(u_{i+1}, v_{j+1})) = (2m-3)i + m(2n+1) - 2k - 1$ and $\varphi_6((u_{i+1}, v_j)(u_{i+1}, v_{j+1})) = (2m-3)(i-1) + 2(mn-k)$ for $j \in [1, m-1]$ and $i \in [\left\lceil \frac{n}{2} \right\rceil + 1, n-2]$. For edge weights we have $w_{\varphi_6}((u_i, v_j)(u_{i+1}, v_{j+1})) = (4m-3)i + 2m + 2j$ and $w_{\varphi_6}((u_{i+1}, v_j)(u_{i+1}, v_{j+1})) = (4m-3)i + 2m + 2j + 1$ for $j \in [1, m-1]$ and $i \in [\left\lceil \frac{n}{2} \right\rceil + 1, n-2]$.

(xii) $\varphi_6((u_{n-1}, v_j)(u_n, v_j)) = \varphi_6((u_n, v_j)(u_{n-1}, v_{j+1})) = m(4n-3) - 2k - 3n + 4 + j$ for all j. Then $w_{\varphi_6}((u_{n-1}, v_j)(u_n, v_j)) = (4m-3)n - 4m + 2j + 4$ for $j \in [1, m]$ and $w_{\varphi_6}((u_{i+1}, v_j)(u_i, v_{j+1})) = (4m-3)n - 4m + 2j + 5$ for $j \in [1, m-1]$.

(xiii) $\varphi_6((u_{n-1}, v_j)(u_n, v_{j+1})) = n(4m-3) - 2k - m + 4$ and $\varphi_6((u_n, v_j)(u_n, v_{j+1})) = n(4m-3) - 2k - m + 3 + j$ for $j \in [1, m-1]$. Then $w_{\varphi_6}((u_{n-1}, v_j)(u_n, v_{j+1})) = (4m-3)n - 2m + j + 4$ and $w_{\varphi_6}((u_n, v_j)(u_n, v_{j+1})) = (4m-3)n - m + 3 + j$ for $j \in [1, m-1]$.

One can see that the edge weights under the total labeling φ_6 receive distinct labels from the set $\{3, 4, \ldots, n(4m-3)+2\}$ and that φ_6 is an edge irregular total labeling having the required property. □

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