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# ITERATIVE ALGORITHMS FOR SOLVING A CLASS OF QUASI VARIATIONAL INEQUALITIES

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*In this paper, we introduce and study a new class of quasi variational inequalities, known as multivalued extended general quasi variational inequalities. It is shown that the multivalued extended general quasi variational inequalities are equivalent to the fixed point problems. We use this alternative equivalent formulation to suggest and analyze some iterative methods. We consider the convergence analysis of an iterative method under suitable conditions. We also introduce a new class of Wiener-Hopf equations, known as multivalued extended general implicit Wiener-Hopf equations. We establish the equivalence between the multivalued extended general quasi variational inequalities and multivalued extended general implicit Wiener-Hopf equations. Using this equivalence, we suggest and analyze some iterative methods. Several special cases are also discussed. The ideas and techniques of this paper may stimulate further research in this field.*

**Keywords:** Quasi variational inequalities; Projection operator; Iterative methods; Convergence; Wiener-Hopf equations.

**MSC2010:** 49J40, 90C33.

## 1. Introduction

Variational inequality problem was introduced by Stampacchia [43]. Variational inequality theory is an important branch of mathematics due to its vast applications in both pure and applied sciences. This field is dynamic and is experiencing an explosive growth in both theory and applications; as a consequence, research techniques and problems are drawn from various fields. The ideas and techniques of variational inequalities are being applied in a variety of diverse areas of sciences and prove to be productive and innovative. It has been shown that this theory provides the most natural, direct, simple, unified and efficient framework for a general treatment of a wide class of unrelated linear and nonlinear problems, see [1-44] and references therein.

Noor [20] introduced and studied an important class of variational inequalities using two operators, which is known as general variational inequality. It turned out that the odd-order and nonsymmetric obstacle, free, unilateral and moving boundary value problems can be studied via the general variational inequality. Noor [22] has shown that the general variational inequality are equivalent to the Wiener-Hopf equations, which were introduced and studied by Shi [42]. Noor [22] has used this alternative equivalent formulation to suggest several iterative methods for solving the general variational inequality and the related problems.

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If the convex set, which involved in the variational inequality, also depends upon the solution implicitly or explicitly, then the variational inequality is called the quasi variational inequality. Quasi variational inequalities were introduced and studied by Bensoussan and Lions [3]. They showed that the quasi variational inequality gives a natural and unified framework for studying the impulse control theory. Chan and Pang [4] introduced and studied the generalized quasi variational inequality problems. It has been shown that the generalized quasi variational inequality contains the quasi variational inequality problem as a special case.

Noor [18] has shown that the quasi variational inequality is equivalent to the fixed point problem using the projection technique. Using this equivalence, he suggested an iterative method for solving the quasi variational inequality. Noor [21] introduced and studied a new class of quasi variational inequality, which is called a general quasi variational inequality involving two operators. Noor [21] proved that the general quasi variational inequality is equivalent to a fixed point problem. Using this fixed point equivalence, he proposed an iterative method for solving the general quasi variational inequality and discussed the convergence of the proposed iterative method.

Motivated and inspired by the research going on in this area, Noor [24] introduced and studied a new class of quasi variational inequality, which known as generalized multivalued quasi variational inequality. It has been shown that this class is most general and includes many classes of quasi variational inequalities as a special cases. He has considered the existence of a solution of problem using the projection technique and proposed a number of iterative methods for solving the generalized multivalued quasi variational inequality. He also established an equivalence relation between generalized multivalued quasi variational inequality and the implicit Wiener-Hopf equations. He used this equivalence to suggest a class of iterative methods for solving generalized multivalued quasi variational inequality, also see [23, 25] and references therein.

It is well known that the convexity plays an important role in the study of variational inequality and its variant forms. The optimality of a differentiable convex function on a convex set can be characterized by the variational inequality. In recent years, the concept of convexity has been generalized in many dimensions, see [5] and references therein. Youness [44] introduced a concept of nonconvex sets and nonconvex functions. For the properties of the nonconvex functions see [26, 37]. Noor [31] has considered and studied the nonconvex function relative to two arbitrary functions. It has shown [31] that the minimum of a differentiable nonconvex function involving three functions can be characterized by a class of variational inequalities, which is called the extended general variational inequality. It also has been shown that many classes of variational inequalities are the special cases of extended general variational inequality. It has been proved that the extended general variational inequalities are equivalent to the fixed point problems. This alternative fixed point formulation is used to study the existence of a solution of extended general variational inequality as well as to develop some iterative methods. Noor [28] has also studied the existence of a solution of extended general variational inequality via auxiliary principle technique. Liu and Cao [15] proposed a recurrent neural network based on the projection operator for solving the extended general variational inequality.

Noor and Noor [36] considered a new class of quasi variational inequality involving three nonlinear operators, called extended general quasi variational inequality. Noor et al. [37] proposed some iterative schemes for solving the extended general quasi variational inequality and discussed the convergence of the iterative methods. Recently, Antipin et al. [1] proposed a second-order iterative method using projection technique for solving quasi variational inequalities. They have considered the convergence criteria of the proposed iterative method. For the dynamical systems associated with quasi variational inequalities and related problems, see Noor et al. [39].

In this paper, we introduce and consider a new class of quasi variational inequality, known as multivalued extended general quasi variational inequality. It is shown that several known classes of variational and quasi variational inequalities, which have been studied by many researchers, are the special cases of this new class. In section 2, we give some basic results, facts, formulate problem, and discuss some special cases of the multivalued extended general quasi variational inequality. In section 3, the equivalence between multivalued quasi variational inequalities and the fixed point problem is established using the projection techniques. This alternative equivalent formulation is used to discuss the existence theory of multivalued extended general quasi variational inequalities. It is observed that for the existence of a solution of the concerned problem, the operators should be strongly monotone and Lipschitz continuous. In section 4, we introduce some new iterative methods for solving the new class of quasi variational inequalities using the equivalent fixed point formulation. We discuss the convergence of the iterative methods for solving the variational inequality under the same conditions which we imposed on the operators in section 3. In section 5, we introduce a new class of Wiener Hopf equation, called multivalued extended general implicit Wiener Hopf equation. We establish the equivalence between the new class of quasi variational inequality and Weiner Hopf equations. This alternative equivalent formulation is used to suggest a number of new iterative methods for solving the multivalued extended general quasi variational inequality. The convergence analysis of these methods is also investigated under some suitable conditions. Several special cases are discussed. Since the multivalued extended general quasi variational inequalities include several new and known classes of (quasi) variational inequalities as special cases, results derived in this paper continue to hold for these problems. The comparison of these new methods with other methods is an interesting and challenging problem for future research.

## 2. **Formulation and Basic Results**

Let $H$ be a real Hilbert space, whose norm and inner product are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. Let $C(H)$ be a family of all nonempty compact subsets of $H$. Let $T,\ V : H \to C(H)$ be the multivalued operators. Let $h_1,\ h_2 : H \to H$ and $N(\cdot,\cdot) : H \times H \to H$ be the single valued operators.

Given a point-to-set mapping $\Omega : u \to \Omega(u)$, which associates a closed convex valued set $\Omega(u)$ with any element $u \in H$, we consider problem of finding $u,\ w,\ y \in H : w \in T(u),\ y \in V(u),\ h_1(u),\ h_2(u) \in \Omega(u)$, and

$$\langle \rho N(w,y) + h_2(u) - h_1(u), h_1(v) - h_2(u)\rangle \geq 0, \quad \forall v \in H : h_1(v) \in \Omega(u) \tag{1}$$

where $\rho > 0$, is a constant. Problem (1) is called the multivalued extended general quasi-variational inequality. It has many applications in the field of mechanics, physics, pure and applied sciences, see [6, 7, 12, 14, 15] and references therein. Furthermore, there are problems arising in structural analysis, which can be studied by the problem (1) only.

**Example 2.1.** For simplicity, and to convey an idea of the applications of the problem (1), we consider an elastoplasticity problem, which is mainly due to Panagiotopoulos and Stavroulakis [40] and Noor [25]. It is assumed that a general hyperelastic material law holds for the elastic behavior of the elastoplastic material under consideration. Moreover, a nonconvex yield function $\sigma \to F(\sigma)$ is introduced for the plasticity. For the basic definitions and concepts, see [40]. Let us assume the decomposition

$$E = E^e + E^p, \tag{2}$$

where $E^e$ denotes the elastic and $E^p$, the plastic deformation of the three-dimensional elastoplastic body. We write the complementarity virtual work expression for the body in the form

$$\langle E^e, \tau - \sigma\rangle + \langle E^p, \tau - \sigma\rangle = \langle f, \tau - \sigma\rangle, \quad \text{for all } \tau \in Z. \tag{3}$$

Here we have assumed that the body on a part $\Gamma_F = \Gamma - \Gamma_U$, the boundary tractions are given, that is, $S_i = F_i$ on $\Gamma_F$, where

$$\langle E, \sigma \rangle = \int_{\Omega} \epsilon_{ij} \sigma_{ij} d\Omega, \tag{4}$$

$$\langle f, \sigma \rangle = \int_{\Gamma_U} U_i S_i d\Gamma, \tag{5}$$

$$Z = \{\tau : \tau_{i,j} + f_i = 0 \text{ on } \Omega,\ i,\ j = 1,2,3,\ T_i = F_i \text{ on } \Gamma_F,\ i = 1,2,3\}, \tag{6}$$

is the set of statically admissible stresses and $\Omega$ is the structure of the body.

Let us assume that the material of the structure $\Omega$ is hyperelastic such that

$$\langle E^e, \tau - \sigma \rangle \leq \langle W_m' (\sigma), \tau - \sigma \rangle, \quad \text{for all } \tau \in \mathfrak{R}^6, \tag{7}$$

where $W_m$ is the superpotential which produces the constitutive law of the hyperelastic material and is assumed to be quasidifferentiable [40], that is, there exist convex and compact subsets $\underline{\partial}' W_m$ and $\overline{\partial}' W_m$ such that

$$\langle W_m' (\sigma), \tau - \sigma \rangle = \max_{W_1^e \in \underline{\partial}' W_m} \langle W_1^e, \tau - \sigma \rangle + \max_{W_2^e \in \overline{\partial}' W_m} \langle W_2^e, \tau - \sigma \rangle. \tag{8}$$

We also introduce the generally nonconvex implicit yield function $P(\sigma) \subset Z$, which is denoted by means of the general quasi-differentiable function $F(\sigma)$, that is,

$$P(\sigma) = \{\sigma \in Z; F(\sigma) \leq \sigma\}. \tag{9}$$

Here $W_m$ is generally nonconvex and nonsmooth, but quasi-diffrentiable function for the case of plasticity with convex yield surface and hyperelasticity. Combining $(2)-(9)$ and using the technique of Panagiotopoulos and Stavroulakis [40], we can obtain the following multivalued variational inequality problem: find $\sigma \in P(\sigma)$ such that $W_1^e \in \underline{\partial}' W_m$, $W_2^e \in \overline{\partial}' W_m$, and

$$\langle W_1^e + W_2^e, \tau - \sigma \rangle \geq \langle f, \tau - \sigma \rangle, \quad \text{for all } \tau \in P(\sigma),$$

which is exactly problem (1) with $N(w, y) = W_1^e + W_2^e$, $h_1 = h_2 = I$, the identity operator,

$$T(u) = \underline{\partial}' W_m, \quad V(u) = \overline{\partial}' W_m, \quad \text{and} \quad \Omega(u) = P(\sigma).$$

For other applications of problem (1), see [6, 7, 12, 14, 15] and the references therein.

We now discuss some special cases of problem (1).

I. If $\Omega(u) \equiv \Omega$, that is, the convex set $\Omega$ is independent of a solution $u$, then problem (1) is equivalent to finding $u,\ w,\ y \in H : w \in T(u),\ y \in V(u),\ h_1(u),\ h_2(u) \in \Omega$, and

$$\langle \rho N(w, y) + h_2(u) - h_1(u), h_1(v) - h_2(u) \rangle \geq 0, \tag{10}$$

for all $\forall v \in H : h_1(v) \in \Omega$, This problem is known as multivalued extended general variational inequality.

II. If $h_1 \equiv h_2$, then problem (1) is equivalent to finding $u,\ w,\ y \in H : w \in T(u),\ y \in V(u),\ h_1(u) \in \Omega(u)$, and

$$\langle \rho N(w, y), h_1(v) - h_1(u) \rangle \geq 0, \quad \forall v \in H : h_1(v) \in \Omega(u) \tag{11}$$

which is known as generalized multivalued quasi-variational inequality. The problem (11) was introduced and studied by Noor [25].

III. If $h_1 \equiv h_2$, and $\Omega(u) = \Omega$, then problem (1) is equivalent to finding $u,\ w,\ y \in H : w \in T(u),\ y \in V(u),\ h_1(u) \in \Omega$, and

$$\langle \rho N(w, y), h_1(v) - h_1(u) \rangle \geq 0, \quad \forall v \in H : h_1(v) \in \Omega, \tag{12}$$

which is known as generalized multivalued variational inequality.

IV. If $h_1 \equiv h_2 \equiv I$, then problem (1) is equivalent to finding $u,\ w,\ y \in H : w \in T(u)$, $y \in V(u)$, $u \in \Omega(u)$, and

$$\langle \rho N(w, y), v - u \rangle \geq 0, \quad \forall v \in \Omega(u), \tag{13}$$

which is known as multivalued quasi-variational inequality, see Noor [24].

V. If the operator $V \equiv 0$ and $T,\ h_1,\ h_2 : H \to H$ are single valued operators, then problem (1) is equivalent to finding $u \in H : h_1(u) \in \Omega(u)$ and

$$\langle \rho (T(u)) + h_2(u) - h_1(u), h_1(v) - h_2(u) \rangle \geq 0, \tag{14}$$

for all $v \in H : h_1(v) \in \Omega(u)$, which is known as the extended general quasi-variational inequality, introduced and studied by Noor and Noor [36] and Noor et al. [38, 39].

VI. If $h_1 \equiv h_2$ and the operators $T,\ V,\ h_1 : H \to H$ are single valued operators, then problem (1) is equivalent to finding $u \in H : h_1(u) \in \Omega(u)$ and

$$\langle \rho (T(u) + V(u)), h_1(v) - h_1(u) \rangle \geq 0, \quad \forall v \in H : h_1(v) \in \Omega(u). \tag{15}$$

It is known as the mildly nonlinear quasi-variational inequality and is due to Noor, see [24].

VII. For the single valued operators $T,\ V : H \to H$, $h_1 \equiv h_2 \equiv I$, the identity operator, and $\Omega(u) = \Omega$, then problem (1) is equivalent to finding $u \in \Omega$ such that

$$\langle \rho (T(u) + V(u)), v - u \rangle \geq 0, \quad \forall v \in \Omega. \tag{16}$$

which is called mildly nonlinear variational inequality, introduced and studied by Noor [17] in 1975.

VIII. If $V = 0, h_1 \equiv h_2 \equiv I$, the identity operator, and for the single valued operator $T : H \to H$, and $\Omega(u) = \Omega$, then problem (1) is equivalent to finding $u \in \Omega$ such that

$$\langle \rho (Tu), v - u \rangle \geq 0, \quad \forall v \in \Omega, \tag{17}$$

which is the well known original variational inequality. It was introduced and studied by Stampacchia [43]. For the recent applications, generalization, numerical methods and other aspects of quasi variational inequalities and related optimization problems, see $[1 - 43]$.

**Lemma 2.1.** *For a given $z \in H,\ u \in \Omega$ satisfies the inequality*

$$\langle u - z, v - u \rangle \geq 0, \quad \forall v \in \Omega,$$

*if and only if*

$$u = P_\Omega [z],$$

*where $P_\Omega$ is the projection of $H$ into a closed and convex set $\Omega$.*

It is well known that projection operator $P_\Omega$ is nonexpansive, that is,

$$\|P_\Omega [u] - P_\Omega [v]\| \leq \|u - v\|, \quad \forall u, v \in H.$$

We now define the concept of strongly monotonicity for the bifunction operator $N(\cdot,\cdot)$, which was introduced by Noor [25].

**Definition 2.1.** *The single valued operator $N(\cdot,\cdot)$ is said to be strongly monotone with respect to the first argument if, for all $u_1,\ u_2 \in H$, there exists a constant $\alpha > 0$, such that*

$$\langle N(w_1,\cdot) - N(w_2,\cdot), u_1 - u_2 \rangle \geq \alpha \|u_1 - u_2\|^2, \quad \forall w_1 \in T(u_1),\ w_2 \in T(u_2).$$

**Definition 2.2.** *The single valued operator $N(\cdot,\cdot)$ is said to be Lipschitz continuous with respect to the first argument, if there exists a constant $\beta > 0$, such that*

$$\|N(u_1,\cdot) - N(u_2,\cdot)\| \leq \beta \|u_1 - u_2\|, \quad \forall u_1,\ u_2 \in H.$$

Similarly, we can define the strongly monotonicity and Lipschitz continuity of the operator $N(\cdot,\cdot)$ with respect to the second argument.

**Definition 2.3.** *The set valued operator $V : H \to C(H)$ is said to be M-Lipschitz continuous, if there exists a constant $\xi > 0$ such that*

$$M(V(u_1), V(u_2)) \leq \xi \|u_1 - u_2\|, \quad \forall u_1,\ u_2 \in H,$$

*where $C(H)$ is the family of all nonempty compact subsets of $H$ and $M(\cdot,\cdot)$ is the Hausdorff metric on $C(H)$, that is for any two nonempty subsets $A$ and $B$ of $H$,*

$$M(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(A,y)\right\},$$

*where*

$$d(x,B) = \inf_{y\in B}\|x-y\| \quad \text{and} \quad d(A,y) = \inf_{x\in A}\|x-y\|.$$

In order to prove our main results, the next lemma is very important.

**Lemma 2.2** ([16])**.** *Let $(H,d)$ be a complete metric space, $T : H \to CB(H)$ be a set-valued mapping. Then, for all $x,\ y \in H$, $u \in T(x)$, there exists $v \in T(y)$ such that*

$$\|u - v\| \leq M(T(x), T(y)).$$

## 3. **Existence Theory**

In this section, we show that the multivalued extended general quasi-variational inequality (1) is equivalent to a fixed point problem using Lemma 2.1. We use this alternative equivalent formulation to discuss the existence of a solution of problem (1).

**Lemma 3.1.** *Let $\Omega(u)$ be a closed and convex valued set in $H$. Then $u,w,y \in H$ is a solution of* (1) *if and only if $u,w,y \in H$ satisfies the relation*

$$h_2(u) = P_{\Omega(u)}[h_1(u) - \rho N(w,y)],$$

*where $\rho > 0$ is a constant and $P_{\Omega(u)}$ is the projection of $H$ onto the closed convex-valued set $\Omega(u)$.*

Thus from Lemma 3.1, we see that problem (1) is equivalent to a fixed point problem. We remark that the implicit projection operator $P_{\Omega(u)}$ is not nonexpansive. However it is known [35, 36] that the implicit projection operator $P_{\Omega(u)}$ satisfies the Lipschitz continuity type condition. This condition plays an important role in the existence theory of problem (1) and the convergence analysis of the iterative algorithms.

**Assumption 3.1.** *For a constant $\nu > 0$, the implicit projection operator $P_{\Omega(u)}$ satisfies the condition*

$$\|P_{\Omega(u)}[w] - P_{\Omega(v)}[w]\| \leq \nu\|u - v\|, \quad \text{for all } u,v,w \in H. \tag{18}$$

We now show that Assumption 3.1 holds for some special cases. We consider the case, for which the convex-valued set $\Omega(u)$ can be defined as:

$$\Omega(u) = m(u) + \Omega,$$

where $\Omega$ is a closed and convex set and $m$ is a point-to-point mapping. In this case, we have

$$P_{\Omega(u)}[w] = P_{m(u)+\Omega}[w] = m(u) + P_\Omega[w - m(u)],$$

where $P_\Omega$ is the projection operator of $H$ onto the convex set $\Omega$.

If the mapping $m(u)$ is a Lipschitz continuous with constant $\gamma > 0$, then for all $u, v, w \in H$, we have

$$\begin{aligned}
\|P_{\Omega(u)}[w] - P_{\Omega(v)}[w]\| &= \|P_{m(u)+\Omega}[w] - P_{m(v)+\Omega}[w]\| \\
&= \|m(u) + P_\Omega[w - m(u)] - m(v) - P_\Omega[w - m(v)]\| \\
&\leq \|m(u) - m(v)\| + \|P_\Omega[w - m(u)] - P_\Omega[w - m(v)]\| \\
&\leq 2\|m(u) - m(v)\| \leq 2\gamma\|u - v\|,
\end{aligned}$$

which shows that the Assumption 3.1 holds for $\nu = 2\gamma > 0$.

We now discuss the existence of a solution of problem (1) and this is the main motivation of our next result.

**Theorem 3.1.** *Let $\Omega(u)$ be a closed and convex valued set in $H$. Let the operator $N(\cdot,\cdot)$ be strongly monotone with respect to the first argument with constant $\alpha > 0$ and Lipschitz continuous with respect to the first argument with constant $\beta > 0$. Let operators $h_1, h_2 \colon H \to H$ be strongly monotone with constants $\sigma_1 > 0$, $\sigma_2 > 0$ and Lipschitz continuous with constants $\delta_1 > 0$, $\delta_2 > 0$, respectively. Assume that the operator $N(\cdot,\cdot)$ is Lipschitz continuous with respect to the second argument with constant $\eta > 0$. Let $T$, $V \colon H \to C(H)$ are M-Lispchitz continuous mappings with constants $\mu > 0$ and $\xi > 0$ respectively. If Assumption* 3.1 *holds and*

$$\begin{aligned}
\theta &= k + t(\rho) + \rho\eta\xi < 1, &(19)\\
k &= \nu + \sqrt{1 - 2\sigma_1 + \delta_1^2} + \sqrt{1 - 2\sigma_2 + \delta_2^2}, &(20)\\
t(\rho) &= \sqrt{1 - 2\rho\alpha + \rho^2\beta^2\mu^2}, &(21)
\end{aligned}$$

*then there exists a solution $u$, $w$, $y \in H : w \in T(u)$, $y \in V(u)$, and $h_1(u)$, $h_2(u) \in \Omega(u)$ satisfying the problem* (1).

*Proof.* Let $u \in H$ be a solution of problem (1), then by Lemma 3.1, we have

$$h_2(u) = P_{\Omega(u)}[h_1(u) - \rho N(w, y)], \tag{22}$$

which can be used to define the mapping $F(u)$ as:

$$F(u) = u - h_2(u) + P_{\Omega(u)}[h_1(u) - \rho N(w, y)]. \tag{23}$$

To prove the existence of a solution of $(1)$, it is enough to show that problem (23) has a fixed point. Thus, for all $u_1 \neq u_2 \in H$, let $T(u_1) \ni w_1 \neq w_2 \in T(u_2)$ and $V(u_1) \ni y_1 \neq y_2 \in V(u_2)$ such that $\|w_1 - w_2\| \leq M(T(u_1), T(u_2))$ and $\|y_1 - y_2\| \leq M(V(u_1), V(u_2))$, consider

$$\begin{aligned}
&\|F(u_1) - F(u_2)\| \\
\leq\ &\|(u_1 - u_2) - \{h_2(u_1) - h_2(u_2)\}\| \\
&+ \left\|P_{\Omega(u_1)}[h_1(u_1) - \rho N(w_1, y_1)] - P_{\Omega(u_2)}[h_1(u_2) - \rho N(w_2, y_2)]\right\| \\
\leq\ &\|(u_1 - u_2) - \{h_2(u_1) - h_2(u_2)\}\| \\
&+ \left\|P_{\Omega(u_1)}[h_1(u_1) - \rho N(w_1, y_1)] - P_{\Omega(u_2)}[h_1(u_1) - \rho N(w_1, y_1)]\right\| \\
&+ \left\|P_{\Omega(u_2)}[h_1(u_1) - \rho N(w_1, y_1)] - P_{\Omega(u_2)}[h_1(u_2) - \rho N(w_2, y_2)]\right\| \\
\leq\ &\|(u_1 - u_2) - \{h_2(u_1) - h_2(u_2)\}\| + \nu\|u_1 - u_2\| \\
&+ \|\{h_1(u_1) - h_1(u_2)\} - \rho\{N(w_1, y_1) - N(w_2, y_2)\}\| \\
\leq\ &\nu\|u_1 - u_2\| + \|(u_1 - u_2) - \{h_2(u_1) - h_2(u_2)\}\| \\
&+ \|(u_1 - u_2) - \{h_1(u_1) - h_1(u_2)\}\| \\
&+ \|(u_1 - u_2) - \rho\{N(w_1, y_1) - N(w_2, y_1)\}\| \\
&+ \rho\|N(w_2, y_1) - N(w_2, y_2)\|,
\end{aligned} \tag{24}$$

where we have used Assumption 3.1.

Since $N(\cdot,\cdot)$ is strongly monotone with respect to the first argument with constant $\alpha > 0$ and Lipschitz continuous with respect to the first argument with constant $\beta > 0$, and $T$ is M-Lipschitz continuous operator with constant $\mu > 0$, therefore

$$\begin{aligned} & \|(u_1 - u_2) - \rho\{N(w_1, y_1) - N(w_2, y_1)\}\|^2 \\ = \; & \|u_1 - u_2\|^2 - 2\rho\langle N(w_1, y_1) - N(w_2, y_1), u_1 - u_2\rangle + \rho^2\|N(w_1, y_1) - N(w_2, y_1)\|^2 \\ \leq \; & \|u_1 - u_2\|^2 - 2\rho\alpha\|u_1 - u_2\|^2 + \rho^2\beta^2\|w_1 - w_2\|^2 \\ \leq \; & \|u_1 - u_2\|^2 - 2\rho\alpha\|u_1 - u_2\|^2 + \rho^2\beta^2\{M(T(u_1), T(u_2))\}^2 \\ \leq \; & \|u_1 - u_2\|^2 - 2\rho\alpha\|u_1 - u_2\|^2 + \rho^2\beta^2\mu^2\|u_1 - u_2\|^2 \\ = \; & \left(1 - 2\rho\alpha + \rho^2\beta^2\mu^2\right)\|u_1 - u_2\|^2. \end{aligned} \tag{25}$$

Since $h_1$ and $h_2$ are strongly monotone with constants $\sigma_1 > 0$, $\sigma_2 > 0$ and Lipschitz continuous with constants $\delta_1 > 0$, $\delta_2 > 0$ respectively, therefore we have

$$\|(u_1 - u_2) - \{h_1(u_1) - h_1(u_2)\}\|^2 \leq \left(1 - 2\sigma_1 + \delta_1^2\right)\|u_1 - u_2\|^2. \tag{26}$$

and

$$\|(u_1 - u_2) - \{h_2(u_1) - h_2(u_2)\}\|^2 \leq \left(1 - 2\sigma_2 + \delta_2^2\right)\|u_1 - u_2\|^2. \tag{27}$$

Since $N(\cdot,\cdot)$ is also Lipschitz continuous with respect to the second argument with constant $\eta > 0$ and $V$ is M-Lipschitz continuous with constant $\xi > 0$, therefore we have

$$\begin{aligned} \|N(w_2, y_1) - N(w_2, y_2)\| & \leq \eta\|y_1 - y_2\| \\ & \leq \eta M(V(u_1), V(u_2)) \\ & \leq \eta\xi\|u_1 - u_2\|. \end{aligned} \tag{28}$$

Combining (19) and (24) – (28), we have

$$\begin{aligned} \|F(u_1) - F(u_2)\| & \leq \{\nu + \sqrt{1 - 2\sigma_1 + \delta_1^2} + \sqrt{1 - 2\sigma_2 + \delta_2^2} \\ & \quad + \sqrt{1 - 2\rho\alpha + \rho^2\beta^2\mu^2} + \eta\xi\rho\}\|u_1 - u_2\| \\ & = \{k + t(\rho) + \eta\xi\rho\}\|u_1 - u_2\| \\ & = \theta\|u_1 - u_2\|, \end{aligned} \tag{29}$$

From condition (19), it follows that $\theta < 1$. This shows that mapping $F(u)$ defined by (23) is a contraction mapping and consequently it has a unique fixed point $u, w, y \in H$ satisfying problem (1).This completes the proof. □

## 4. **Iterative Methods**

In this section, we develop and discuss some iterative methods for solving problem (1). We also consider the convergence analysis of these iterative methods.

From Lemma 3.1, we have

$$u = u - h_2(u) + P_{\Omega(u)}[h_1(u) - \rho N(w, y)].$$

For a controlling parameter $0 < \lambda < 1$, we have

$$u = (1 - \lambda)u + \lambda\left\{u - h_2(u) + P_{\Omega(u)}[h_1(u) - \rho N(w, y)]\right\}. \tag{30}$$

This fixed point formulation enables us to suggest the following iterative algorithm, for solving problem (1).

**Algorithm 4.1.** *Assume that $T$, $V: H \to C(H)$, are multivalued operators. Let $N: H \times H \to H$, $h_1$, $h_2: H \to H$ are single valued operators. Let $\Omega(u)$ be a closed convex valued* *set in a real Hilbert space $H$. For given $u_0$, $w_0$, $y_0 \in H$, let $w_0 \in T(u_0)$, $y_0 \in V(u_0)$, $h_1(u_0) \in \Omega(u_0)$, $h_2(u_0) \in \Omega(u_0)$ and*

$$u_1 = (1-\lambda)u_0 + \lambda\left\{u_0 - h_2(u_0) + P_{\Omega(u_0)}\left[h_1(u_0) - \rho N(w_0, y_0)\right]\right\}.$$

*Using Lemma 2.2; since $w_0 \in T(u_0)$, $y_0 \in V(u_0)$, then there exist $w_1 \in T(u_1)$, $y_1 \in V(u_1)$ such that*

$$\|w_0 - w_1\| \leq M(T(u_0), T(u_1))$$
$$\|y_0 - y_1\| \leq M(V(u_0), V(u_1)),$$

*where $M(\cdot,\cdot)$ is the Hausdorff metric on $C(H)$. Let*

$$u_2 = (1-\lambda)u_1 + \lambda\left\{u_1 - h_2(u_1) + P_{\Omega(u_1)}\left[h_1(u_1) - \rho N(w_1, y_1)\right]\right\}.$$

*By continuing this process, we can obtain the sequences $\{u_n\}$, $\{w_n\}$, $\{y_n\}$ such that*

$$w_n \in T(u_n) : \|w_{n+1} - w_n\| \leq M(T(u_{n+1}), T(u_n)) \tag{31}$$

$$y_n \in V(u_n) : \|y_{n+1} - y_n\| \leq M(V(u_{n+1}), V(u_n)) \tag{32}$$

$$u_{n+1} = (1-\lambda)u_n + \lambda\left\{u_n - h_2(u_n) + P_{\Omega(u_n)}\left[h_1(u_n) - \rho N(w_n, y_n)\right]\right\}, \tag{33}$$

*for $n = 0, 1, 2, \ldots$.*

We now discuss some special cases of Algorithm 4.1, which are used to solve some important classes of variational and quasi variational inequalities.

I. If $T, V : H \to C(H)$ are multivalued operators and $h_2 = h_1$, then Algorithm 4.1 reduces to the following algorithm which is used to find a solution of problem (11).

**Algorithm 4.2.** *For given $u_0$, $w_0$, $y_0 \in H : w_0 \in T(u_0)$, $y_0 \in V(u_0)$, $h_1(u_0) \in \Omega(u_0)$, compute the sequences $\{u_n\}$, $\{w_n\}$, and $\{y_n\}$ from the iterative schemes*

$$w_n \in T(u_n) : \|w_{n+1} - w_n\| \leq M(T(u_{n+1}), T(u_n))$$
$$y_n \in V(u_n) : \|y_{n+1} - y_n\| \leq M(V(u_{n+1}), V(u_n))$$
$$u_{n+1} = (1-\lambda)u_n + \lambda\left\{u_n - h_1(u_n) + P_{\Omega(u_n)}\left[h_1(u_n) - \rho N(w_n, y_n)\right]\right\}, \quad n = 0, 1, 2, \ldots.$$

II. If $T, V : H \to C(H)$ are multivalued operators, and $h_2 = h_1 = I$, the identity operator, then Algorithm 4.1 reduces to the following algorithm.

**Algorithm 4.3.** *For given $u_0$, $w_0$, $y_0 \in \Omega(u_0) : w_0 \in T(u_0)$, $y_0 \in V(u_0)$, compute the sequences $\{u_n\}$, $\{w_n\}$, and $\{y_n\}$ from the iterative schemes*

$$w_n \in T(u_n) : \|w_{n+1} - w_n\| \leq M(T(u_{n+1}), T(u_n))$$
$$y_n \in V(u_n) : \|y_{n+1} - y_n\| \leq M(V(u_{n+1}), V(u_n))$$
$$u_{n+1} = (1-\lambda)u_n + \lambda P_{\Omega(u_n)}\left[h_1(u_n) - \rho N(w_n, y_n)\right], \quad n = 0, 1, 2, \ldots.$$

III. If $\Omega(u) = m(u) + \Omega$, where $m$ is a point-to-point mapping and $\Omega$ is a closed and convex set in the real Hilbert space $H$, then Algorithm 4.1 reduces to the following algorithm.

**Algorithm 4.4.** *For given $u_0$, $w_0$, $y_0 \in H : w_0 \in T(u_0)$, $y_0 \in V(u_0)$, and $h_1(u_0)$, $h_2(u_0) \in \Omega(u_0) = m(u_0) + \Omega$, then compute the sequences $\{u_n\}$, $\{w_n\}$, and $\{y_n\}$ from the iterative schemes*

$$w_n \in T(u_n) : \|w_{n+1} - w_n\| \leq M(T(u_{n+1}), T(u_n))$$
$$y_n \in V(u_n) : \|y_{n+1} - y_n\| \leq M(V(u_{n+1}), V(u_n))$$
$$u_{n+1} = (1-\lambda)u_n + \lambda\{u_n - h_2(u_n) + m(u_n) + P_\Omega\left[h_1(u_n) - \rho N(w_n, y_n) - m(u_n)\right]\},$$

*for $n = 0, 1, 2, \ldots$.*

IV. If $\Omega(u) = m(u) + \Omega$ and $h_2 = h_1$, then Algorithm 4.1 reduces to the following algorithm.

**Algorithm 4.5.** *For given $u_0$, $w_0$, $y_0 \in H : w_0 \in T(u_0)$, $y_0 \in V(u_0)$, and $h_1(u_0) \in \Omega(u_0) = m(u_0) + \Omega$, then compute the sequences $\{u_n\}$, $\{w_n\}$, and $\{y_n\}$ from the iterative schemes*

$$w_n \in T(u_n) : \|w_{n+1} - w_n\| \leq M(T(u_{n+1}), T(u_n))$$
$$y_n \in V(u_n) : \|y_{n+1} - y_n\| \leq M(V(u_{n+1}), V(u_n))$$
$$u_{n+1} = (1-\lambda) u_n + \lambda \{u_n - h_1(u_n) + m(u_n) + P_K[h_1(u_n) - \rho N(w_n, y_n) - m(u_n)]\},$$

*for $n = 0, 1, 2, \ldots$.*

V. If $\Omega(u) = m(u) + \Omega$ and $h_2 = h_1 = I$, the identity operator, then Algorithm 4.1 reduces to the following algorithm.

**Algorithm 4.6.** *For given $u_0$, $w_0$, $y_0 \in \Omega(u_0) = m(u_0) + \Omega : w_0 \in T(u_0)$, $y_0 \in V(u_0)$, then compute the approximate solutions $\{u_n\}$, $\{w_n\}$, and $\{y_n\}$ from the iterative schemes*

$$w_n \in T(u_n) : \|w_{n+1} - w_n\| \leq M(T(u_{n+1}), T(u_n))$$
$$y_n \in V(u_n) : \|y_{n+1} - y_n\| \leq M(V(u_{n+1}), V(u_n))$$
$$u_{n+1} = (1-\lambda) u_n + \lambda \{m(u_n) + P_\Omega[u_n - \rho N(w_n, y_n) - m(u_n)]\},$$

*for $n = 0, 1, 2, \ldots$.*

For suitable and appropriate choice of the operators and spaces, one can suggest several iterative methods for solving problem (1).

In the next theorem, we show that the approximate solution obtained from the iterative Algorithm 4.1 converges strongly to $u, w, y \in H$, the exact solution of problem (1).

**Theorem 4.1.** *Let $\Omega(u)$ be any closed and convex valued set in $H$. Let the operator $N(\cdot,\cdot)$ be strongly monotone with respect to the first argument with constant $\alpha > 0$ and Lipschitz continuous with respect to the first argument with constant $\beta > 0$. Let the operators $h_1$, $h_2 \colon H \to H$ be strongly monotone with constant $\sigma_1 > 0$, $\sigma_2 > 0$ and Lipschitz continuous with constants $\delta_1 > 0$, $\delta_2 > 0$, respectively. Assume that the operator $N(\cdot,\cdot)$ is Lipschitz continuous with respect to the second argument with constant $\eta > 0$. Let $T$, $V \colon H \to C(H)$ be M-Lispchitz continuous mappings with constants $\mu > 0$ and $\xi > 0$ respectively. If Assumption 3.1 and relation (19) hold, then there exists a solution $u$, $w$, $y \in H : w \in T(u)$, $y \in V(u)$, and $h_1(u)$, $h_2(u) \in \Omega(u)$ satisfying problem (1), and the sequences $\{u_n\}$, $\{w_n\}$, and $\{y_n\}$ generated by Algorithm 4.1 converges to $u$, $w$ and $y$ strongly in $H$, respectively.*

*Proof.* From Theorem 3.1 it is clear that there exists a solution $u \in H$ of problem (1). Then from (30) and (33), we have

$$\begin{aligned}
\|u_{n+1} - u\| &= \|(1-\lambda)(u_n - u) + \lambda\{(u_n - u) - (h_2(u_n) - h_2(u)) \\
&\quad + P_{\Omega(u_n)}[h_1(u_n) - \rho N(w_n, y_n)] - P_{\Omega(u)}[h_1(u) - \rho N(w, y)]]\}\| \\
&\leq (1-\lambda)\|u_n - u\| + \lambda\|(u_n - u) - \{h_2(u_n) - h_2(u)\}\| \\
&\quad + \lambda\left\|P_{\Omega(u_n)}[h_1(u_n) - \rho N(w_n, y_n)] - P_{\Omega(u)}[h_1(u) - \rho N(w, y)]\right\| \\
&\leq (1-\lambda)\|u_n - u\| + \lambda\|(u_n - u) - \{h_2(u_n) - h_2(u)\}\| \\
&\quad + \lambda\left\|P_{\Omega(u_n)}[h_1(u_n) - \rho N(w_n, y_n)] - P_{\Omega(u)}[h_1(u_n) - \rho N(w_n, y_n)]\right\| \\
&\quad + \lambda\left\|P_{\Omega(u)}[h_1(u_n) - \rho N(w_n, y_n)] - P_{\Omega(u)}[h_1(u) - \rho N(w, y)]\right\| \\
&\leq (1-\lambda)\|u_n - u\| + \lambda\|(u_n - u) - \{h_2(u_n) - h_2(u)\}\| \\
&\quad + \lambda\nu\|u_n - u\| + \lambda\|(u_n - u) - \{h_1(u_n) - h_1(u)\}\| \\
&\quad + \lambda\|(u_n - u) - \rho\{N(w_n, y_n) - N(w, y_n)\}\| \\
&\quad + \lambda\rho\|N(w, y_n) - N(w, y)\|,
\end{aligned}$$

where we have used Assumption 3.1.

Using (25)-(28), we have

$$\begin{aligned}\|u_{n+1}-u\| &\leq (1-\lambda)\|u_n-u\|+\lambda\sqrt{1-2\sigma_2+\delta_2^2}\,\|u_n-u\| \\ &\quad +\lambda\nu\|u_n-u\|+\lambda\sqrt{1-2\sigma_1+\delta_1^2}\,\|u_n-u\| \\ &\quad +\lambda\sqrt{1-2\rho\alpha+\rho^2\beta^2\mu^2}\,\|u_n-u\|+\lambda\rho\eta\xi\|u_n-u\| \\ &= [(1-\lambda)+\lambda\theta]\|u_n-u\| = [1-\lambda(1-\theta)]\|u_n-u\|, \end{aligned} \tag{34}$$

where $\theta$ is defined in (19). Thus

$$\|u_{n+1}-u\| \leq \theta_1 \|u_n-u\|, \tag{35}$$

where $\theta_1 = 1-\lambda(1-\theta)$.

Since from (19), we have $0<\theta<1$, therefore $\theta_1<1$.

Hence from (35), we know that the sequence $\{u_n\}$ is a Cauchy sequence in $H$, so there exists $u \in H$ with $u_{n+1} \to u$.

Now since $N(\cdot,\cdot)$ is Lipschitz continuous with respect to the second argument with constant $\eta>0$ and $V$ is M-Lipschitz continuous with constant $\xi>0$, we have that, by using Lemma 2.2, an $y \in V(u)$ exists such that $\|y_{n+1}-y\| \leq M(V(u_{n+1}), V(u))$, then

$$\begin{aligned}\|N(w,y_{n+1})-N(w,y)\| &\leq \eta\|y_{n+1}-y\| \\ &\leq \eta M(V(u_{n+1}),V(u)) \\ &\leq \eta\xi\|u_{n+1}-u\|,\end{aligned}$$

which implies that the sequence $\{y_n\}$ is also a Cauchy sequence in $H$, so there exists $y \in H$ such that $y_{n+1} \to y$.

Similarly for the first argument of $N(\cdot,\cdot)$,we have

$$\|N(w_{n+1},y)-N(w,y)\| \leq \beta\mu\|u_{n+1}-u\|,$$

which shows that the sequence $\{w_n\}$ is also a Cauchy sequence in $H$, so there exists $w \in H$ such that $w_{n+1} \to w$.

Thus by Lemma 3.1, it follows that $u,\, w,\, y \in H$ such that $w \in T(u),\ y \in V(u)$ and $h_1(u),\ h_2(u) \in \Omega(u)$ satisfies the the multivalued extended general quasi-variational inequality (1) and $u_n \to u$, $w_n \to w$ and $y_n \to y$ strongly in $H$. This completes the proof. □

If $\Omega(u) = m(u) + \Omega$, where $m$ is a point-to-point Lipschitz continuous with constant $\nu > 0$ and $\Omega$ is a closed and convex set in the real Hilbert space $H$ in the Theorem 4.1, then we have the following result.

**Corollary 4.1.** *Let $N$, $T$, $V$, $h_1$, and $h_2$ be the same as defined in Theorem 4.1. Assume that the point-to-point mapping $m$ is Lipschitz continuous with constant $\gamma > 0$ and a relation (19) with $k = 2\gamma + \sqrt{1-2\sigma_1+\delta_1^2} + \sqrt{1-2\sigma_2+\delta_2^2}$ hold. Then problem (1) has a solution $u,\, w,\, y \in H : w \in V(u),\ h_1(u) \in \Omega(u),\ h_2(u) \in \Omega(u)$ and $u_n \to u$, $w_n \to w$, $y_n \to y$ strongly in $H$, where the sequences $\{u_n\}, \{w_n\}, \{y_n\}$ are generated by Algorithm 4.4.*

## 5. **Wiener-Hopf Equations Technique**

In this section, we introduce a new class of Wiener-Hopf equations, which is called the multivalued extended general implicit Wiener-Hopf equations. We establish the equivalence between the multivalued extended general implicit Wiener-Hopf equations and problem (1). By using this equivalence, we suggest a number of new iterative methods for solving the different classes of problem (1) and its variant forms.

For given nonlinear multivalued operators $T$, $V\colon H \to C(H)$ and single valued operators $N(\cdot,\cdot): H \times H \to H$, and $h_1$, $h_2\colon H \to H$. Suppose that inverse of the operator $h_2$ exists, we consider problem of finding $z$, $u$, $w$, $y \in H : w \in T(u)$, $y \in V(u)$, and

$$N(w,y) + \rho^{-1} Q_{\Omega(u)}[z] = 0, \tag{36}$$

where $Q_{\Omega(u)} = I - h_1\left(h_2^{-1} P_{\Omega(u)}\right)$, $I$ is the identity operator and $\rho > 0$ is a constant. The equation (36) is known as multivalued extended general implicit Wiener-Hopf equations.

We now discuss some special cases of problem (36).

I. If $h_2 = h_1$, then problem (36) reduces to problem of finding $z$, $u$, $w$, $y \in H : w \in T(u)$, $y \in V(u)$, and

$$N(w,y) + \rho^{-1} Q_{\Omega(u)}[z] = 0,$$

where $Q_{\Omega(u)} = I - P_{\Omega(u)}$, $I$ is the identity operator and $\rho > 0$ is a constant. These problems are known as the multivalued general implicit Wiener-Hopf equations.

II. If $h_2 = h_1$, and $\Omega(u) = \Omega$, then problem (36) reduces to problem of finding $z$, $u$, $w$, $y \in H : w \in T(u)$, $y \in V(u)$, and

$$N(w,y) + \rho^{-1} Q_{\Omega}[z] = 0,$$

where $Q_\Omega = I - P_\Omega$, $I$ is the identity operator, and $\rho > 0$ is a constant. These problems are known as the generalized multivalued Wiener-Hopf equations.

III. If $h_2 = h_1 = I$, the identity operator, $T, V : H \to H$ are single valued operator, and $\Omega(u) = \Omega$, then problem (36) is equivalent to finding $z \in H$ such that

$$TP_\Omega[z] + \rho^{-1} Q_\Omega[z] = 0,$$

which are known as Wiener-Hopf equations. These equations were introduced and studied by Shi [42] and Robinson [41], independently.

Using the technique of Noor et al. [37], one can prove the following result.

**Lemma 5.1.** *The problem* (1) *has a solution* $u$, $w$, $y \in H : w \in T(u)$, $y \in V(u)$, *and* $h_1(u)$, $h_2(u) \in \Omega(u)$, *if and only if the problem* (36) *have a solution* $z$, $u$, $w$, $y \in H : w \in T(u)$, $y \in V(u)$, *provided*

$$h_2(u) = P_{\Omega(u)}[z], \tag{37}$$

*and*

$$z = h_1(u) - \rho N(w,y), \tag{38}$$

*where* $\rho > 0$ *is a constant.*

Lemma 5.1 implies that problem (1) and problem (36) are equivalent. This equivalent formulation is used to suggest and analyze some iterative methods for solving (1).

I. Equation (36) can be written as:

$$\begin{aligned} \rho N(w,y) &= -Q_{\Omega(u)}[z] \\ &= h_1\left(h_2^{-1} P_{\Omega(u)}\right)[z] - z = h_1(u) - z, \end{aligned}$$

which implies that

$$z = h_1(u) - \rho N(w,y). \tag{39}$$

This fixed point formulation enables us to suggest the following iterative method for solving problem (1).

**Algorithm 5.1.** *For given* $z_0$, $u_0$, $w_0$, $y_0 \in H : w_0 \in T(u_0)$, $y_0 \in V(u_0)$, *compute the sequences* $\{z_n\}$, $\{u_n\}$, $\{w_n\}$, *and* $\{y_n\}$ *by the iterative schemes*

$$h_2(u_n) = P_{\Omega(u_n)}[z_n] \tag{40}$$

$$w_n \in T(u_n) : \|w_{n+1} - w_n\| \le M(T(u_{n+1}), T(u_n)) \tag{41}$$

$$y_n \in V(u_n) : \|y_{n+1} - y_n\| \le M(V(u_{n+1}), V(u_n)) \tag{42}$$

$$z_{n+1} = h_1(u_n) - \rho N(w_n, y_n), \quad n = 0, 1, 2, \ldots. \tag{43}$$

We now discuss some special cases of Algorithm 5.1.

(i). If $h_2 = h_1$, then Algorithm 5.1 reduces to the following algorithm which is developed and studied by Noor [24, 25] for solving the generalized multivalued quasi variational problem (11).

**Algorithm 5.2.** *For given $z_0$, $u_0$, $w_0$, $y_0 \in H : w_0 \in T(u_0)$, $y_0 \in V(u_0)$, compute the sequences $\{z_n\}$, $\{u_n\}$, $\{w_n\}$, and $\{y_n\}$ by the iterative schemes*

$$h_1(u_n) = P_{\Omega(u_n)}[z_n]$$

$$w_n \in T(u_n) : \|w_{n+1} - w_n\| \leq M(T(u_{n+1}), T(u_n))$$

$$y_n \in V(u_n) : \|y_{n+1} - y_n\| \leq M(V(u_{n+1}), V(u_n))$$

$$z_{n+1} = h_1(u_n) - \rho N(w_n, y_n), \quad n = 0, 1, 2, \ldots.$$

(ii). If $h_2 = h_1$, and $\Omega(u) = \Omega$, then Algorithm 5.1 reduces to the following algorithm.

**Algorithm 5.3.** *For given $z_0$, $u_0$, $w_0$, $y_0 \in H : w_0 \in T(u_0)$, $y_0 \in V(u_0)$, compute the sequences $\{z_n\}$, $\{u_n\}$, $\{w_n\}$, and $\{y_n\}$ by the iterative schemes*

$$h_1(u_n) = P_{\Omega}[z_n]$$

$$w_n \in T(u_n) : \|w_{n+1} - w_n\| \leq M(T(u_{n+1}), T(u_n))$$

$$y_n \in V(u_n) : \|y_{n+1} - y_n\| \leq M(V(u_{n+1}), V(u_n))$$

$$z_{n+1} = h_1(u_n) - \rho N(w_n, y_n), \quad n = 0, 1, 2, \ldots.$$

II. Equation (36) can be written as:

$$\rho^{-1} Q_{\Omega(u)}[z] = -N(w, y),$$

which implies that

$$\begin{aligned} N(w, y) - \left(1 - \rho^{-1}\right) Q_{\Omega(u)}[z] &= -Q_{\Omega(u)}[z] \\ &= h_1\left(h_2^{-1} P_{\Omega(u)}\right)[z] - z = h_1(u) - z. \end{aligned}$$

This implies

$$z = h_1(u) - N(w, y) + \left(1 - \rho^{-1}\right) Q_{\Omega(u)}[z]. \tag{44}$$

Using this fixed point formulation, we can suggest the following iterative scheme for solving problem (1).

**Algorithm 5.4.** *For given $z_0$, $u_0$, $w_0$, $y_0 \in H : w_0 \in T(u_0)$, $y_0 \in V(u_0)$, compute the sequences $\{z_n\}$, $\{u_n\}$, $\{w_n\}$, and $\{y_n\}$ by the iterative schemes*

$$h_2(u_n) = P_{\Omega(u_n)}[z_n]$$

$$w_n \in T(u_n) : \|w_{n+1} - w_n\| \leq M(T(u_{n+1}), T(u_n))$$

$$y_n \in V(u_n) : \|y_{n+1} - y_n\| \leq M(V(u_{n+1}), V(u_n))$$

$$z_{n+1} = h_1(u_n) - N(w_n, y_n) + \left(1 - \rho^{-1}\right) Q_{\Omega(u_n)}[z_n], \quad n = 0, 1, 2, \ldots.$$

We now discuss some special cases of Algorithm 5.4 which were due to Noor [23, 25].

(i). If we take $h_2 = h_1$, then Algorithm 5.4 reduces to the following algorithm which is studied by Noor [23, 25] for solving problem (11).

**Algorithm 5.5.** *For given* $z_0,\ u_0,\ w_0,\ y_0 \in H : w_0 \in T(u_0),\ y_0 \in V(u_0)$, *compute the sequences* $\{z_n\},\ \{u_n\},\ \{w_n\}$, *and* $\{y_n\}$ *by the iterative schemes*

$$\begin{gathered} h_1(u_n) = P_{\Omega(u_n)}[z_n] \\ w_n \in T(u_n) : \|w_{n+1} - w_n\| \leq M(T(u_{n+1}), T(u_n)) \\ y_n \in V(u_n) : \|y_{n+1} - y_n\| \leq M(V(u_{n+1}), V(u_n)) \\ z_{n+1} = h_1(u_n) - N(w_n, y_n) + \left(1 - \rho^{-1}\right) Q_{\Omega(u_n)}[z_n], \quad n = 0, 1, 2, \ldots. \end{gathered}$$

(ii). If we take $h_2 = h_1$, and $\Omega(u) = \Omega$, then Algorithm 5.4 reduces to the following algorithm.

**Algorithm 5.6.** *For given* $z_0,\ u_0,\ w_0,\ y_0 \in H : w_0 \in T(u_0),\ y_0 \in V(u_0)$, *compute the sequences* $\{z_n\},\ \{u_n\},\ \{w_n\}$, *and* $\{y_n\}$ *by the iterative schemes*

$$\begin{gathered} h_1(u_n) = P_{\Omega}[z_n] \\ w_n \in T(u_n) : \|w_{n+1} - w_n\| \leq M(T(u_{n+1}), T(u_n)) \\ y_n \in V(u_n) : \|y_{n+1} - y_n\| \leq M(V(u_{n+1}), V(u_n)) \\ z_{n+1} = h_1(u_n) - N(w_n, y_n) + \left(1 - \rho^{-1}\right) Q_{\Omega}[z_n], \quad n = 0, 1, 2, \ldots. \end{gathered}$$

We now discuss the convergence analysis of Algorithm 5.1 and this is the main motivation of our next result.

**Theorem 5.1.** *With the same conditions as in Theorem* 3.1, *then there exists* $z,\ u,\ w,\ y \in H : w \in T(u)$, *and* $y \in V(u)$ *satisfying problem* (36) *and the sequences* $\{z_n\}$, $\{u_n\}$, $\{w_n\}$, *and* $\{y_n\}$ *generated by Algorithm* 5.1 *converge to* $z$, $u$, $w$, *and* $y$ *strongly in* $H$, *respectively.*

*Proof.* Using (39) and (43), we have

$$\begin{aligned} \|z_{n+1} - z\| &= \|h_1(u_n) - h_1(u) - \rho\{N(w_n, y_n) - N(w, y)\}\| \\ &\leq \|(u_n - u) - \{h_1(u_n) - h_1(u)\}\| \\ &\quad + \|(u_n - u) - \rho\{N(w_n, y_n) - N(w, y_n)\}\| \\ &\quad + \rho\|N(w, y_n) - N(w, y)\|. \end{aligned} \tag{45}$$

Combining (25)-(28), and (45), we have

$$\begin{aligned} \|z_{n+1} - z\| \leq\ & \{k - \nu - \sqrt{1 - 2\sigma_2 + \delta_2^2} + \rho\eta\xi \\ & + \sqrt{1 - 2\rho\alpha + \rho^2\beta^2\mu^2}\} \|u_n - u\|. \end{aligned} \tag{46}$$

Now using (37) and (40), we have

$$\begin{aligned} \|u_n - u\| &= \left\|(u_n - u) - \{h_2(u_n) - h_2(u)\} + \left[P_{\Omega(u_n)}[z_n] - P_{\Omega(u)}[z]\right]\right\| \\ &\leq \|(u_n - u) - \{h_2(u_n) - h_2(u)\}\| \\ &\quad + \left\|P_{\Omega(u_n)}[z_n] - P_{\Omega(u_n)}[z]\right\| + \left\|P_{\Omega(u_n)}[z] - P_{\Omega(u)}[z]\right\|, \end{aligned}$$

from $(20), (27)$ and using Assumption 3.1, we have

$$\|u_n - u\| \leq \left\{k - \nu - \sqrt{1 - 2\sigma_1 + \delta_1^2}\right\} \|u_n - u\| + \|z_n - z\| + \nu\|u_n - u\|,$$

which implies

$$\|u_n - u\| \leq \frac{1}{1 - k + \sqrt{1 - 2\sigma_1 + \delta_1^2}} \|z_n - z\|. \tag{47}$$

Thus from (46) ,and (47), we have

$$\|z_{n+1} - z\| \leq \theta\|z_n - z\|, \tag{48}$$

where
$$\theta = \frac{k - \nu - \sqrt{1 - 2\sigma_2 + \delta_2^2} + \rho\eta\xi + \sqrt{1 - 2\rho\alpha + \rho^2\beta^2\mu^2}}{1 - k + \sqrt{1 - 2\sigma_1 + \delta_1^2}}.$$
From (19), we see that $\theta < 1$, and consequently, from (48), it is clear that $\{z_n\}$ is a Cauchy sequence in $H$, that is, $z_{n+1} \to z \in H$ as $n \to \infty$. From (47), we know that $\{u_n\}$ is also a Cauchy sequence in $H$, that is $u_{n+1} \to u \in H$ as $n \to \infty$. From (28), it follows that $\{y_n\}$ is also a Cauchy sequence in $H$, that is $y_{n+1} \to y \in H$ as $n \to \infty$.

Using the continuity of the operators $N$, $h_1$, $h_2$, and Algorithm 5.1, we have
$$z = h_1(u) - \rho N(w, y) \in H.$$
From the technique of the Theorem 4.1, we can easily show that $y \in V(u)$. From Lemma 5.1, we see that $z, u, w, y \in H$, such that $w, \in T(u)$, and $y \in V(u)$ is a solution of the multivalued extended general implicit Wiener-Hopf equation (36), and consequently, $z_{n+1} \to z, u_{n+1} \to u, w_{n+1} \to w$, and $y_{n+1} \to y$ strongly in $H$. □

## 6. Conclusion

In this paper, we have introduced and studied a new class of quasi variational inequalities, which is called multivalued extended general quasi variational inequalities. Using the projection operator technique, it is shown that the multivalued extended general quasi variational inequalities are equivalent to the fixed point problems. This alternative equivalent fixed point formulation is used to discuss the existence of a solution of new class of quasi variational inequalities. We have used this equivalent fixed point formulation to develop several iterative methods for solving the multivalued extended general quasi variational inequalities. We have also consider the convergence criteria of iterative method under suitable conditions. It has been shown that the multivalued extended general quasi variational inequalities are also equivalent to the multivalued extended general implicit Wiener-Hopf equations. This alternative equivalent formulation has been used to develop several iterative methods for solving multivalued extended general quasi variational inequalities and its variant forms. The convergence analysis of these methods is considered. We expect that this work will motivate and inspire interested readers to explore the novel and innovative applications of multivalued extended general quasi variational inequalities in various fields of pure and applied science.

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