A SINGULAR FRACTIONAL INTEGRO-DIFFERENTIAL EQUATION

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We investigate the existence of solutions for a singular fractional differential equation with some integral boundary condition.

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1. **Introduction**

As you know it has been published many papers about fractional differential equations and inclusions which have been applied in modeling of many problems of engineering sciences, physics, nano technology, etc. Some researchers have been investigated the existence of solutions for some singular fractional differential equations (see for example, [4], [5], [7] and [10]). In 2010, the fractional problem $D^{\alpha}u(t) + f(t, u(t), D^{\mu}u(t)) = 0$ with boundary conditions $u(0) = u(1) = 0$ investigated, where $0 < t < 1$, $1 < \alpha < 2$, $0 < \mu \leq \alpha - 1$, D^{α} is the standard Riemann-Liouville fractional derivative, f satisfies the Carateodory conditions on $[0, 1]\times(0, \infty)\times R$, f is positive and f(t,x,y) is singular at $t = 0$ ([1]). In 2012, the fractional differential equation $D^{\alpha}u(t) + f(t, u(t)) = 0$ with boundary conditions $u(0) = u''(0) = 0$ and $u(1) = \lambda \int_0^1 u(s)ds$ investigated, where $0 < t < 1$, $2 < \alpha < 3$, $0 < \lambda < 2$, D^{α} is the Caputo fractional derivative and $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is a continuous function ([2]). In 2014, the singular fractional problem ${}^cD^q_{0+}u(t) + f(t, u(t), {}^cD^{\sigma}_{0+}u(t)) = 0$ with boundary conditions $u(0) = u'(0) = 0$ and $u'(1) = {}^cD^{\sigma}_{0+}u(1)$ investigated, where $0 < t < 1$, $2 < q < 3$, $0 < \sigma < 1$, $f : (0, 1] \times R \times R \to R$ is continuous with $f(t, x, y)$ may be singular at some points of $t \in [0, 1]$ and ${}^cD^q_{0+}$ is the Caputo derivative ([3]). By using main idea of the works, we investigate the singular fractional integro-differential equation

$$D^{\alpha}x(t) + f(t, x(t), x'(t), D^{\beta}x(t), \int_0^t h(\xi)x(\xi)d\xi) = 0 \quad (1.1)$$

with boundary conditions $x(0) = 0$ and $x(1) = D^{\gamma}x(\mu)$, where $0 < t < 1$, $x \in C^1[0, 1]$, $\alpha \geq 2$, $0 < \beta < 1$, $0 < \gamma < 1$, $0 < \mu < 1$, $h \in L^1[0, 1]$ is nonnegative with $\|h\|_1 = m$, D^q is

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the Caputo fractional derivative of order q and $f(t, x_1, x_2, x_3, x_4)$ is singular at some points $t \in [0,1]$. In this paper, we use $\|.\|_1$ as the norm of $L^1[0,1]$, $\|.\|$ as the norm $Y = C[0,1]$ and $\|.\|_*$ as the norm of $X = C^1[0,1]$. As we know, Riemann-Liouville integral of order p with the lower limit $a \geq 0$ for a function $f : (a, \infty) \to \mathbb{R}$ is defined by $I^p_{a+} f(t) = \frac{1}{\Gamma(p)} \int_a^t (t-s)^{p-1} f(s) ds$ whenever the right-hand side is pointwise define on (a, ∞) ([6]). we denote $I^p_{0+} f(t)$ by $I^p f(t)$. Also, the Caputo fractional derivative of order $\alpha > 0$ of a function $f : (a, \infty) \to \mathbb{R}$ is defined by ${}^cD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t \frac{f^n(s)}{(t-s)^{\alpha+1-n}} ds$, where $n = [\alpha] + 1$ ([6]). We need the following results.

Lemma 1.1. *([8]) Suppose that $0 < n-1 \leq \alpha < n$ and $x \in C(0,1) \cap L^1(0,1)$. Then $I^\alpha D^\alpha x(t) = x(t) + \sum_{i=0}^{n-1} c_i t^i$ for some constants $c_0, \ldots, c_{n-1} \in \mathbb{R}$.*

Lemma 1.2. *([9]) If E is a closed, bounded and convex subset of a Banach space X and $F : E \to E$ is completely continuous, then F has a fixed point in E.*

Lemma 1.3. *([11]) Let X be a Banach space, C a closed and convex subset of X, Ω a relatively open subset of C with $0 \in \Omega$ and $F : \Omega \to C$ a continuous and compact map. Then either F has a fixed point in $\bar{\Omega}$ or there exist $y \in \partial\Omega$ and lambda $\in (0,1)$ such that $y = \lambda F y$.*

2. Main Results

We first prove the following key result.

Lemma 2.1. *Let $\alpha \geq 2$, $0 < \gamma < 1$, $0 < \mu < 1$, $\Delta = 1 - \frac{\mu^{1-\gamma}}{\Gamma(2-\gamma)}$ and $y \in L^1[0,1]$. Then unique solution of the problem $D^\alpha u(t) + y(t) = 0$ with boundary conditions $u(1) = D^\gamma u(\mu)$ and $u(0) = 0$ is $u_0(t) = \int_0^1 G(t,s) y(s) ds$, where*

$$G(t,s) = \begin{cases} \frac{-\Delta(t-s)^{\alpha-1}\Gamma(\alpha-\gamma) + t\Gamma(\alpha-\gamma)(1-s)^{\alpha-1} - t\Gamma(\alpha)(\mu-s)^{\alpha-\gamma-1}}{\Delta\Gamma(\alpha)\Gamma(\alpha-\gamma)} & 0 \leq s \leq t \leq 1,\ s \leq \mu, \\ \frac{-\Delta(t-s)^{\alpha-1} + t(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} & 0 \leq \mu \leq s \leq t \leq 1, \\ \frac{t\Gamma(\alpha-\gamma)(1-s)^{\alpha-1} - t\Gamma(\alpha)(\mu-s)^{\alpha-\gamma-1}}{\Delta\Gamma(\alpha)\Gamma(\alpha-\gamma)} & 0 \leq t \leq s \leq \mu \leq 1, \\ \frac{t(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} & 0 \leq t \leq s \leq 1, \mu \leq s. \end{cases}$$

Proof. Let u be a solution for the problem. By using Lemma 1.1, we get

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds + c_1 t.$$

Hence, $D^\gamma u(t) = -\frac{1}{\Gamma(\alpha-\gamma)} \int_0^\mu (\mu-s)^{\alpha-\gamma-1} y(s) ds + c_1 \frac{\mu^{1-\gamma}}{\Gamma(2-\gamma)}$ and $u(1) = -\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s) ds + c_1$. Since $u(1) = D^\gamma u(\mu)$, we conclude that

$$c_1 \left(1 - \frac{\mu^{1-\gamma}}{\Gamma(2-\gamma)}\right) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s) ds - \frac{1}{\Gamma(\alpha-\gamma)} \int_0^\mu (\mu-s)^{\alpha-\gamma-1} y(s) ds$$

and so $c_1 = \frac{1}{\Delta\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s) ds - \frac{1}{\Delta\Gamma(\alpha-\gamma)} \int_0^\mu (\mu-s)^{\alpha-\gamma-1} y(s) ds$. Thus, we obtain $u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds + \frac{t}{\Delta\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s) ds - \frac{t}{\Delta\Gamma(\alpha-\gamma)} \int_0^\mu (\mu-s)^{\alpha-\gamma-1} y(s) ds$.

Hence,

$$u(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + \frac{t}{\Delta\Gamma(\alpha)}(\int_0^t + \int_t^\mu + \int_\mu^1)((1-s)^{\alpha-1}y(s)ds)$$

$$-\frac{t}{\Delta\Gamma(\alpha-\gamma)}(\int_0^t + \int_t^\mu)((\mu-s)^{\alpha-\gamma-1}y(s)ds)$$

$$= \int_0^t \frac{-\Delta(t-s)^{\alpha-1}\Gamma(\alpha-\gamma) + t\Gamma(\alpha-\gamma)(1-s)^{\alpha-1} - t\Gamma(\alpha)(\mu-s)^{\alpha-\gamma-1}}{\Delta\Gamma(\alpha)\Gamma(\alpha-\gamma)}y(s)ds$$

$$+\int_t^\mu \frac{t\Gamma(\alpha-\gamma)(1-s)^{\alpha-1} - t\Gamma(\alpha)(\mu-s)^{\alpha-\gamma-1}}{\Delta\Gamma(\alpha)\Gamma(\alpha-\gamma)}y(s)ds + \int_\mu^1 \frac{t(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)}y(s)ds$$

whenever $t \leq \mu$ and

$$u(t) = -\frac{1}{\Gamma(\alpha)}(\int_0^\mu + \int_\mu^t)((t-s)^{\alpha-1}y(s)ds)$$

$$+\frac{t}{\Delta\Gamma(\alpha)}(\int_0^\mu + \int_\mu^t + \int_t^1)((1-s)^{\alpha-1}y(s)ds) - \frac{t}{\Delta\Gamma(\alpha-\gamma)}\int_0^\mu (\mu-s)^{\alpha-\gamma-1}y(s)ds$$

$$= \int_0^\mu \frac{-\Delta(t-s)^{\alpha-1}\Gamma(\alpha-\gamma) + t\Gamma(\alpha-\gamma)(1-s)^{\alpha-1} - t\Gamma(\alpha)(\mu-s)^{\alpha-\gamma-1}}{\Delta\Gamma(\alpha)\Gamma(\alpha-\gamma)}y(s)ds$$

$$+\int_\mu^t \frac{-\Delta(t-s)^{\alpha-1} + t(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)}y(s)ds + \int_t^1 \frac{t(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)}y(s)ds$$

whenever $t \geq \mu$. This implies that, $u(t) = \int_0^1 G(t,s)y(s)ds = u_0(t)$ for all t. □

Define the map $F : X \to X$ by

$$F_x(t) = \int_0^1 G(t,s)f(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi)ds$$

$$= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi)ds$$

$$+\frac{t}{\Delta\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}f(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi)ds$$

$$-\frac{t}{\Delta\Gamma(\alpha-\gamma)}\int_0^\mu (\mu-s)^{\alpha-\gamma-1}f(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi)ds. \quad (2.1)$$

If $x \in C^1[0,1]$, then$D^\beta x(t) = \frac{1}{\Gamma(1-\beta)}\int_0^t \frac{x'(s)ds}{(t-s)^\beta}$ and so

$$|D^\beta x(t)| \leq \frac{\|x'\|}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}ds = \frac{\|x'\|}{\Gamma(2-\beta)}t^{1-\beta}.$$

Thus, $D^\beta x \in C[0,1]$ and $|D^\beta x| \leq \frac{\|x'\|}{\Gamma(2-\beta)}$. Since $\int_0^1 h(z)dz = m > 0$, we get $|g(t)| \leq \|x\|\int_0^t h(z)dz \leq m\|x\|$, where $g(t) = \int_0^t h(z)x(z)dz$. Now, we give our main result.

Theorem 2.1. *Suppose that there exist the maps $a_1, a_2, a_3, a_4 : [0,1] \to \mathbb{R}$ with $\int_0^1 a_i(t)dt < \infty$ for all $i = 1, 2, 3, 4$ such that*

$$|f(t, x_1, y_1, z_1, w_1) - f(t, x_2, y_2, z_2, w_2)|$$
$$\leq a_1(t)|x_1 - x_2| + a_2(t)|y_1 - y_2| + a_3(t)|z_1 - z_2| + a_4(t)|w_1 - w_2|$$

for all $(x_1, y_1, z_1, w_1), (x_2, y_2, z_2, w_2) \in \mathbb{R}^4$ and $t \in [0,1]$. Assume that there exist $b \in L^1[0,1]$ and $H \in Y^4$ such that $|f(t, x_1, x_2, x_3, x_4)| \leq b(t)H(x_1, x_2, x_3, x_4)$ for all $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ and almost all $t \in [0,1]$ and $\|H\|_Y := Sup\{|H(x_1, x_2, x_3, x_4)| : (x_1, x_2, x_3, x_4) \in \mathbb{R}^4\} < \infty$. Then the singular problem (1.1) has a solution.

Proof. If $x_1, x_2 \in X$, then

$$|F_{x_1}(t) - F_{x_2}(t)| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, x_1(s), ..., \int_0^s h(\xi)x_1(\xi)d\xi)$$
$$-f(s, x_2(s), ..., \int_0^s h(\xi)x_2(\xi)d\xi)|ds + \frac{t}{\Gamma(\alpha)}$$
$$\times \int_0^1 (1-s)^{\alpha-1} |f(s, x_1(s), ..., \int_0^s h(\xi)x_1(\xi)d\xi) - f(s, x_2(s), ..., \int_0^s h(\xi)x_2(\xi)d\xi)|ds$$
$$+\frac{t}{\Delta\Gamma(\alpha-\gamma)} \int_0^\mu (\mu-s)^{\alpha-\gamma-1} |f(s, x_1(s), ..., \int_0^s h(\xi)x_1(\xi)d\xi)$$
$$-f(s, x_2(s), ..., \int_0^s h(\xi)x_2(\xi)d\xi)|ds$$
$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (a_1(s)|x_1 - x_2| + a_2(s)|x_1' - x_2'| + a_3(s)|D^\beta x_1 - D^\beta x_2|$$
$$+a_4(s)|\int_0^s h(\xi)(x_1(\xi) - x_2(\xi))d\xi|)ds$$
$$+\frac{t}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} (a_1(s)|x_1 - x_2| + a_2(s)|x_1' - x_2'| + a_3(s)|D^\beta x_1 - D^\beta x_2|$$
$$+a_4(s)|\int_0^s h(\xi)(x_1(\xi) - x_2(\xi))d\xi|)ds$$
$$+\frac{t}{\Delta\Gamma(\alpha-\gamma)} \int_0^\mu (\mu-s)^{\alpha-\gamma-1} (a_1(s)|x_1 - x_2| + a_2(s)|x_1' - x_2'|$$
$$+a_3(s)|D^\beta x_1 - D^\beta x_2| + a_4(s)|\int_0^s h(\xi)(x_1(\xi) - x_2(\xi))d\xi|)ds$$
$$\leq \|x_1 - x_2\|(\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}(a_1(s) + a_4(s).m)ds)$$
$$+\|x_1' - x_2'\|(\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}(a_2(s) + \frac{a_3(s)}{\Gamma(2-\beta)})ds)$$
$$+\|x_1 - x_2\|(\frac{t}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}(a_1(s) + a_4(s).m)ds)$$
$$+\|x_1' - x_2'\|(\frac{t}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}(a_2(s) + \frac{a_3(s)}{\Gamma(2-\beta)})ds)$$
$$+\|x_1 - x_2\|(\frac{t}{\Delta\Gamma(\alpha-\gamma)} \int_0^\mu (\mu-s)^{\alpha-\gamma-1}(a_1(s) + a_4(s).m)ds)$$

$$
\begin{aligned}
&+\|x_1'-x_2'\|(\frac{t}{\Delta\Gamma(\alpha-\gamma)}\int_0^\mu(\mu-s)^{\alpha-\gamma-1}(a_2(s)+\frac{a_3(s)}{\Gamma(2-\beta)})ds)\\
&\le \|x_1-x_2\|\int_0^1(1-s)^{\alpha-\gamma-1}(\frac{2a_1(s)+2ma_4(s)}{\Gamma(\alpha)}+\frac{a_1(s)+ma_4(s)}{\Delta\Gamma(\alpha-\gamma)})ds\\
&+\|x_1'-x_2'\|\int_0^1(1-s)^{\alpha-\gamma-1}(\frac{2a_2(s)}{\Gamma(\alpha)}+\frac{a_2(s)}{\Delta\Gamma(\alpha-\gamma)}\\
&+\frac{2a_3(s)}{\Gamma(\alpha)\Gamma(2-\beta)}+\frac{a_3(s)}{\Delta\Gamma(\alpha-\gamma)\Gamma(2-\beta)})ds\\
&\le \theta_1(\|x_1-x_2\|+\|x_1'-x_2'\|)=\theta_1\|x_1-x_2\|_*,
\end{aligned}
$$

where

$$
\begin{aligned}
\theta_1 = max\{&\int_0^1(1-s)^{\alpha-\gamma-1}(\frac{2a_1(s)+2ma_4(s)}{\Gamma(\alpha)}+\frac{a_1(s)+ma_4(s)}{\Delta\Gamma(\alpha-\gamma)})ds,\\
&\int_0^1(1-s)^{\alpha-\gamma-1}(\frac{2a_2(s)}{\Gamma(\alpha)}+\frac{a_2(s)}{\Delta\Gamma(\alpha-\gamma)}+\frac{2a_3(s)}{\Gamma(\alpha)\Gamma(2-\beta)}+\frac{a_3(s)}{\Delta\Gamma(\alpha-\gamma)\Gamma(2-\beta)})ds\}<\infty.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&|F_{x_1}'(t)-F_{x_2}'(t)|\\
&\le \int_0^1\frac{\partial G(t,s)}{\partial t}|f(s,x_1(s),...,\int_0^s h(\xi)x_1(\xi)d\xi)-f(s,x_2(s),...,\int_0^s h(\xi)x_2(\xi)d\xi)|ds\\
&\le \|x_1-x_2\|\int_0^1(1-s)^{\alpha-2}(\frac{a_1(s)}{\Gamma(\alpha-1)}+\frac{a_1(s)}{\Gamma(\alpha)}+\frac{a_1(s)}{\Delta\Gamma(\alpha-\gamma)}\\
&+\frac{ma_4(s)}{\Gamma(\alpha-1)}+\frac{ma_4(s)}{\Gamma(\alpha)}+\frac{ma_4(s)}{\Delta\Gamma(\alpha-\gamma)})ds\\
&+\|x_1'-x_2'\|\int_0^1(1-s)^{\alpha-2}(\frac{a_2(s)}{\Gamma(\alpha-1)}+\frac{a_2(s)}{\Gamma(\alpha)}+\frac{a_2(s)}{\Delta\Gamma(\alpha-\gamma)}+\frac{a_3(s)}{\Gamma(\alpha-1)\Gamma(2-\beta)}\\
&+\frac{a_3(s)}{\Gamma(\alpha)\Gamma(2-\beta)}+\frac{a_3(s)}{\Delta\Gamma(\alpha-\gamma)\Gamma(2-\beta)})ds\le\theta_1(\|x_1-x_2\|+\|x_1'-x_2'\|)=\theta_2\|x_1-x_2\|_*,
\end{aligned}
$$

where

$$
\begin{aligned}
\theta_2 = max\{&\int_0^1(1-s)^{\alpha-2}(\frac{a_1(s)}{\Gamma(\alpha-1)}+\frac{a_1(s)}{\Gamma(\alpha)}+\frac{a_1(s)}{\Delta\Gamma(\alpha-\gamma)}\\
&+\frac{ma_4(s)}{\Gamma(\alpha-1)}+\frac{ma_4(s)}{\Gamma(\alpha)}+\frac{ma_4(s)}{\Delta\Gamma(\alpha-\gamma)})ds,\\
&\int_0^1(1-s)^{\alpha-2}(\frac{a_2(s)}{\Gamma(\alpha-1)}+\frac{a_2(s)}{\Gamma(\alpha)}+\frac{a_2(s)}{\Delta\Gamma(\alpha-\gamma)}\\
&+\frac{a_3(s)}{\Gamma(\alpha-1)\Gamma(2-\beta)}+\frac{a_3(s)}{\Gamma(\alpha)\Gamma(2-\beta)}+\frac{a_3(s)}{\Delta\Gamma(\alpha-\gamma)\Gamma(2-\beta)})ds\}<\infty.
\end{aligned}
$$

If $\theta_0 = max\{\theta_1,\theta_2\}$, then we obtain $\|Fx_1-Fx_2\|_*\le\theta_0\|x_1-x_2\|_*$ and so $\|Fx_1-Fx_2\|_*\to 0$ as $\|x_1-x_2\|_*\to 0$. Now, put $M_1=\frac{1}{\Gamma(\alpha)}+\frac{1}{\Delta\Gamma(\alpha)}+\frac{1}{\Delta\Gamma(\alpha-\gamma)}$, $M_2=\frac{1}{\Gamma(\alpha-1)}+\frac{1}{\Delta\Gamma(\alpha)}+\frac{1}{\Delta\Gamma(\alpha-\gamma)}$, $m_0=\int_0^1(1-s)^{\alpha-\gamma-1}b(s)ds$ and $r_0=max\{\|H\|_Y.M_1.m_0,\|H\|_Y.M_2.m_0\}$. Note that, $m_0<\infty$ because $b\in L^1$. Let $x\in X$ and $E=\{x\in X:\|x\|_*\le r_0\}$. Then, we have

$$
\begin{aligned}
|F_x(t)|\le&\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}b(s)H(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)d\xi)ds\\
&+\frac{1}{\Delta\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}b(s)H(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)d\xi)ds
\end{aligned}
$$

$$+\frac{1}{\Delta\Gamma(\alpha-\gamma)}\int_0^{\mu}(\mu-s)^{\alpha-\gamma-1}b(s)H(s,x(s),x'(s),D^{\beta}x(s),\int_0^s h(\xi)x(\xi)d\xi)ds$$
$$\leq \|H\|_Y(\frac{1}{\Gamma(\alpha)}+\frac{1}{\Delta\Gamma(\alpha)}+\frac{1}{\Delta\Gamma(\alpha-\gamma)})(\int_0^1(1-s)^{\alpha-\gamma-1}b(s)ds)=\|H\|_Y M_1 m_0$$

for all $t\in[0,1]$. Note that, $m_0\geq\int_0^1(1-s)^{\alpha-1}b(s)ds$. Also, we have

$$F_x'(t)=\int_0^1\frac{\partial G(t,s)}{\partial t}f(s,x(s),x'(s),D^{\beta}x(s),\int_0^s h(\xi)x(\xi)d\xi)ds$$
$$=\frac{-1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}f(s,x(s),x'(s),D^{\beta}x(s),\int_0^s h(\xi)x(\xi)d\xi)ds$$
$$+\frac{1}{\Delta\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}f(s,x(s),x'(s),D^{\beta}x(s),\int_0^s h(\xi)x(\xi)d\xi)ds$$
$$+\frac{1}{\Delta\Gamma(\alpha-\gamma)}\int_0^{\mu}(\mu-s)^{\alpha-\gamma-1}f(s,x(s),x'(s),D^{\beta}x(s),\int_0^s h(\xi)x(\xi)d\xi)ds$$

and so $|F_x'(t)|\leq\|H\|_Y(\frac{1}{\Gamma(\alpha-1)}+\frac{1}{\Delta\Gamma(\alpha)}+\frac{1}{\Delta\Gamma(\alpha-\gamma)})(\int_0^1(1-s)^{\alpha-\gamma-1}b(s)ds)=\|H\|_Y M_2 m_0$. Hence, $\|F_x\|_*=max\{\|F_x\|,\|F_x'\|\}\leq r_0$. Thus, F maps E into E. Similarly one can check that F maps bounded sets into bounded sets. Let $t_1,t_2\in[0,1]$ with $t_1\leq t_2$. Then, we have

$$|F_x(t_1)-F_x(t_2)|$$
$$\leq\frac{1}{\Gamma(\alpha)}\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]|f(s,x(s),x'(s),D^{\beta}x(s),\int_0^s h(\xi)x(\xi)d\xi)|ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_1-s)^{\alpha-1}|f(s,x(s),x'(s),D^{\beta}x(s),\int_0^s h(\xi)x(\xi)d\xi)|ds$$
$$+\frac{|t_2-t_1|}{\Delta\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}|f(s,x(s),x'(s),D^{\beta}x(s),\int_0^s h(\xi)x(\xi)d\xi)|ds$$
$$+\frac{|t_2-t_1|}{\Delta\Gamma(\alpha-\gamma)}\int_0^{\mu}(\mu-s)^{\alpha-\gamma-1}|f(s,x(s),x'(s),D^{\beta}x(s),\int_0^s h(\xi)x(\xi)d\xi)|ds$$
$$\leq\|H\|_Y(\frac{1}{\Gamma(\alpha)}\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]b(s)ds+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_1-s)^{\alpha-1}b(s)ds$$
$$+|t_2-t_1|(\frac{1}{\Delta\Gamma(\alpha)}+\frac{1}{\Delta\Gamma(\alpha-\gamma)})(\int_0^1(1-s)^{\alpha-\gamma-1}b(s)ds)).$$

Since $b\in L^1[0,1]$, $\int_0^1(1-s)^{\alpha-\gamma-1}b(s)ds<\infty$. Also, we have

$$\sup\{\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]b(s)ds:t_1,t_2\in[0,1]\}\leq\int_0^1(1-s)^{\alpha-1}b(s)ds<\infty.$$

Since $(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\to0$ as $t_2\to t_1$, for each $\epsilon>0$ there exists $\delta>0$ such that $|t_2-t_1|<\delta$ implies $(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}<\epsilon$. If $0<\delta<\epsilon$ and $|t_2-t_1|<\delta$, then $\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]b(s)ds\leq\epsilon\int_0^1 b(s)ds$ and so

$$\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]b(s)ds\to0$$

as $t_2\to t_1$. Similarly we conclude that $\int_{t_1}^{t_2}(t_1-s)^{\alpha-1}b(s)ds$ and

$$|t_2-t_1|\int_0^1(1-s)^{\alpha-\gamma-1}b(s)ds\to0$$

tend as $t_2 \to t_1$. Thus, $|F_x(t_2) - F_x(t_1)| \to 0$ as $t_2 \to t_1$ and so F is equi-continuous on E. Hence, $F : E \to E$ is completely continuous. Now by using Lemma 1.2, F has a fixed point on E and so the problem (1.1) has a solution. □

Note that in Theorem 2.1, the map $f(t,.,.,.,.)$ could be discontinuous at points of a subset of $[0,1]$ of measure zero. One can obtain solutions of the problem (1.1) under some different conditions. For example in next result, the map $f(t,.,.,.,.)$ could be discontinuous at $t = 0$.

Theorem 2.2. *Suppose that $f : [0,1] \times X^4 \to \mathbb{R}$ is a map such that $f(t, x_1,.,.,x_4) \geq 0$ for all $(x_1,.,.,x_4) \in X^4$ and almost all $t \in [0,1]$, $f(t,.,.,.,.) : [0,1] \to \mathbb{R}$ is continuous for almost all $t \in [0,1]$, there exist $b \in L^1[0,1]$ and $H, K : \mathbb{R}^4 \to [0,\infty)$ such that H and K are nondecreasing in all components, $\lim_{x\to\infty} \frac{H(x,x,x,x)}{x} = 0$, $\lim_{z\to\infty} K(z,z,z,z) = \Lambda < \infty$ and*

$$f(t, x_1,.,.,x_4) \leq b(t)H(x_1,.,.,x_4) + K(x_1,.,.,x_4)$$

for all $(x_1,.,.,x_4) \in X^4$ and $t \in [0,1]$. Then the problem (1.1) has a solution.

Proof. For each $x \in X$ and $n \geq 1$ define $(x)_n(t) = max\{\frac{1}{n}, x(t)\}$ whenever $x(t) \geq 0$ and $(x)_n(t) = min\{\frac{-1}{n}, x(t)\}$ whenever $x(t) < 0$. Put $f_n(t, x_1, x_2, x_3, x_4) = f(t,(x_1)_n,.,.,(x_4)_n)$ for all n, t and x_1, x_2, x_3, x_4. It is clear that $(x)_n(t) \to x(t)$ and each f_n is a regular function on $[0,1]$. For each n, consider the regular fractional differential equation

$$D^\alpha x(t) + f_n(t, x(t), x'(t), D^\beta x(t), \int_0^t h(\xi)x(\xi)d\xi) = 0 \quad (2.2)$$

under the boundary condition of the problem (1.1). Suppose that $\|b\|_1 > 0$ and $\epsilon_0 > 0$. Choose $r_1 > 0$ such that $|\frac{\Lambda}{x}| < \frac{\epsilon_0}{2}$ for all $|x| > r_1$ and choose $r_2 > 0$ such that $\frac{H(x,x,x,x)}{x} < \frac{\epsilon_0}{2\|b\|_1}$ for all $|x| > r_2$. Thus, $\frac{\Lambda+\|b\|_1 H(x,x,x,x)}{x} < \epsilon_0$ for all $|x| > r_0 := max\{r_1, r_2\}$. Put

$$\theta_0 := max\{\frac{1}{\Gamma(\alpha)} + \frac{1}{\Delta\Gamma(\alpha)} + \frac{1}{\Delta\Gamma(\alpha-\gamma)}, \frac{1}{\Gamma(\alpha-1)} + \frac{1}{\Delta\Gamma(\alpha)} + \frac{1}{\Delta\Gamma(\alpha-\gamma)}\}$$

and $\epsilon_0 = \frac{1}{\theta_0}$. If $r > max\{r_0, \frac{r_0}{\Gamma(2-\beta)}, m.r_0\}$, then $\frac{\Lambda+\|b\|_1 H(r,r,\frac{r}{\Gamma(2-\beta)},m.r)}{r} < \frac{1}{\theta_0}$. Now, consider the set $\Omega = \{x \in X : \|x\|_* < r\}$. For each $n \geq 1$, define $F_n : \bar{\Omega} \to X$ as (2.1) in which we replaced f by f_n. If $\{x_k\}$ is a convergent sequence in $\bar{\Omega}$, then $x_k \to x$ and $x'_k \to x'$ uniformly on $[0,1]$. Since $\|D^\beta x_k - D^\beta x\| \leq \frac{\|x_k - x'\|}{\Gamma(2-\beta)}$, $D^\beta x_k \to D^\beta x$. Also, we have

$$|\int_0^t h(\xi)x_k(\xi)d\xi - \int_0^t h(\xi)x(\xi)d\xi| \leq \int_0^t h(\xi)|x_k(\xi) - x(\xi)|d\xi \leq \|x_k - x\|.m$$

and so $\lim_{k\to\infty} \int_0^t h(\xi)x_k(\xi)d\xi = \int_0^t h(\xi)x(\xi)d\xi$. Thus,

$$\lim_{k\to\infty} f_n(t, x_k(t), ..., \int_0^t h(\xi)x_k(\xi)d\xi) = f_n(t, x(t), ..., \int_0^t h(\xi)x(\xi)d\xi).$$

Note that,

$$|F_n x_k(t) - F_n x(t)| \leq \int_0^1 [\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{t(t-s)^{\alpha-1}}{\Delta\Gamma(\alpha)}$$

$$+\frac{t(\mu-s)^{\alpha-\gamma-1}}{\Delta\Gamma(\alpha-\gamma)}]|f_n(s, x_k(s), ..., \int_0^s h(\xi)x_k(\xi)d\xi) - f_n(s, x(s), ..., \int_0^s h(\xi)x(\xi)d\xi)|ds$$

$$\le (\frac{1}{\Gamma(\alpha)} + \frac{1}{\Delta\Gamma(\alpha)} + \frac{1}{\Delta\Gamma(\alpha-\gamma)})$$

$$\times \int_0^1 |f_n(s, x_k(s), ..., \int_0^s h(\xi)x_k(\xi)d\xi) - f_n(s, x(s), ..., \int_0^s h(\xi)x(\xi)d\xi)|ds.$$

By using a similar method, we have

$$|F_n' x_k(t) - F_n' x(t)| \le (\frac{1}{\Gamma(\alpha-1)} + \frac{1}{\Delta\Gamma(\alpha)} + \frac{1}{\Delta\Gamma(\alpha-\gamma)})$$

$$\times \int_0^1 |f_n(s, x_k(s), ..., \int_0^s h(\xi)x_k(\xi)d\xi) - f_n(s, x(s), ..., \int_0^s h(\xi)x(\xi)d\xi)|ds.$$

Thus, $\|F_n x_k - F_n x\|_* \to 0$ as $x_k \to x$. Hence, $\{F_n(x_k)\}_{k=1}^{\infty}$ is relatively compact in $\bar{\Omega}$ and so F_n is a completely continuous operator on $\bar{\Omega}$ for all n. Suppose that $n \ge 1$ be given and there exist $y \in \partial\Omega$ and $\lambda \in (0,1)$ such that $y = \lambda F_n y$. Since $\|y\|_* = r$, $\|y\| \le r$, $\|y'\| \le r$ and also $\|D^\beta y\| \le \frac{\|y'\|}{\Gamma(2-\beta)} \le \frac{r}{\Gamma(2-\beta)}$ and $\|\int h(z)y(z)dz\| \le mr$. By using the assumption, we have

$$|y(t)| = |\lambda F_n y(t)| = |\lambda \int_0^1 G(t,s) f_n(s, y(s), ..., \int_0^s h(\xi)y(\xi)d\xi)ds|$$

$$< [\frac{1}{\Gamma(\alpha)} + \frac{1}{\Delta\Gamma(\alpha)} + \frac{1}{\Delta\Gamma(\alpha-\gamma)}]$$

$$\times \int_0^1 K(r, r, \frac{r}{\Gamma(2-\beta)}, m.r)ds + \int_0^1 b(s)H(y(s), ..., \int_0^s h(\xi)y(\xi)d\xi)ds$$

$$\le (\frac{1}{\Gamma(\alpha)} + \frac{1}{\Delta\Gamma(\alpha)} + \frac{1}{\Delta\Gamma(\alpha-\gamma)})(\Lambda + \|b\|_1.H(r, r, \frac{r}{\Gamma(2-\beta)}, mr))$$

and

$$|y'(t)| = |\lambda F_n' y(t)| = |\lambda \int_0^1 \frac{\partial G(t,s)}{\partial t} f_n(s, y(s), ..., \int_0^s h(\xi)y(\xi)d\xi)ds|$$

$$< [\frac{1}{\Gamma(\alpha-1)} + \frac{1}{\Delta\Gamma(\alpha)} + \frac{1}{\Delta\Gamma(\alpha-\gamma)}]$$

$$\times(\Lambda + \|b\|_1.H(r, r, \frac{r}{\Gamma(2-\beta)}, mr)).$$

Hence,

$$\|y\|_* < max\{\frac{1}{\Gamma(\alpha)} + \frac{1}{\Delta\Gamma(\alpha)} + \frac{1}{\Delta\Gamma(\alpha-\gamma)}, \frac{1}{\Gamma(\alpha-1)} + \frac{1}{\Delta\Gamma(\alpha)} + \frac{1}{\Delta\Gamma(\alpha-\gamma)}\}$$

$$\times(\Lambda + \|b\|_1.H(r, r, \frac{r}{\Gamma(2-\beta)}, mr))$$

and so $r < \theta_0(\Lambda + \|b\|_1.H(r, r, \frac{r}{\Gamma(2-\beta)}, mr))$. Thus, $\frac{r}{\Lambda + \|b\|_1.H(r, r, \frac{r}{\Gamma(2-\beta)}, m.r)} < \theta_0$ which is a contradiction to (4.2). This implies that $y \notin \partial\Omega$. Now by using Lemma 1.3, F_n has a fixed point $x_n \in \bar{\Omega}$ for each n, that is the problem (2.2) has a solution. Let $(x)_n$ be the solution of the problem (2.2). As we proved, $\{(x)_n\}$ is relatively compact and $(x)_n \to x$ for some $x \in X$. Thus, $x \in \bar{\Omega}$. Similar to last result, we can show that $lim_{n\to\infty} D^\beta x_n(t) = D^\beta x(t)$, $lim_{n\to\infty} x_n'(t) = x'(t)$ and $lim_{n\to\infty} \int_0^t h(z)x_n(z)dz = \int_0^t h(z)x(z)dz$ for all $t \in [0,1]$. Consequently, we get

$$lim_{n\to\infty} f_n(t, x(t), ..., \int_0^t h(z)x(z)dz) = f(t, x(t), ..., \int_0^t h(z)x(z)dz)$$

and

$$|G(t,s)f_n(t,x(t),...,\int_0^t h(z)x(z)dz)$$
$$\leq (\frac{1}{\Gamma(\alpha)}+\frac{1}{\Delta\Gamma(\alpha)}+\frac{1}{\Delta\Gamma(\alpha-\gamma)})[b(t)K(r,r,\frac{r}{\Gamma(2-\beta)},m.r)]<\infty$$

for almost all $t\in[0,1]$. By using the Lebesgue dominated theorem, we obtain

$$x(t)=\int_0^1 G(t,s)f(s,x(s),...,\int_0^s h(\xi)x(\xi)d\xi)ds$$

for all $t\in[0,1]$. This completes the proof. □

Now we give the following examples to illustrate our results.

Example 2.1. *Let* $\alpha\geq 2$, $\mu,\gamma,\beta\in(0,1)$, $k\geq 1$, $q_1,\dots,q_{k+1}\in(0,1)$, $\delta_1,\delta_2,...,\delta_k$ *be real numbers such that* $0<\delta_1<\delta_2<...<\delta_k<1$ *and* $h\in L^1[0,1]$. *Put* $b(t)=\frac{1}{t_1^q}$ *whenever* $t\in(0,\delta_1)$, $b(t)=\frac{1}{(t-\delta_1)_2^q}$ *whenever* $t\in(\delta_1,\delta_2)$ *and finally* $b(t)=\frac{1}{(t-\delta_k)_k^q+1}$ *whenever* $t\in(\delta_k,1)$. *Also, let* $K(x_1,x_2,x_3,x_4)=\sum_{i=1}^4\frac{|x_i|}{1+|x_i|}$ *and* $y_x(t)=\int_0^t h(\xi)x(\xi)d\xi$. *By using Theorem 2.2, the fractional deferential equation*

$$D^\alpha x(t)+b(t)(\frac{|x(t)|}{1+|x(t)|}+\frac{|x'(t)|}{1+|x'(t)|}+\frac{|D^\beta x(t)|}{1+|D^\beta x(t)|}+\frac{|y_x(t)|}{1+|y_x(t)|})=0$$

with boundary condition $x(0)=0$ *and* $x(1)=D^\gamma x(\mu)$ *has a solution.*

Example 2.2. *Let* $\alpha\geq 2$, $\mu,\gamma,\beta,q\in(0,1)$, $p_1,\dots,p_4\in[0,1)$, $\alpha_1,\dots,\alpha_4\in[0,\infty)$ *and* $h\in L^1[0,1]$. *Consider the problem*

$$D^\alpha x+\frac{1}{t^q}[\alpha_1|x|^{p_1}+\alpha_2|x'|^{p_1}+\alpha_3|D^\beta x|^{p_3}+\alpha_1|y_x|^{p_1}]$$
$$+\frac{1}{1+x^2(t)}+\frac{1}{1+(x')^2}+\frac{1}{1+(D^\beta x)^2}+\frac{1}{1+(y_x)^2}=0, \quad (2.3)$$

with boundary conditions $x(0)=0$ *and* $x(1)=D^\gamma x(\mu)$, *where* $y_x=\int_0^t h(\xi)x(\xi)d\xi$. *Put* $H(x_1,x_2,x_3,x_4)=\sum_{i=1}^4\alpha_i|x_i|^{p_i}$, $K(x_1,x_2,x_3,x_4)=\sum_{i=1}^4\frac{1}{1+x_i^2}$ *and* $b(t)=\frac{1}{t^q}$. *One can check that* H, K *and* b *satisfy the conditions of Theorem 2.3. Thus, the problem (2.3) has a solution.*

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