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A NEW CONCEPT FOR NUMERICAL RADIUS: THE SIGN-REAL NUMERICAL RADIUS

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We define and investigate a new type of numerical radius for real matrices, the sign-real numerical radius, and derive some properties. We extend the Perron-Frobenius theory for the numerical radius of nonnegative matrices to real matrices.

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1. Introduction and preliminaries

By the Perron-Frobenius theory, if A is a square nonnegative matrix, then its spectral radius $\rho(A)$ is an eigenvalue of A and there is a corresponding nonnegative eigenvector. It has numerous applications, not only in many branches of mathematics, such as Markov chains, graph theory, game theory and etc. [1], but in various fields of science and technology, e.g. control theory [7, 12] and the population dynamics [6]. In [11] a new quantity for real matrices, the sign-real spectral radius, is defined. For $A \in M_n(\mathbb{R})$, the real spectral radius of A is defined by $\rho_0(A) = \max\{|\lambda| : \lambda \textit{ a real eigenvalue of } A\}$, where $\rho_0(A) := 0$ if A has no real eigenvalues. A signature matrix is a diagonal matrix with diagonal entries $+1$ or -1. Note that there are 2^n signature matrices of dimension n. Let φ denote the set of signature matrices. The sign-real spectral radius of a real matrix A is defined by

$$\rho_0^S(A) = \max_{S\in\varphi} \rho_0(SA).$$

The sign-real spectral radius of a real matrix A has similar properties to the spectral radius of a nonnegative matrix (cf. [11]). It has also been applied to engineering problems (see, for example, [9, 8, 10] and the references therein).

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Let $M_n(\mathbb{C})$ $(M_n(\mathbb{R}))$ be the set of $n \times n$ complex (real) matrices. For $A \in M_n(\mathbb{C})$, the numerical range of A is defined and denoted by

$$W(A) = \{x^*Ax : \ x \in \mathbb{C}^n, \ x^*x = 1\},$$

which is a useful concept in studying matrices and operators (see, for example, [3, Chapter 1]). The numerical radius of A is $\omega(A) = \max\{|z| : z \in W(A)\}$. The Perron-Frobenius theory has been extended to the numerical range of a nonnegative matrix by Issos in his unpublished Ph.D. thesis [4] and then completed in [5]. In the course of proving results of Issos for real matrices, the sign-real numerical radius occurs.

Definition 1.1. *For $A \in M_n(\mathbb{R})$ the sign-real numerical radius is defined and denoted by*

$$\omega_0^S(A) = \max_{S \in \varphi} \omega_0(SA),$$

where $\omega_0(A) = \max\{\, |z| \, : \, z \in W(A) \cap \mathbb{R}\,\}$.

The sign-real numerical radius of a real matrix has similar properties to the numerical radius of a nonnegative matrix. For example, in parallel to the Perron-Frobenius theory, we show that there exists some $S \in \varphi$ such that $\omega_0^S(A) \in W(SA)$ (Theorem 2.1) and also there is a unit nonnegative vector x such that $\omega_0^S(A) = x^t S_1 A S_2 x$ for some $S_1, S_2 \in \varphi$ (Corollary 2.1). In addition, the relation between the sign-real numerical radius and the sign-real spectral radius is characterized in Theorem 2.2.

We always use $A = (a_{rs})$ to denote an $n \times n$ complex matrix. The following notations will be adopted:

M_n	the set of all $n \times n$ complex matrices;
$\mathbb{R}_+^n$	the nonnegative orthant of $\mathbb{R}^n$;
$W(A)$	the (classical) numerical range of A;
$\omega(A)$	the numerical radius of A;
$\omega_0^S(A)$	the sign-real numerical radius of real matrix A;
$\sigma(A)$	the spectrum of A;
$\rho(A)$	the spectral radius of A;
$\rho_0^S(A)$	the sign-real spectral radius of real matrix A;
A^t	the transpose of A;
A^*	the conjugate transpose of A;
$H(A)$	the Hermitian part of A , i.e. $(A + A^*)/2$;
$\lambda_{\max}(A)$	the largest eigenvalue of the Hermitian matrix A;
$\|A\|$	the matrix $(\|a_{rs}\|)$ for all r, s;
$\|x\|$	the vector $(\|x_1\|, \|x_2\|, \cdots, \|x_n\|)^t$;
$A \leq B$	$a_{rs} \leq b_{rs}$ for all r, s.

For a vector $x \in \mathbb{C}^n$, we denote by $\|x\|$ and $\|x\|_1$ the Euclidean norm and the sum norm of x, respectively, i.e., $\|x\| = (x^*x)^{1/2}$, $\|x\|_1 = |x_1| + \cdots + |x_n|$. For a matrix $A \in M_n$, we denote by $\|A\|$ and $\|A\|_2$ the operator norm and the spectral norm of A, respectively, i.e., $\|A\| = \max_{\|x\|=1} \|Ax\|$, $\|A\|_2^2 = \rho(A^*A)$, where $\|\cdot\|$ is the vector norm.

2. Main results

In the following Lemma, we investigate some properties of the sign-real numerical radius.

Lemma 2.1. *Let $A \in M_n(\mathbb{R})$, signature matrices $S_1, S_2, T \in \varphi$, a real diagonal matrix D, a real orthogonal matrix U and a permutation matrix P be given. Then*

(a) $\omega_0^S(A) = \omega_0^S(S_1AS_1) = \omega_0^S(S_1AS_2) = \omega_0^S(A^t) = \omega_0^S(P^tAP)$;
(b) $\omega_0^S(\alpha A) = |\alpha|\omega_0^S(A)$, *for all* $\alpha \in \mathbb{R}$;
(c) $\omega_0^S(UD) = \omega_0^S(DU)$;
(d) $\omega_0^S(A) \leq \|A\|$;
(e) $\rho_0^S(A) \leq \omega_0^S(A)$;
(f) $\omega_0^S(D) = \omega_0^S(U^tDU)$;
(g) *if $A = A^t$, and V is a real orthogonal matrix, then* $\omega_0^S(A) = \omega_0^S(V^tAV)$.

Proof. (a) For every unitary matrix $U_1 \in M_n$, we have

$$W(AU_1) = W(U_1(AU_1)U_1^*) = W(U_1A). \tag{1}$$

Therefore $\omega_0(TA) = \omega_0(AT)$, and then, for all $S_1 \in \varphi$,

$$\omega_0(S_1AS_1) = \omega_0(A). \tag{2}$$

By using the equation (2), for all $S_1 \in \varphi$ and for some $T_1, T_2 \in \varphi$, we have

$$\begin{aligned} \omega_0^S(S_1AS_1) &= \omega_0(T_1S_1AS_1) = \omega_0(S_1T_1AS_1) = \omega_0(T_1A) \\ &\leq \omega_0^S(A) = \omega_0(T_2A) = \omega_0(T_2S_1AS_1) \leq \omega_0^S(S_1AS_1). \end{aligned} \tag{3}$$

Thus, $\omega_0^S(A) = \omega_0^S(S_1AS_1)$, for all $S_1 \in \varphi$. Again, by using the equation (2), for all $S_1, S_2 \in \varphi$, we see that

$$\begin{aligned} \omega_0^S(A) &= \max_{T\in\varphi} \omega_0(TA) = \max_{T\in\varphi} \omega_0(S_2(S_1S_1TA)S_2) \\ &= \max_{T_1\in\varphi} \omega_0(T_1(S_1AS_2)) = \omega_0^S(S_1AS_2). \end{aligned}$$

In view of (1), for every permutation matrix P, we have $\omega_0(AP) = \omega_0(PA)$. Since PSP^t is a signature matrix for every signature matrix S, and then by using the same method in (3), we conclude that $\omega_0^S(A) = \omega_0^S(P^tAP)$. Also, $\omega_0^S(A^t) = \omega_0^S(A)$, since $\omega_0(A^t) = \omega_0(A)$.

(b) It is trivial.

(c) In view of (1), we have $\omega_0(AU) = \omega_0(UA)$, and thus for some $T_1, T_2 \in \varphi$, we see that

$$\begin{aligned}\omega_0^S(DU) &= \omega_0(T_1DU) = \omega_0(DT_1U) = \omega_0(T_1UD) \leq \omega_0^S(UD)\\ &= \omega_0(T_2UD) = \omega_0(DT_2U) = \omega_0(T_2DU) \leq \omega_0^S(DU).\end{aligned}$$

(d) For any nonzero vector $x \in \mathbb{C}^n$, we have $|x^*Ax| \leq \|Ax\|\,\|x\|$ (Cauchy-Schwarz inequality), and by Definition 1.1, we obtain $\omega_0(A) \leq \|A\|$ and hence $\omega_0^S(A) \leq \|SA\| = \|A\|$ for any $S \in \varphi$.

(e) By using the spectral containment property [3, Property 1.2.6], we conclude that, there exists some $T_1 \in \varphi$ such that $\rho_0^S(A) = \rho_0(T_1A) \leq \omega_0(T_1A) \leq \omega_0^S(A)$.

(f) Again in view of (1) and for some $T_1 \in \varphi$, we have

$$\omega_0^S(D) = \omega_0(T_1D) = \omega_0(D) = \omega_0(UU^tD) = \omega_0(U^tDU) \leq \omega_0^S(U^tDU). \tag{4}$$

By part (d), we have

$$\omega_0^S(U^tDU) \leq \|U^tDU\|_2 = \|D\|_2 = \omega_0(D) \leq \omega_0^S(D). \tag{5}$$

Then by (4) and (5), $\omega_0^S(U^tDU) = \omega_0^S(D)$.

(g) By assumption, there is a real orthogonal matrix $Q \in M_n(\mathbb{R})$ such that $A = Q^t\Lambda Q$, where Λ is a real diagonal matrix with elements $\lambda_i \in \sigma(A)$ for all $i = 1, 2, ..., n$. Therefore, by part (f), $\omega_0^S(A) = \omega_0^S(Q^t\Lambda Q) = \omega_0^S(\Lambda)$, and $\omega_0^S(V^tAV) = \omega_0^S(V^tQ^t\Lambda QV) = \omega_0^S(\Lambda)$. □

We want to emphasize that in the last part of Lemma 2.1 we are assuming that A is a real Hermitian matrix. Our next example will show that this is not true for all $n \times n$ real matrices.

Example 2.1. *Consider the matrices*

$$A = \begin{pmatrix} -1 & -3 & 2\\ 0 & -1 & 2\\ 0 & 0 & 3\end{pmatrix}, \qquad U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0\\ 1 & -1 & 0\\ 0 & 0 & \sqrt{2}\end{pmatrix},$$

where A is a real matrix and U is a real orthogonal matrix. Then $\omega_0^S(A) = 4.1861 \neq \omega_0^S(U^tAU) = 4.4495$.

In [4, Theorem 1], it is shown that if $A \geq 0$, then $\omega(A) \in W(A)$. We obtain a similar result for the sign-real numerical radius of real matrix A.

Theorem 2.1. *If $A \in M_n(\mathbb{R})$, then for all $T \in \varphi$, there exists some $S \in \varphi$ and $0 \neq x \in \mathbb{R}^n$ such that x is nonnegative and $\left(\frac{Tx}{\|x\|}\right)^t (SA)\left(\frac{Tx}{\|x\|}\right) = \lambda \in W(SA)$ for some $0 \leq \lambda \in \mathbb{R}$.*

Proof. Let $T \in \varphi$ be given. In the case $ATx = 0$ for some $x \neq 0$, $x \geq 0$, the proof is trivial. Suppose $ATx \neq 0$ for all nonzero $x \geq 0$, we define $f(x) := \frac{|ATx|}{\|ATx\|_1}$. It is readily seen that $f(x)$ is a well-defined continuous function from the nonempty, compact and convex set $E = \{x \in \mathbb{R}^n \ : \ x \geq 0, \ \|x\|_1 = 1\}$ into itself. Due to Brouwer's fixed point theorem, there is some $x \in E$ such that $f(x) = x$. For suitable $S \in \varphi$, we have $SATx = T|ATx|$, and hence $SATx = \|ATx\|_1 Tx = T\,|ATx|$ for all $x \in E$; equivalently,

$$(Tx)^t SA \ (Tx) = \|ATx\|_1 \|Tx\|^2 = \|ATx\|_1 \|x\|^2,$$

where $\left(\frac{Tx}{\|x\|}\right)^t \left(\frac{Tx}{\|x\|}\right) = 1.$ □

As a result from Theorem 2.1, we can conclude that for $A \in M_n(\mathbb{R})$, there exists some $S \in \varphi$ such that $\omega_0^S(A) \in W(SA)$.

Corollary 2.1. *If $A \in M_n(\mathbb{R})$, then there exist $S_1, S_2 \in \varphi$, and a nonnegative unit vector x such that $\omega_0^S(A) = x^t S_1 A S_2 x$.*

Corollary 2.2. *If $A \in M_n(\mathbb{R})$, then $r = \omega_0^S(A)$ if and only if the matrix $T = rI - H(SA)$ is positive semi-definite for some $S \in \varphi$.*

Proof. By Theorem 2.1, $r = \omega_0^S(A)$ if and only if $rx^t x \geq x^t SAx$ for every $x \in \mathbb{R}^n$, and for some $S \in \varphi$. It is clear that $x^t SAx = x^t H(SA)x$ for all $x \in \mathbb{R}^n$. Therefore $r = \omega_0^S(A)$ if and only if $x^t Tx \geq 0$ for all $x \in \mathbb{R}^n$. □

Corollary 2.3. *Let $A \in M_n(\mathbb{R})$, and $D = diag(\lambda_1, \cdots, \lambda_n)$ be congruent to the matrix $T = rI - H(SA)$ for some $S \in \varphi$. then $r = \omega_0^S(A)$ if and only if all the λ_i's are nonnegative and at least one of them vanishes.*

Proof. By Corollary 2.2, $r = \omega_0^S(A)$ if and only if the eigenvalues of the symmetric matrix T are nonnegative and at least one of them vanishes. By Sylvester's law of inertia (cf. [2, Theorem 4.5.8]) the Corollary follows. □

It was shown in [11, Theorem 3.1] that $\rho(A) = \rho_0^S(A)$ for any nonnegative matrix A. Similarly, we have the following result.

Lemma 2.2. *If A is a real matrix, then $\omega_0^S(A) \leq \omega_0^S(|A|)$. Suppose, in addition, that A is a nonnegative matrix, then*

$$\omega(A) = \omega_0^S(A) = \max\{\, z^t A\, z : z \in \mathbb{R}_+^n, \ \|z\| = 1\}.$$

Proof. There exists some $S \in \varphi$ such that

$$\begin{aligned}
\omega_0^S(A) &= \max\ \{\ |x^* SAx| : (x^* SAx) \in \mathbb{R},\ x \in \mathbb{C}^n,\ \|x\| = 1\} \\
&\leq \max\ \left\{\ |x|^t\, |SA|\, |x| : (x^* SAx) \in \mathbb{R},\ x \in \mathbb{C}^n,\ \|x\| = 1\right\} \\
&= \max\ \left\{\ z^t\, |A|\ z : \left(z^t SAz\right) \in \mathbb{R},\ z \in \mathbb{R}^n_+,\ \|z\| = 1\right\} \\
&= \max\ \left\{\ z^t\, |A|\ z :,\ z \in \mathbb{R}^n_+,\ \|z\| = 1\right\} \\
&\leq \max\ \{\ |z^* |A|\ z| : (z^*\, |A|\, z) \in \mathbb{R},\ z \in \mathbb{C}^n,\ \|z\| = 1\} \\
&= \omega_0\left(|A|\right) \leq \omega_0^S(|A|).
\end{aligned}$$

If A is a nonnegative matrix, then for any unit vector $x \in \mathbb{C}^n$, $|x^* Ax| \leq |x|^t A |x|$; hence $\omega(A) = \max\{|x^* Ax| : x \in \mathbb{C}^n, \|x\| = 1\} = \max\left\{z^t Az : z \in \mathbb{R}^n_+, \|z\| = 1\right\}$. So, the above inequalities all become equalities and the proof is complete. □

Notice that in general for real matrices A, B such that $A \leq B$, we do not have $\omega_0^S(A) \leq \omega_0^S(B)$. However, it is true for nonnegative matrices A, B, because by the above Lemma we have

$$\begin{aligned}
\omega_0^S(A) &= \max\ \{\ z^t A\ z : z \in \mathbb{R}^n_+,\ \|z\| = 1\} \\
&\leq \max\ \{\ z^t B\ z : z \in \mathbb{R}^n_+,\ \|z\| = 1\} = \omega_0^S(B).
\end{aligned}$$

A relation between the sign-real numerical radius of A and the sign-real spectral radius of $H(S_1AS_2)$ for some $S_1, S_2 \in \varphi$, is observed in the following Theorem.

Theorem 2.2. *If $A \in M_n(R)$, then there exist some $S_1, S_2 \in \varphi$ such that $\omega_0^S(A) = \rho_0^S(H(S_1AS_2)) = \lambda_{\max}(H(S_1AS_2))$. Moerover, if $\lambda_{\max}(H(S_1AS_2))$ is a simple eigenvalue of $H(S_1AS_2)$, and $x \geq 0$ is the unit vector of Corollary* 2.1, *then y is a unit vector with $y^* S_1AS_2 y = \omega_0^S(A)$ if and only if $y = e^{i\theta}x$ for some $\theta \in [0, 2\pi)$.*

Proof. By Corollary 2.1, we can find a nonzero unit vector $x \geq 0$ so that $x^t(S_1AS_2)x = \omega_0^S(A)$, and also $x^t(S_1AS_2)^t x = \omega_0^S(A)$ for some $S_1, S_2 \in \varphi$. Adding the two equations, we obtain

$$x^t\left(\omega_0^S(A)\, I - H(S_1AS_2)\right) x = 0. \tag{6}$$

Since $H(S_1AS_2)$ is Hermitian matrix, therefore $\|H(S_1AS_2)\|_2 = \rho_0(H(S_1AS_2)) = \omega_0(H(S_1AS_2)) \leq \omega_0^S(H(S_1AS_2)) \leq \|H(S_1AS_2)\|_2$, where the last inequality follows from Lemma 2.1. Therefore $\|H(S_1AS_2)\|_2 = \omega_0^S(H(S_1AS_2))$. Also, we have $\|H(S_1AS_2)\|_2 = \rho_0^S(H(S_1AS_2))$ (cf. [11, Theorem 2.15]), which implies that

$$\omega_0^S(H(S_1AS_2)) = \rho_0^S(H(S_1AS_2)). \tag{7}$$

It follows from the property of $\omega_0^S(A)$ and [3, Property 1.2.7] that

$$\omega_0^S\left(A + A^t\right) \leq \omega_0^S(A) + \omega_0^S\left(A^t\right) = 2\omega_0^S(A). \tag{8}$$

Thus by (7) and (8), $\omega_0^S(A) = \omega_0^S(S_1AS_2) \geq \omega_0^S(H(S_1AS_2)) = \rho_0^S(H(S_1AS_2)) \geq \lambda_{max}(H(S_1AS_2))$. In view of the above inequality and (6), $\left(\omega_0^S(A)I - H(S_1AS_2)\right)$ is positive semi-definite. It follows that $(\omega_0^S(A)I - H(S_1AS_2))x = 0$, so x is an eigenvector of $H(S_1AS_2)$ corresponding to $\omega_0^S(A)$. Hence the above inequalities all become equalities, i.e., $\omega_0^S(A) = \rho_0^S(H(S_1AS_2)) = \lambda_{\max}(H(S_1AS_2))$. Suppose now that $\lambda_{\max}(H(S_1AS_2))$ is a simple eigenvalue of $H(S_1AS_2)$ and $y \in \mathbb{C}^n$ is a unit vector such that $y^*S_1AS_2y = \omega_0^S(A)$. Similarly to the proof of the first part, $\omega_0^S(A)$ is an eigenvector of $H(S_1AS_2)$ corresponding to $\lambda_{\max}(H(S_1AS_2))$. Thus, as $\lambda_{\max}(H(S_1AS_2))$ is simple, $y = e^{i\theta}x$ for some $\theta \in [0, 2\pi)$. □

Remark 2.1. *It will be clear from the proof of Theorem* 2.2 *that, if* $A \in M_n(R)$, *then* $\omega_0^S(A) = \rho_0^S(H(SA)) = \lambda_{\max}(H(SA))$ *for some* $S \in \varphi$.

We illustrate Theorems 2.1 and 2.2 in the following Example.

Example 2.2. *Consider the real matrix A of Example* 2.1. *Then there exist the signature matrices* $S_1 = diag(+1, +1, +1), S_2 = diag(-1, -1, +1)$, *and a nonnegative unit vector* $x = (0.4544, 0.4544, 0.7662)^t$ *such that* $\omega_0^S(A) = 4.1861 = \omega_0(S_2A) = x^tS_1AS_2x$. *Furthermore, for the signature matrix* $S_3 = diag(+1, +1, -1)$ *we have the relation* $\omega_0^S(A) = 4.1861 = \rho_0^S(H(S_3A)) = \lambda_{\max}(H(S_3A))$.

In view of the proof of Theorem 2.2, we immediately obtain the following corresponding result.

Corollary 2.4. *If* $A \in M_n(\mathbb{R})$ *and* $A = A^t$, *then* $\rho(A^k) = \rho_0^S(A^k) = \omega_0^S(A^k) = \omega(A^k)$ *for* $k = 1, 2, \ldots$.

3. Conclusions

In this paper, we presented an extension of Perron-Frobenius theory to the numerical range of real matrices. This extension is interesting since it leads to a relation between the sign-real spectral radius - which is used in engineering [9, 10] - and the sign-real numerical radius (Theorem 2.2 and Remark 2.1). Further applications of the sign-real spectral radius will be given in a forthcoming paper [8].

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