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DETERMINING AND DISTINGUISHING NUMBER OF HYPERGRAPHS

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In this paper, we extend the study of determining number and distinguishing number to hypergraphs. We give sharp lower bounds for the determining and distinguishing number of hypergraphs in general and give exact values with specified conditions.

Keywords: Graph automorphism, determining number, distinguishing number, hypergraph

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1. Introduction

An *automorphism* of a graph G is a permutation ρ of the vertex set $V(G)$ of G with the property that for any two vertices u and v, $\rho(u) \sim \rho(v)$ (form an edge) in G if and only if $u \sim v$ (form an edge) in G. The set of all automorphisms of G, with the operation of composition of permutations, is a *permutation group on* $V(G)$ (a subgroup of the symmetric group on $V(G)$). This is called the *automorphism group of* G and is denoted by $Aut(G)$. Every automorphism is also an isometry, that is, for $u, v \in V(G)$ and $\phi \in Aut(G)$, $d(u,v) = d(\phi(u), \phi(v))$, where $d(\cdot,\cdot)$ denotes the length of a shortest path between two vertices in G.

A set of vertices $D \subseteq V(G)$ is called a *determining set* if for $\rho, \phi \in Aut(G)$, and $\rho(v) = \phi(v)$ for all $v \in D$ implies $\rho = \phi$. That is, the image of D under an arbitrary automorphism determines the automorphism group completely. The *determining number* is the size of a smallest determining set and is denoted by $Det(G)$. The concept of determining set was introduced by Boutin in [1]. Every graph has a determining set, since any set containing all but one vertex is determining. The complete graph K_n is a graph for which such a determining set is minimal. Boutin also proved that $Det(P_n) = 1, n \geq 2; Det(C_n) = 2, n \geq 3; Det(P(5,2)) = 3$, where P_n, C_n and $P(5,2)$ represent a path on n vertices, a cycle on n vertices and the standard Petersen graph, respectively [1].

For $v \in G$, the set $\{\phi(v) : \phi \in Aut(G)\}$ is the *orbit* of v under $Aut(G)$ and two vertices in the same orbit are *similar*. An automorphism ρ *fixes* $v \in V(G)$ if

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$\rho(v) = v$ and ρ is said to fix the set D if for every $v \in D$, we have $\rho(v) = v$. The set of automorphisms that fix v is a subgroup of $Aut(G)$ called the *stabilizer* of v, denoted by $Stab(v)$. The set of automorphisms that fix a set D is a subgroup of $Aut(G)$ and is called the *point-wise stabilizer* of D, denoted by $Stab(D)$, and is defined as follows:

$$Stab(D) = \{g \in Aut(G)|\ g(v) = v, \forall\ v \in D\} = \bigcap_{v \in D} Stab(v).$$

Erwin and Harary, independently, studied the notion of determining set and used the term fixing set [2], defined as follows: If D is a set of vertices for which $Stab(D) = S_1$, where S_1 is the trivial group, then D fixes the graph G and we say that D is a *fixing set* of G. The minimum cardinality of a set of vertices that fixes G is called the fixing number of G, denoted by $fix(G)$.

Using the concept of point wise stabilizer of D, an equivalent definition of determining set was provided by Boutin [1] as follows: a set D of vertices is a determining set for G if and only if $Stab(D) = S_1$. Thus form the definition of fixing set, another equivalent definition is as follows: a set of vertices is a determining set of a graph G if and only if it is a fixing set of G [3].

Notice that, by the definition, the images of the vertices in a determining set under the trivial automorphism uniquely determine the positions of the remaining vertices. Thus a determining set not only uniquely identifies each automorphism, but also uniquely identifies each vertex in the graph by its graph properties and its relationship to the determining set. The notion of determining set has its origin in the idea of distinguishing the vertices in a graph, and more concretely, in the concept of symmetry breaking which was introduced by Albertson and Collins [4] and, independently, by Harary [5, 6]. Symmetry breaking has several applications; among them those related to the problem of programming a robot to manipulate objects [7]. Determining sets have been since then widely studied. There exists by now an extensive literature on this topic. Besides the above mentioned references, see for instance [8, 9].

To see some uses of the determining set, Albertson and Collins [4] introduced the concept of distinguishing labeling (coloring) defined as follows: A labeling $\lambda : V(G) \to \{1, 2, \ldots, l\}$ is called *l-distinguishing* if it is invariant only under the trivial automorphism. The *distinguishing number* of a graph G, $dist(G)$, is the least integer l such that G has a l-distinguishing labeling. Recent work shows that, in many infinite families, all large graphs are 2-distinguishable. These include hypercubes Q_n with $n \geq 4$ [10], 3-connected planar graphs [11] and nontrivial cartesian powers of a connected graph, $G \neq K_2, K_3$ [12]. Since distinguishing labelings involve graph automorphisms, determining sets provide a useful tool for studying them. Albertson and Boutin used determining sets to show that certain types of geometric cliques [13] and the Kneser graphs $K_{n:k}$ with $n \geq 6, k \geq 2$ [8] belong to the list of 2-distinguishable graphs. Boutin used determining sets to answer a question of Wilfried Imrich regarding the size of a smallest label class in a 2-distinguishing labeling of Q_n [14].

A *hypergraph* H is a triple $(V(H), E(H), I(H))$, where $V(H)$ is a finite set of elements called *vertices*, $E(H)$ is a finite set of elements called *hyperedges*, and $I(H) \subseteq V(H) \times E(H)$ is the *incidence relation*. $|V(H)|$ and $|E(H)|$ are called the order and the size of H, denoted by m and k, respectively. A *subhypergraph* K

of a hypergraph H is a hypergraph with vertex set $V(K) \subseteq V(H)$ and edge set $E(K) \subseteq E(H)$. A hypergraph H is *linear* if for two hyperedges $E_i, E_j \in E(H)$, $|E_i \cap E_j| \leq 1$, so for a linear hypergraph there may be no repeated hyperedges of cardinality greater than one. A hypergraph H with no hyperedge is a subset of any other is called *Sperner*.

Hypergraphs are used in clustering the data of high dimensional spaces. In a hypergraph model, each data item is represented as a vertex and related data items are connected with weighted hyperedges. A hyperedge presents a relationship (affinity) among subsets of data and the weight of the hyperedge reflects the strength of this affinity. Then a hypergraph partitioning algorithm is used to find a partitioning of the vertices such that the corresponding data items in each partition are highly related and the weight of the hyperedges cut by the partitioning is minimized [15]. Inspired by such an application of hypergraphs, we study the determining sets of hypergraphs which, in view of the above mentioned application, uniquely identifies each data item in hypergraphs by its graph properties and its relationship to the determining set.

A vertex $v \in V(H)$ is *incident* with a hyperedge E of H if $v \in E$. If v is incident with exactly n hyperedges, then we say that the *degree* of v is n; if all the vertices $v \in V(H)$ have degree n, then H is *n-regular*. Similarly, if there are exactly n vertices incident with a hyperedge E, then we say that the size of E is n; if all the hyperedges $E \in E(H)$ have size n, then H is *n-uniform*. A graph is simply a 2-uniform hypergraph. A *path* from a vertex v to another vertex u, in a hypergraph, is a finite sequence of the form $v, E_1, w_1, E_2, w_2, \ldots, E_{l-1}, w_{l-1}, E_l, u$, having *length* l such that $v \in E_1$, $w_i \in E_i \cap E_{i+1}$ for $i = 1, 2, \ldots l-1$ and $u \in E_l$. A hypergraph H is called *connected* if there is a path between any two vertices of H. All the hypergraphs considered in this paper are connected Sperner hypergraphs.

The primal graph, $prim(H)$, of a hypergraph H is a graph with vertex set $V(H)$ and vertices x and y of $prim(H)$ are adjacent if and only if x and y are contained in a hyperedge. The *middle graph*, $M(H)$, of H is a subgraph of $prim(H)$ formed by deleting all loops and parallel edges. The *dual* of $H = (\{v_1, v_2, \ldots, v_m\}, \{E_1, E_2, \ldots, E_k\})$, denoted by H^*, is the hypergraph whose vertices are $\{e_1, e_2, \ldots, e_k\}$ corresponding to the hyperedges of H and with hyperedges $V_i = \{e_j \ : \ v_i \in E_j \text{ in } H\}$, where $i = 1, 2, \ldots, m$. In other words, the dual H^* swaps the vertices and hyperedges of H.

A connected hypergraph H with no hypercycle is called a *hypertree*. A sub-hypertree of a hypertree H with edge set, say $\{E_{p_1}, E_{p_2}, \ldots, E_{p_l}\} \subset E(H)$, is called a *branch* of H if E_{p_1} (say) is the only hyperedge such that, for $E_i, E_j \in E(H) \setminus \{E_{p_1}, E_{p_2}, \ldots, E_{p_l}\}$, $E_{p_1} \cap E_i \neq \emptyset$ and $E_{p_1} \cap E_j \neq \emptyset$ implies $(E_{p_1} \cap E_i) \cap (E_{p_1} \cap E_j) \neq \emptyset$. The hyperedge E_{p_1} is called the *joint* of the branch.

A hypergraph H is said to be a *hyperstar* if $E_i \cap E_j = C \neq \emptyset$, for any $E_i, E_j \in E(H)$. We will call C, the *center* of the hyperstar. If there exist a sequence of hyperedges $E_1, E_2, \ldots, E_k$ in a hypergraph H, then H is said to be (1) a *hyperpath* if $E_i \cap E_j \neq \emptyset$ if and only if $|i - j| = 1$; (2) a *hypercycle* if, $E_i \cap E_j \neq \emptyset$ if and only if $|i - j| = 1 \pmod{k}$. A hyperedge E of H is called a *pendant hyperedge* if for $E_i, E_j \in E(H)$, $E \cap E_i \neq \emptyset$ and $E \cap E_j \neq \emptyset$ implies $(E \cap E_i) \cap (E \cap E_j) \neq \emptyset$. Let $v \in \bigcap_{i \in I} E_i$ then v is called a *supporting vertex* of E_j if and only if some E_j is a pendant hyperedge. Set of all supporting vertices of E_j is called the *supporting set* of E_j.

An automorphism of a hypergraph H is a pair (ϕ, ψ), where ϕ is a permutation of $V(H)$ and ψ is a permutation of $E(H)$ such that $(v, E) \in I(H)$ if and only if $(\phi(v), \psi(E)) \in I(H)$ for all $v \in V(H)$ and for all $E \in E(H)$. The set of all the automorphisms of H forms a group under composition of functions, denoted by $Aut(H)$. The image of the projection $(\phi, \psi) \mapsto \phi$ is denoted by $Aut_V(H)$, while the image of the projection $(\phi, \psi) \mapsto \psi$ is denoted by $Aut_E(H)$. The members of $Aut_V(H)$ and $Aut_E(H)$ are called vertex automorphisms and edge automorphisms, respectively. Any automorphism (ϕ, ψ) fixes $(v, E) \in I(H)$ if $(\phi(v), \psi(E)) = (v, E)$. The set of automorphisms that fix $(v, E) \in I(H)$ is a subgroup of $Aut(H)$ called *stabilizer* of (v, E), denoted by $Stab\{(v, E)\}$. We say that a vertex v is fixed if $\phi(v) = v$ under all vertex automorphisms of H. Similarly, an edge E is fixed if $\psi(E) = E$ under all edge automorphisms of H. In a hypercycle H, an edge automorphism $\psi \in Aut_E(H)$ such that $\psi(E_i) = E_{i+j}, 1 \leq j \leq k$ and for all E_i is called a *rotation*. A vertex automorphism $\phi \in Aut_V(H)$ such that $\phi(v_j) = v_j, \phi(v_{j-1}) = v_{j+1}, \ldots, \phi(v_1) = v_n$, is called a *flipping*. Definition of hypergraph and some of the relevant terminology is taken from [16, 17, 18].

In this paper, we study the determining and distinguishing number of hypergraphs. We give the sharp lower bounds for the determining and distinguishing number of hypergraphs. Also, we study the determining number of some well-known families of hypergraphs such as hyperpaths, hypertrees and n-uniform linear hypercycles. Further, we study the distinguishing number of hyperpaths, n-uniform linear hypercycles and n-uniform linear hypertrees.

2. Determining Number of Hypergraphs

If we denote the set of all vertices of degree d in $E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_d}$ by $C(i_1, i_2, \ldots, i_d)$, then the collection of all such sets gives a partition of $V(H)$. We denote $n(i_1, i_2, \ldots, i_d) = |C(i_1, i_2, \ldots, i_d)| - 1$ if $C(i_1, i_2, \ldots, i_d) \neq \emptyset$, otherwise take $n(i_1, i_2, \ldots, i_d) = 0$.
From the definition of $C(i_1, i_2, \ldots, i_d)$, we see that for any two vertices $u, v \in C(i_1, i_2, \ldots, i_d)$, there exists a non-trivial vertex automorphism $\phi \in Aut_V(H)$ such that $\phi(u) = v$. Thus, we have the following lemma:

Lemma 2.1. *If $u, v \in C(i_1, i_2, \ldots, i_d)$ and $D \subseteq V(H)$ be a determining set of H, then either u or v is in D. Moreover, if $u \in D$ and $v \notin D$, then $(D \setminus \{u\}) \cup \{v\}$ is also a determining set of H.*

Proposition 2.1. *For any hypergraph H with k hyperedges,*

$$Det(H) \geq \sum_{j=1}^{k} \sum_{i_1 < .. < i_j}^{k} n(i_1, i_2, \ldots, i_j).$$

Proof. It follows from the fact that, if there are $|C(i_1, i_2, \ldots, i_d)|$ number of vertices of degree d in $E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_d}$, then by Lemma 2.1, at least $n(i_1, i_2, \ldots, i_d)$ vertices should belong to a determining set D.

Remark 2.1. *By Proposition 2.1, it is clear that in order to obtain a determining set of any hypergraph H, we need $|C(i_1, i_2, \ldots, i_d)| - 1$ vertices from each class*

$C(i_1, i_2, \ldots, i_d)$, if $C(i_1, i_2, \ldots, i_d) \neq \emptyset$, that is, we can left one vertex from each class $C(i_1, i_2, \ldots, i_d)$. We call this vertex a representative vertex of $C(i_1, i_2, \ldots, i_d)$ and denote the set of all representative vertices in a hypergraph H by $R(H)$. Hence we always have $V(H) \setminus R(H) \subseteq D$ for any determining set D.

At first we are interested to discuss only those hypergraphs H for which the equality holds in Proposition 2.1.

Theorem 2.1. *For any hypergraph H with k hyperedges, if $n(i) \neq 0$ for all $E_i \in E(H)$ except one, then $Det(H) = \sum_{j=1}^{k} \sum_{i_1<..<i_j}^{k} n(i_1, i_2, \ldots, i_j)$. Moreover, there are $\prod_{j=1}^{k} \prod_{i_1<..<i_j}^{k} (n(i_1, i_2, \ldots, i_j) + 1)$ determining sets of cardinality $Det(H)$ of H.*

Proof. Consider $D = V(H) \setminus R(H)$, we have to show that D is a determining set for H. For any non-trivial vertex automorphism ϕ, there exist representative vertices v_i and v_j such that $\phi(v_i) = v_j$. But one of $n(i)$ or $n(j)$ is non-zero, say $n(i) \neq 0$, by Lemma 2.1, we must have at least one vertex $v \in E_i \cap D$ and hence E_i is fixed, a contradiction.

Further, by Lemma 2.1, there are $\prod_{j=1}^{k} \prod_{i_1<..<i_j}^{k} (n(i_1, i_2, \ldots, i_j) + 1)$ such determining sets.

For all $n \geq 4$, if H is an n-uniform linear hypergraph with k hyperedges, then $n(i) \neq 0$ for all $E_i \in E(H)$ and $n(i, i+1) = 0$ for all i. Thus, we have the following corollary:

Corollary 2.1. *For $n \geq 4$, let H be an n-uniform linear hypergraph with k hyperedges. Then $Det(H) = \sum_{i=1}^{k} n(i)$.*

We give a simple example which shows that the condition in Theorem 2.1 cannot be relaxed generally.

Example 2.1. *Let H be a hypergraph with vertex set $V(H) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and edge set $E(H) = \{E_1 = \{v_1, v_2\}, E_2 = \{v_2, v_3, v_4, v_5\}, E_3 = \{v_5, v_6\}\}$. Clearly, $n(1) = n(3) = 0$ and $n(2) = 1$. Without loss of generality, we can take $R(H) = \{v_1, v_2, v_3, v_5, v_6\}$ and hence $D = \{v_4\}$. But D is not a determining set for H, since there exists $\phi \in Aut_V(H)$ such that $\phi(v_1) = v_6$. In fact $Det(H) = 2 > 1$.*

However, the condition in Theorem 2.1 can be reduced in some special cases as shown in the following results.

Lemma 2.2. *Let H be a hyperpath with k hyperedges, say, $E_1, E_2, \ldots E_k$ in a canonical way. Then $Det(H) = \sum_{i=1}^{k} n(i) + \sum_{i=1}^{k-1} n(i, i+1)$, when*

(1) *k is even and $n(i) \neq 0$ for some $1 \leq i \leq k$ or $n(i, i+1) \neq 0$ for some $1 \leq i < k$ except $i = \frac{k}{2}$.*
(2) *k is odd and $n(i) \neq 0$, for any $1 \leq i \leq k$ except $i = \lceil \frac{k}{2} \rceil$ or $n(i, i+1) \neq 0$, for any $1 \leq i < k$.*

Proof. It is easy to see that the only possible non-trivial vertex automorphism for any hyperpath is, $\phi(v_i) = v_{k-i+1}$, for all $1 \leq i \leq k$ and $\phi(v_{i,i+1}) = v_{k-i,k-i+1}$ for all $1 \leq i < k$, where v_i and $v_{i,i+1}$ are representative vertices. If k is even, then this automorphism is possible only when $n(i) = 0$, for all $1 \leq i \leq k$ and $n(i, i+1) = 0$, for all $1 \leq i < k$ except $i = \frac{k}{2}$, since $\phi(v_{\frac{k}{2},\frac{k}{2}+1}) = v_{\frac{k}{2},\frac{k}{2}+1}$. If k is odd, then this automorphism is possible only when $n(i, i+1) = 0$, for any $1 \leq i < k$, and $n(i) = 0$, for all $1 \leq i \leq k$, except $i = \lceil\frac{k}{2}\rceil$, since $\phi(v_{\lceil\frac{k}{2}\rceil}) = v_{\lceil\frac{k}{2}\rceil}$.

Example 2.2. *Let H be a hypergraph with vertex set $V(H) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}\}$ and edge set $E(H) = \{E_1 = \{v_1, v_2\}, E_2 = \{v_2, v_3, v_4\}, E_3 = \{v_4, v_5, v_6, v_7\}, E_4 = \{v_7, v_8, v_9\}, E_5 = \{v_9, v_{10}\}\}$. Clearly, $n(3) \neq 0$. Without loss of generality, we can take $R(H) = \{v_1, v_2, v_3, v_4, v_5, v_7, v_8, v_9, v_{10}\}$ and hence $D = \{v_6\}$. But D is not a determining set for H, since there exists $\phi \in Aut_V(H)$ such that $\phi(v_1) = v_{10}, \phi(v_2) = v_9, \phi(v_3) = v_8, \phi(v_4) = v_7$. In fact, $Det(H) = 2 > 1$.*

Theorem 2.2. *Let H be a hypertree. Let $S_1, S_2, \ldots, S_s$ be the supporting sets and $E_{p_1^j}, E_{p_2^j}, \ldots, E_{p_i^j}$ be the pendant hyperedges with respect to supporting set $S_j (1 \leq j \leq s)$. Then $Det(H) = \sum_{j=1}^{k} \sum_{i_1<..<i_j}^{k} n(i_1, i_2, \ldots, i_j)$, if $n(p_l^j) \neq 0$ for all j and for all $l = 1, 2, \ldots, i$ except one.*

Proof. Consider $D = V(H) \setminus R(H)$, we have to show that D is a determining set for H. For any non-trivial vertex automorphism ϕ, there exist representative vertices v_i and v_j such that $\phi(v_i) = v_j$. Consider a hyperpath containing hyperedges, say, E_i and E_j together with pendant hyperedges say, E_{p_1} and E_{p_2}, respectively. But one of $n(p_1)$ and $n(p_2)$ is non-empty and hence proof follows by Theorem 2.1.

An n-uniform linear hyperstar ($n \geq 3$) is a special case of the hypertree in which $n(i) \neq 0$ for all $E_i \in E(H)$, so we have the following corollary:

Corollary 2.2. *For $n \geq 3$, let H be an n-uniform linear hyperstar with k (≥ 3) hyperedges. Then $Det(H) = k(n-2)$.*

Consider an n-uniform linear hypercycle $\mathcal{C}_{k,n}$ with k hyperedges. When $n \geq 4$, then $Det(\mathcal{C}_{k,n}) = k(n-3)$, by Corollary 2.1.

For the case $n = 3$, we have $n(i) = 0$ for all $E_i \in E(H)$, hence the lower bound given in Proposition 2.1 is zero and every vertex in $\mathcal{C}_{k,3}$ is the representative vertex. We discuss this case in the following result:

Theorem 2.3. *Let $\mathcal{C}_{k,3}$ be a 3-uniform linear hypercycle with k hyperedges. Then $Det\,(\mathcal{C}_{k,3}) = 2$ for all $k \geq 3$.*

Proof. Let $V(\mathcal{C}_{k,3}) = \{v_i, v_{i,i+1}\ ;\ 1 \leq i \leq k\}$ and edge set $E(\mathcal{C}_{k,3}) = \{E_1, E_2, \ldots, E_k\}$, where $v_i \in E_i$ is a vertex of degree one and $v_{i,i+1} \in E_i \cap E_{i+1}$ is a vertex of degree two with $v_{k,k+1} = v_{k,1} \in E_k \cap E_{k+1} = E_k \cap E_1$. Then one can easily see that if we fix only one vertex in $\mathcal{C}_{k,3}$, then all the rotations are destroyed and the fixing of one more vertex destroys the flipping. Hence the proof follows.

From the definition of the primal graph of a hypergraph H, we note that $Aut(H)$ is automorphism group of H if and only if $Aut(H)$ is an automorphism group for the primal graph of H. Thus, we have the following straightforward result:

Theorem 2.4. *Let H be a hypergraph and $prim(H)$ be the primal graph of H. Then $Det(H) = Det(prim(H))$.*

The primal graph of a hypergraph H is a simple graph (without loops and parallel edges), which is also the middle graph. But, the primal graph of the dual hypergraph H^* of H is not a simple graph, in this case, the middle graph of H^* is a simple graph. We discuss the determining number of dual hypergraphs separately in the following result:

Theorem 2.5. *Let H^* be the dual of a hypergraph H and let $M(H^*)$ be the middle graph of H^*. Then $Det(H^*) = Det(M(H^*))$.*

Proof. By the definition of middle graph, $Aut(H^*)$ is automorphism group of H^* if and only if $Aut(H^*)$ is an automorphism group for the middle graph of H^*. Thus a set $D \subseteq V(H^*)$ is a determining set for H^* if and only if D is a determining set for $M(H^*)$.

3. Distinguishing Number of Hypergraphs

In this section, we study the distinguishing number of hypergraphs.

From Lemma 2.1, we have the following result:

Lemma 3.1. *Let λ be a distinguishing labeling of the vertices of H. If $u, v \in C(i_1, i_2, ..., i_d)$, then $\lambda(u)$ is different from $\lambda(v)$.*

The following result gives the lower bound for the distinguishing number of hypergraphs.

Proposition 3.1. *Let H be a hypergraph with k hyperedges. Then $dist(H) \geq \eta$, where $\eta = \max |C(i_1, i_2, \ldots, i_d)|$ in H.*

Proof. Since $\eta = \max |C(i_1, i_2, \ldots, i_d)|$ in H, by Lemma 3.1, we have at least η distinct labels to distinguish the labeling of the hypergraph. Thus $dist(H) \geq \eta$.

The lower bound given in Proposition 3.1 is sharp for an n-uniform linear hyperpath with k hyperedges $(n, k \geq 3)$ as we have shown in the following result:

Theorem 3.1. *Let H be an n-uniform linear hyperpath with k hyperedges. Then*

$$dist\,(H) = \begin{cases} n & when \;\; k = 2, n \geq 2 \;\; or \;\; n = 2, k \geq 2, \\ n - 1 & when \;\; n, k \geq 3. \end{cases}$$

Proof. When $n = 2$ and $k \geq 2$, then H is a simple path P_{k+1} on $k + 1$ vertices and $dist(P_{k+1}) = dist(H) = 2 = n$.

When $k = 2$ and $n \geq 2$, then since there are only two hyperedges, say E_1 and E_2, in H having only one vertex in common. If we consider a labeling $\lambda : V(H) \to \{1, 2, \ldots, n-1\}$, then there exists a non-trivial automorphism (ϕ, ψ) of H such that $\psi(E_1) = E_2$ and $\phi(\lambda(v)) = \lambda(\phi(v)) = \lambda(u)$, for all $v \in E_1$, $u \in E_2$. Thus $dist(H) = n$.

When $n, k \geq 3$. Let us denote the vertices of a hyperedge E_j $(1 \leq j \leq k)$ in H by $v_{i,j}$ $(1 \leq i \leq n)$ with $v_{n,j} = v_{1,j+1}$ $(1 \leq j \leq k - 1)$. Then the vertices

$v_{1,1}, v_{n,k}, v_{i,j}$ $(2 \le i \le n-1;\ 1 \le j \le k)$ are of degree one and the vertices $v_{1,j+1}$ $(1 \le j \le k-1)$ are of degree two. Consider a labeling $\lambda : V(H) \to \{1, 2, \ldots, n-1\}$ defined as: $v_{i,j} \mapsto i$ (for $1 \le i \le n-1$ and for each j; $1 \le j \le k-1$); $v_{1,j+1} \mapsto 1$ $(1 \le j \le k-2)$; $v_{1,k} \mapsto n-1$ and $v_{i,k} \mapsto i-1$ $(2 \le i \le n)$. Then one can easily see that there is no non-trivial automorphism which preserves this labeling. It follows that $dist(H) = n-1$.

In the following example, we give a hypergraph with $dist(H) > \eta$.

Example 3.1. *Consider a hypergraph H with vertex set $V(H) = \{v_1, v_2, \ldots, v_{16}\}$ and edge set $E(H) = \{E_1 = \{v_1, v_2, v_3, v_4\}, E_2 = \{v_4, v_5, v_6, v_7\}, E_3 = \{v_7, v_8, v_9, v_{10}\}, E_4 = \{v_{10}, v_{11}, v_{12}, v_{13}\}, E_5 = \{v_{10}, v_{14}, v_{15}, v_{16}\}\}$. Here E_4 and E_5 are the pendant hyperedges in H with one vertex in common and $\eta = 3$. But $dist(H) \neq \eta$ since otherwise there exists a non-trivial automorphism (ϕ, ψ) of H such that $\psi(E_4) = E_5$ and $\phi(\lambda(v)) = \lambda(\phi(v)) = \lambda(u)$, for all $v \in E_4, u \in E_5$. In fact, $dist(H) = 4 > \eta$.*

Theorem 3.2. *Let H be a hyperpath with k hyperedges. Then $\eta \le dist(H) \le \eta + 1$, where $\eta = \max |C(i_1, i_2, \ldots, i_d)|$ in H.*

Proof. If H is an n-uniform linear, then the result follows from Theorem 3.1. If H is not an n-uniform linear, then left hand side of the inequality follows from Proposition 3.1. For the right side, consider a labeling $\lambda : V(H) \to \{1, 2, \ldots, \eta+1\}$. Since $\eta = \max |C(i_1, i_2, \ldots, i_d)|$, so it is straightforward to see that there does not exist any non-trivial vertex automorphism $\phi \in Aut_V(H)$ such that $\phi(\lambda(v)) = \lambda\phi(v)$. Thus $dist(H) \le \eta + 1$.

Let $\mathcal{C}_{k,n}$ be an n-uniform linear hypercycle with $k \ge 3$ hyperedges. We denote the vertices of $\mathcal{C}_{k,n}$ by $v_{i,j}$, where j $(1 \le j \le k)$ represents the hyperedge number of $\mathcal{C}_{k,n}$ and i $(1 \le i \le n)$ represents the vertex number of the jth hyperedge. Each $v_{i,j} \in E_j$, $2 \le i \le n-1$, represents a vertex of degree one and $v_{n,j} = v_{1,j+1} \in E_j \cap E_{j+1}$ represents the vertex of degree two with $v_{n,k} = v_{1,1}$.

Theorem 3.3. *Let $\mathcal{C}_{k,n}$ be an n-uniform linear hypercycle with $k \ge 3$ hyperedges. Then for $n \ge 5$, we have $dist(\mathcal{C}_{k,n}) = n-2$ and*

$$dist\,(\mathcal{C}_{k,n}) = \begin{cases} 3 & \text{when } n = 2, 4, \\ 2 & \text{when } n = 3. \end{cases}$$

Proof. When $n = 2$, then $\mathcal{C}_{k,2}$ is a simple cycle C_k on k vertices and $dist(\mathcal{C}_{k,2}) = dist(C_k) = 3$ [4].

When $n = 3$ and 4, then it is straightforward to see that there is no automorphism of $\mathcal{C}_{k,3}$ and of $\mathcal{C}_{k,4}$ which preserves the following labeling respectively:
$\lambda : V(\mathcal{C}_{k,3}) \to \{1, 2\}$ defined as: $v_{2,j} \mapsto j$ for all $j = 1, 2$; $v_{2,3} \mapsto 2$; $v_{2,j} \mapsto 1$ for all $4 \le j \le k$; $v_{1,1} \mapsto 1$; $v_{1,2} \mapsto 2$; $v_{1,3} \mapsto 2$ and $v_{1,j+1} \mapsto 1$ for all $3 \le j \le k-1$, and
$\mu : V(\mathcal{C}_{k,4}) \to \{1, 2, 3\}$ defined as: for all $1 \le j \le k$, $v_{2,j} \mapsto 1$, $v_{3,j} \mapsto 2$ when j is odd and $v_{3,j} \mapsto 3$ when j is even; $v_{1,1} \mapsto 1$; $v_{1,2} \mapsto 2$; $v_{1,3} \mapsto 2$; $v_{1,4} \mapsto 3$ and $v_{1,j+1} \mapsto 1$ for all $3 \le j \le k-1$.

When $n \ge 5$, then since $\max |C(i_1, i_2, \ldots, i_d)| = n-2$ in $\mathcal{C}_{k,n}$ so, by Proposition 3.1, $dist(\mathcal{C}_{k,n}) \ge n-2$. Now, if we consider a labeling $\nu : V(\mathcal{C}_{k,n}) \mapsto \{1, 2, \ldots, n-2\}$

defined as: $v_{i,j} \mapsto i-1$ for all $1 \leq j \leq k$ and $2 \leq i \leq n-1$; $v_{1,1} \mapsto 1$; $v_{1,j+1} \mapsto j+1$ for $1 \leq j \leq n-3$; $v_{1,j+1} \mapsto 1$ for all $n-2 \leq j \leq k-1$, then there is no automorphism of $\mathcal{C}_{k,n}$ which preserves this labeling. This implies that $dist(\mathcal{C}_{k,n}) \leq n-2$.

The following result gives the distinguishing number of an n-uniform linear hypertree.

Theorem 3.4. *Let H be an n-uniform linear hypertree with k hyperedges $(n, k \geq 3)$. Let $s_1, s_2, \ldots, s_q$ be the supporting vertices in H and n_p denotes the number of pendant hyperedges with respect to the supporting vertex s_p $(1 \leq p \leq q)$. If $r = max(n_1, n_2, \ldots, n_q)$, then H is l-distinguishing, where*

$$l = \begin{cases} n-1 & \text{if } r = 1, \\ n & \text{if } 2 \leq r \leq n, \\ r & \text{if } r > n. \end{cases}$$

Proof. Suppose that $E_{p_1^t}, E_{p_2^t}, \ldots, E_{p_r^t}$ be the pendant hyperedges with respect to the supporting vertex s_t. If $\lambda : V(H) \to \{1, 2, \ldots, l-1\}$ be a labeling of the vertices of H, then there exists a non-trivial automorphism (ϕ, ψ) of H such that for two pendant hyperedges, say $E_{p_i^t}$ and $E_{p_j^t}$, with respect to the supporting vertex s_t, we have $\psi(E_{p_i^t}) = E_{p_j^t}$ and $\phi(\lambda(u)) = \lambda(\phi(u)) = \lambda(v)$ for all $u \in E_{p_i^t}, v \in E_{p_j^t}$.

4. Conclusion

In this paper, we have studied the determining and distinguishing number of hypergraphs. We gave the sharp lower bounds for the determining and distinguishing number of hypergraphs. Also, we studied the determining number of some well-known families of hypergraphs such as hyperpaths, hypertrees and n-uniform linear hypercycles. Moreover, we have computed the distinguishing number of hyperpaths, n-uniform linear hypercycles and n-uniform linear hypertrees.

Future work can be directed towards obtaining the determining and distinguishing number of some other challenging classes of graphs and hypergraphs. Moreover, it can be interesting (like as in the simple graphs) to study the difference between metric dimension (studied in [18]) and determining number of hypergraphs.

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