ON APPROXIMATELY BIFLAT BANACH ALGEBRAS

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In this paper, we study the notion of approximately biflat Banach algebras for second dual Banach algebras and semigroup algebras. We show that for a locally compact group G, if $S(G)^{**}$ is approximately biflat, then G is an amenable group. Also we give some conditions which the second dual of a Triangular Banach algebra is never approximately biflat. For a uniformly locally finite semigroup S, we show that $\ell^1(S)$ is approximately biflat if and only if $\ell^1(S)$ is biflat.

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1. Introduction

Helemskii [6] defined the notion of biflat Banach algebras. In fact a Banach algebra A is biflat if there exists a bounded A-bimodule morphism $\rho : (A \otimes_p A)^* \to A^*$ such that $\rho \circ \pi_A^*(f) = f$ for each $f \in A^*$, where $\pi_A : A \otimes_p A \to A$ is defined by $\pi_A(a \otimes b) = ab$ for each $a, b \in A$. For a group algebra $L^1(G)$, associated with a locally compact group G, $L^1(G)$ is biflat if and only if G is amenable. For the further details of Banach algebra homology see [6]. Recently Ramsden in [12] characterized the biflatness of semigroup algebras associated to a locally finite inverse semigroup. He showed that for a locally finite inverse semigroup S, $\ell^1(S)$ is biflat if and only if each G_p is amenable group, where p is an idempotent and G_p is a maximal subgroup of S. Also biflatness of Triangular Banach algebras have been studied in [10].

Recently, approximate notions in the homology of Banach algebras have been introduced and improved. A Banach algebra A is approximately biflat if there exists a net of A-bimodule morphism (ρ_α) from $(A \otimes_p A)^*$ into A^* such that $\rho_\alpha \circ \pi_A^* \xrightarrow{W^*OT} id_{A^*}$, where W^*OT is denoted the weak-star operator topology and id_{A^*} is the identity map on A^*. In fact for the discrete Heisenberg group G the Fourier algebra $A(G)$ is approximately biflat but $A(G)$ is not biflat, see [14]. Samei *et al.* also showed that if A is an approximately biflat Banach algebra with an approximate identity, then A is pseudo-amenable.

Motivated by these considerations, we study approximate biflatness of $\ell^1(S)$, where S is a uniformly locally finite semigroup. We show that $\ell^1(S)$ is approximately biflat if and only if $\ell^1(S)$ is biflat. Also we show that approximately biflatness of $\ell^1(S)^{**}$ implies the

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pseudo-amenability of $\ell^1(S)$. Also for a locally compact group G, we show that approximately biflatness of $S(G)^{**}$, implies that G is amenable, where $S(G)$ is a Segal algebra with respect to G. Finally we give a criteria to study approximately biflatness of Triangular Banach algebras. We show that some second dual of Triangular Banach algebras related to a locally compact groups are never approximately biflat.

2. **Preliminaries**

Let A be a Banach algebra. We recall that if X is a Banach A-bimodule, then X^* is also a Banach A-bimodule via the following actions

$$(a \cdot f)(x) = f(x \cdot a), \quad (f \cdot a)(x) = f(a \cdot x) \qquad (a \in A, x \in X, f \in X^*).$$

Throughout, the character space of A is denoted by $\Delta(A)$, that is, the set of all non-zero multiplicative linear functionals on A. Let $\phi \in \Delta(A)$. Then ϕ has a unique extension $\tilde{\phi} \in \Delta(A^{**})$ which is defined by $\tilde{\phi}(F) = F(\phi)$ for every $F \in A^{**}$.

Let $\{A_\alpha\}_{\alpha\in\Gamma}$ be a collection of Banach algebras. Then we define the ℓ^1-direct sum of A_α by

$$\ell^1 - \oplus_{\alpha\in\Gamma} A_\alpha = \{(a_\alpha) \in \prod_{\alpha\in\Gamma} A_\alpha : \sum_{\alpha\in\Gamma} ||a_\alpha|| < \infty\}.$$

It is easy to verify that

$$\Delta(\ell^1 - \oplus_{\alpha\in\Gamma} A_\alpha) = \{\oplus\phi_\beta : \phi_\beta \in \Delta(A_\beta), \beta \in \Gamma\},$$

where $\oplus\phi_\beta((a_\alpha)_{\alpha\in\Gamma}) = \phi_\beta(a_\beta)$ for every $(a_\alpha)_{\alpha\in\Gamma} \in \ell^1 - \oplus_{\alpha\in\Gamma} A_\alpha$ and every $\beta \in \Gamma$.

Let A be a Banach algebra and let Λ be a non-empty set. The set of all $\Lambda \times \Lambda$ matrices $(a_{i,j})_{i,j}$ which entries come from A is denoted by $\mathbb{M}_\Lambda(A)$. With the matrix multiplication and the following norm

$$||(a_{i,j})_{i,j}|| = \sum_{i,j} ||a_{i,j}|| < \infty,$$

$\mathbb{M}_\Lambda(A)$ is a Banach algebra. $\mathbb{M}_\Lambda(A)$ belongs to the class of ℓ^1-Munn algebras. The map $\theta : \mathbb{M}_\Lambda(A) \to A \otimes_p \mathbb{M}_\Lambda(\mathbb{C})$ defined by $\theta((a_{i,j})) = \sum_{i,j} a_{i,j} \otimes E_{i,j}$ is an isometric algebra isomorphism, where $(E_{i,j})$ denotes the matrix unit of $\mathbb{M}_\Lambda(\mathbb{C})$. Also it is well-known that $\mathbb{M}_\Lambda(\mathbb{C})$ is a biflat Banach algebra [12, Proposition 2.7].

The main reference for the semigroup theory is [7]. We say that S is an inverse semigroup, if for each $s \in S$ there exists an element $s^* \in S$ with $ss^*s = s$ and $s^*ss^* = s$. An inverse semigroup S is called Clifford if for each $s \in S$, we have $ss^* = s^*s$. Let S be a semigroup and let $E(S)$ be the set of its idempotents. A partial order on $E(S)$ is defined by

$$s \leq t \Longleftrightarrow s = st = ts \quad (s, t \in E(S)).$$

If S is an inverse semigroup, then there exists a partial order on S which coincides with the partial order on $E(S)$. Indeed

$$s \leq t \Longleftrightarrow s = ss^*t \quad (s, t \in S).$$

For every $x \in S$, we denote $(x] = \{y \in S | \, y \leq x\}$. S is called locally finite (uniformly locally finite) if for each $x \in S$, $|(x]| < \infty$ ($\sup\{|(x]| : x \in S\} < \infty$), respectively. Suppose that S is an inverse semigroup. Then the maximal subgroup of S at $p \in E(S)$ is denoted by $G_p = \{s \in S | ss^* = s^*s = p\}$. For an inverse semigroup S, it is well-known that there exists an equivalence relation $\mathfrak{D}$ such that $s\mathfrak{D}t$ if and only if there exists $x \in S$ such that

$ss^* = xx^*$ and $t^*t = x^*x$. We denote $\{\mathfrak{D}_\lambda : \lambda \in \Lambda\}$ for the collection of $\mathfrak{D}$-classes and $E(\mathfrak{D}_\lambda) = E(S) \cap \mathfrak{D}_\lambda$.

3. Approximate biflatness of second dual of Banach algebras

In this section we investigate approximate biflatness dual Banach algebras.

Proposition 3.1. *Let A be a Banach algebra. Then A is approximately biflat if and only if there exists a net (ρ_α) of bounded A-bimodule morphism from A into $(A \otimes_p A)^{**}$ such that $\pi_A^{**} \circ \rho_\alpha(a) \to a$ for every $a \in A$.*

Proof. Let A be approximately biflat. Then there exists a net $\xi_\alpha : (A \otimes_p A)^* \to A^*$ of bounded A-bimodule morphism such that $\xi_\alpha \circ \pi_A^*(f) - f \xrightarrow{w^*} 0$, for every $f \in A^*$. Set $\rho_\alpha = \xi_\alpha^*$, hence for each $a \in A$ and $f \in A^*$ with $||f|| \leq 1$, we have

$$\begin{aligned} ||\pi_A^{**} \circ \rho_\alpha(a) - a|| = ||\pi_A^{**} \circ \xi_\alpha^*(a) - a|| &= ||(\xi_\alpha \circ \pi_A^*)^*(a) - a|| \\ &= ||(\xi_\alpha \circ \pi_A^*)^*(a)(f) - a(f)|| \\ &= ||a(\xi_\alpha \circ \pi_A^*(f)) - a(f)|| \\ &= ||a(\xi_\alpha \circ \pi_A^*(f) - f)|| \to 0. \end{aligned}$$

For the converse, suppose that there exists a net (ρ_α) of bounded A-bimodule morphism from A into $(A \otimes_p A)^{**}$ such that $\pi_A^{**} \circ \rho_\alpha(a) \to a$ for every $a \in A$. Set $\xi_\alpha = \rho_\alpha^*|_{(A\otimes_p A)^*}$.

$$\xi_\alpha \circ \pi_A^*(f)(a) - f(a) = f(\pi_A^{**} \circ \rho_\alpha(a) - a) \to 0,$$

where $f \in A^*, a \in A$. Then A is approximately biflat. □

We recall that a Banach algebra A is called pseudo-amenable if there exists a (not necessarily bounded) net (m_α) in $A \otimes_p A$ such that

$$a \cdot m_\alpha - m_\alpha \cdot a \to 0, \quad \pi_A(m_\alpha)a \to a \quad (a \in A),$$

see [5] for further details.

Theorem 3.1. *Let A be a Banach algebra with an approximate identity. If A^{**} is approximately biflat, then A is pseudo-amenable.*

Proof. Suppose that A^{**} is approximately biflat. Then by Proposition 3.1 there exists a net $(\rho_\alpha)_{\alpha\in I}$ of A^{**}-bimodule morphism from A^{**} into $(A^{**} \otimes_p A^{**})^{**}$ such that $\pi_{A^{**}}^{**} \circ \rho_\alpha(a) \to a$ for each $a \in A^{**}$. It is easy to see that the net $(\rho_\alpha|_A)$ is also a net of A-bimodule morphism satisfies $\pi_{A^{**}}^{**} \circ \rho_\alpha|_A(a) \to a$ for each $a \in A$. We denote $(e_\lambda)_{\lambda\in J}$ for the approximate identity of A. Consider

$$\begin{aligned} \lim_\alpha \lim_\lambda a \cdot \rho_\alpha(e_\lambda) - \rho_\alpha(e_\lambda) \cdot a &= \lim_\alpha \lim_\lambda \rho_\alpha(ae_\lambda - e_\lambda a) \\ &= \lim_\alpha \rho_\alpha(0) = 0 \quad (a \in A). \end{aligned}$$

Also

$$\lim_\alpha \lim_\lambda \pi_{A^{**}}^{**} \circ \rho_\alpha(e_\lambda)a - a = \lim_\alpha \pi_{A^{**}}^{**} \circ \rho_\alpha(a) - a = 0, \qquad (a \in A).$$

Let $E = I \times J^I$ be a directed set with product ordering, that is,

$$(\alpha, \beta) \leq_E (\alpha', \beta') \Leftrightarrow \alpha \leq_I \alpha', \beta \leq_{J^I} \beta' \qquad (\alpha, \alpha' \in I, \quad \beta, \beta' \in J^I),$$

where J^I is the set of all functions from I into J and $\beta \leq_{J^I} \beta'$ means that $\beta(d) \leq_J \beta'(d)$ for each $d \in I$. Suppose that $\gamma = (\alpha, \beta_\alpha) \in E$ and $m_\gamma = \rho_\alpha(e_{\lambda_\alpha}) \in (A^{**} \otimes_p A^{**})^{**}$. Applying iterated limit theorem [9, page 69] and above calculations, we can easily see that

$$a \cdot m_\gamma - m_\gamma \cdot a \to 0, \quad \pi^{**}_{A^{**}}(m_\gamma)a \to a, \quad (a \in A).$$

There exists a bounded linear map $\psi : A^{**} \otimes_p A^{**} \to (A \otimes_p A)^{**}$ such that for $a, b \in A$ and $m \in A^{**} \otimes_p A^{**}$, the following holds;

(i) $\psi(a \otimes b) = a \otimes b$,

(ii) $\psi(m) \cdot a = \psi(m \cdot a), \qquad a \cdot \psi(m) = \psi(a \cdot m)$,

(iii) $\pi^{**}_A(\psi(m)) = \pi_{A^{**}}(m)$,

see [4, Lemma 1.7]. So $\psi^{**}(m_\gamma)$ is a net in $(A \otimes_p A)^{****}$ such that

$$a \cdot \psi^{**}(m_\gamma) - \psi^{**}(m_\gamma) \cdot a \to 0, \quad \pi^{****}_A(\psi^{**}(m_\gamma))a = \pi^{**}_{A^{**}}(m_\gamma)a \to a \qquad (a \in A).$$

Put $n_\gamma = \psi^{**}(m_\gamma)$. Suppose that $\epsilon > 0$ and $F = \{a_1, ..., a_r\} \subseteq A$. Set

$$\begin{aligned} V =& \{(a_1 \cdot n - n \cdot a_1, ..., a_r \cdot n - n \cdot a_r, \pi^{**}_A(n)a_1 - a_1, ..., \\ & \pi^{**}_A(n)a_r - a_r) | n \in (A \otimes_p A)^{**}\} \\ \subseteq & (\prod_{i=1}^{r} (A \otimes_p A)^{**}) \oplus_1 (\prod_{i=1}^{r} A^{**}). \end{aligned}$$

It is easy to see that $(0, 0, ..., 0)$ is a w-limit point of V. Since V is convex set $\overline{V}^{||\cdot||} = \overline{V}^{w}$, then $(0, 0, ..., 0)$ is a $|| \cdot ||$-limit point of V. Hence there exists a net $(n_{(F,\epsilon)})$ in $(A \otimes_p A)^{**}$ such that

$$||a_i \cdot n_{(F,\epsilon)} - n_{(F,\epsilon)} \cdot a_i|| < \epsilon, \quad ||\pi^{**}_A(n_{(F,\epsilon)})a_i - a_i|| < \epsilon, \quad (i \in \{1, 2, ..., r\}).$$

Observe that

$$\Delta = \{(F, \epsilon) : F \text{ is a finite subset of } A, \epsilon > 0\},$$

with the following order

$$(F, \epsilon) \leq (F', \epsilon') \Longrightarrow F \subseteq F', \quad \epsilon \geq \epsilon'$$

is a directed set. It is easy to see that there exists a net $(n_{(F,\epsilon)})_{(F,\epsilon)\in\Delta}$ in $(A \otimes_p A)^{**}$ such that

$$a \cdot n_{(F,\epsilon)} - n_{(F,\epsilon)} \cdot a \to 0, \quad \pi^{**}_A(n_{(F,\epsilon)})a - a \to 0,$$

for every $a \in A$. Using the same method as above we can assume that $(n_{(F,\epsilon)})_{(F,\epsilon)\in\Delta}$ is a subset of $A \otimes_p A$. This means that A is pseudo-amenable. □

Let A be a Banach algebra and $\phi \in \Delta(A)$. We say that A is approximately ϕ-inner amenable if there exists a net $(a_\alpha)_\alpha$ in A such that $aa_\alpha - a_\alpha a \to 0$ and $\phi(a_\alpha) \to 1$, for all $a \in A$. Also A is approximately left ϕ-amenable if there exists a net m_α in A such that $am_\alpha - \phi(a)m_\alpha \to 0$ and $\phi(m_\alpha) \to 1$, see [1].

The proof of following two results are similar to the proof of Theorem 3.1 which we omit them.

Theorem 3.2. *Suppose that A is an approximately ϕ-inner amenable Banach algebra. If A^{**} is approximately biflat, then A is approximately left ϕ-amenable.*

Corollary 3.1. *Suppose that A is an approximately ϕ-inner amenable Banach algebra. If A is approximately biflat, then A is approximately left ϕ-amenable.*

Theorem 3.3. *Let A be a Banach algebra with $\phi \in \Delta(A)$. Suppose that $\overline{A \ker \phi} = \ker \phi$. If A^{**} is approximately biflat, then A is left ϕ-amenable.*

Proof. Since A^{**} is approximately biflat, there exists a net (ρ_α) of A^{**}-bimodule morphism from A^{**} into $(A^{**} \otimes_p A^{**})^{**}$ such that $\pi^{**}_{A^{**}} \circ \rho_\alpha(a) \to a \quad (a \in A^{**})$. We denote $id : A \to A$ for the identity map and $q : A \to \frac{A}{\ker \phi}$ the quotient map. Also it is well-known that there exists a bounded linear map $\psi : A^{**} \otimes_p A^{**} \to (A \otimes_p A)^{**}$ such that for $a, b \in A$ and $m \in A^{**} \otimes_p A^{**}$, the following holds;

(i) $\psi(a \otimes b) = a \otimes b$,
(ii) $\psi(m) \cdot a = \psi(m \cdot a), \qquad a \cdot \psi(m) = \psi(a \cdot m)$,
(iii) $\pi^{**}_A(\psi(m)) = \pi_{A^{**}}(m)$,

see [4, Lemma 1.7]. Set $\eta_\alpha := (id \otimes q)^{****} \circ \psi^{**} \circ \rho_\alpha|_A : A \to (A \otimes_p \frac{A}{\ker \phi})^{****}$ for each α. We claim that $\eta_\alpha(l) = 0$ for each $l \in \ker \phi$. To see this let $l \in \ker \phi$ be an arbitrary element. Since $\overline{A \ker \phi} = \ker \phi$, there exist two nets (a_β) in A and (l_β) in $\ker \phi$ such that $l = \lim_\beta a_\beta l_\beta$. Consider

$$\begin{aligned} \eta_\alpha(l) = (id \otimes q)^{****} \circ \psi^{**} \circ \rho_\alpha(l) &= (id \otimes q)^{****} \circ \psi^{**} \circ \rho_\alpha(\lim_\beta a_\beta l_\beta) \\ &= \lim_\beta (id \otimes q)^{****} \circ \psi^{**} \circ \rho_\alpha(a_\beta l_\beta) \\ &= \lim_\beta (id \otimes q)^{****}(\psi^{**} \circ \rho_\alpha(a_\beta) \cdot l_\beta) = 0. \end{aligned} \tag{1}$$

Hence, for each α, η_α induces a map on $\frac{A}{\ker \phi}$ which we again denote it by η_α. We also denote $\overline{\phi}$ for a character which induced by ϕ on $\frac{A}{\ker \phi}$ given by

$$\overline{\phi}(a + \ker \phi) = \phi(a) \quad (a \in A).$$

Set

$$g_\alpha = (id \otimes \overline{\phi})^{****} \circ \eta_\alpha : \frac{A}{\ker \phi} \to A^{**}.$$

Pick an element $x_0 \in A$ such that $\phi(x_0) = 1$. Define $m_\alpha = g_\alpha(x_0 + \ker \phi)$. We know that (g_α) is a net of left A-module morphisms. Thus

$$\begin{aligned} am_\alpha = a(id \otimes \overline{\phi})^{****} \circ \eta_\alpha(x_0 + \ker \phi) &= (id \otimes \overline{\phi})^{****} \circ \eta_\alpha(ax_0 + \ker \phi) \\ &= \phi(a)(id \otimes \overline{\phi})^{****} \circ \eta_\alpha(x_0 + \ker \phi) \\ &= \phi(a) m_\alpha, \end{aligned} \tag{2}$$

the last equality holds because $ax_0 - \phi(a)x_0 \in \ker \phi$. Since

$$\tilde{\tilde{\phi}} \circ (id \otimes \overline{\phi})^{****} = (\phi \otimes \overline{\phi})^{****}, \quad (\phi \otimes \overline{\phi})^{****} \circ (id \otimes q)^{****} = \tilde{\tilde{\phi}} \circ \pi^{****}_A,$$

we have

$$\begin{aligned} \tilde{\tilde{\phi}}(m_\alpha) &= \tilde{\tilde{\phi}} \circ (id \otimes \overline{\phi})^{****} \circ \eta_\alpha(x_0 + \ker \phi) \\ &= \tilde{\tilde{\phi}} \circ (id \otimes \overline{\phi})^{****} \circ \eta_\alpha(x_0) \\ &= \tilde{\tilde{\phi}} \circ (id \otimes \overline{\phi})^{****} \circ (id \otimes q)^{****} \circ \rho_\alpha(x_0) \\ &= \tilde{\tilde{\phi}} \circ \pi^{****}_A \circ \rho_\alpha(x_0) \to \phi(x_0) = 1. \end{aligned} \tag{3}$$

Replacing (m_α) with $(\frac{m_\alpha}{\phi(m_\alpha)})$ on can find an element $m \in A^{****}$ such that $am = \phi(a)m$ and $\widetilde{\widetilde{\phi}}(m) = 1$ for every $a \in A$. Let $F = \{a_1, a_2, ..., a_r\}$ be an arbitrary finite subset of A and $\epsilon > 0$. Set

$$V = \{(a_1n - \phi(a_1)n, a_2n - \phi(a_2)n, ..., a_rn - \phi(a_r)n, \widetilde{\phi}(n) - 1) | n \in A^{**}, ||n|| \leq ||m||\}.$$

It is easy to see that V is a convex subset of $\prod_{i=1}^r A^{**} \oplus_1 \mathbb{C}$. So $(0, 0, ..., 0) \in \overline{V}^w = \overline{V}^{||\cdot||}$. Thus there exists a bounded net $(n_{(F,\epsilon)})_{(F,\epsilon)}$ in A^{**} such that

$$||a_in_{(F,\epsilon)} - \phi(a_i)n_{(F,\epsilon)}|| < \epsilon, \quad |\tilde{\phi}(n_{(F,\epsilon)}) - 1| < \epsilon, \quad a_i \in F.$$

One can show that

$$\Delta = \{(F, \epsilon) : F \text{ is a finite subset of } A, \epsilon > 0\},$$

with the following order

$$(F, \epsilon) \leq (F', \epsilon') \Longrightarrow F \subseteq F', \quad \epsilon \geq \epsilon'$$

is a directed set. Therefore there exists a bounded net $(n_{(F,\epsilon)})_{(F,\epsilon)\in\Delta}$ in A^{**} such that

$$an_{(F,\epsilon)} - \phi(a)n_{(F,\epsilon)} \to 0, \quad \tilde{\phi}(n_{(F,\epsilon)}) - 1 \to 0, \quad a \in A.$$

Since $(n_{(F,\epsilon)})_{(F,\epsilon)}$ is a bounded net in A^{**}, then $(n_{(F,\epsilon)})_{(F,\epsilon)}$ has a w^*-limit point in A^{**}, say N. It is easy to see that

$$aN = \phi(a)N, \quad \tilde{\phi}(N) = 1 \quad (a \in A).$$

It means that A is left ϕ-amenable. □

The map $\phi_1 : L^1(G) \to \mathbb{C}$ which is specified by

$$\phi_1(f) = \int_G f(x)dx$$

is called the augmentation character. We know that the augmentation character induces a character on $S(G)$ we denote by ϕ_1 again, see [2].

Let G be a locally compact group. A linear subspace $S(G)$ of $L^1(G)$ is said to be a Segal algebra on G if it satisfies the following conditions:

(i) $S(G)$ is a dense in $L^1(G)$,
(ii) $S(G)$ with a norm $|| \cdot ||_{S(G)}$ is a Banach space and $||f||_{L^1(G)} \leq ||f||_{S(G)}$ for every $f \in S(G)$,
(iii) for $f \in S(G)$ and $y \in G$, we have $L_yf \in S(G)$ and the map $y \mapsto L_y(f)$ from G into $S(G)$ is continuous, where $L_y(f)(x) = f(y^{-1}x)$,
(iv) $||L_y(f)||_{S(G)} = ||f||_{S(G)}$ for every $f \in S(G)$ and $y \in G$,

for more information see [13].

Corollary 3.2. *Let G be a locally compact group. If $S(G)^{**}$ is approximately biflat, then G is amenable.*

Proof. It is well-known that every Segal algebra has a left approximate identity. Suppose that $\phi \in \Delta(S(G))$. It is easy to see that $\overline{S(G)\ker\phi} = \ker\phi$. Using the Theorem 3.3, the approximate biflatness of $S(G)^{**}$ implies that $S(G)$ is left ϕ-amenable. Now by [2, Corollary 3.4] G is amenable. □

4. **Approximate biflatness of certain semigroup algebras**

In this section we study approximate biflatness of some semigroup algebras.

Before giving the following proposition we have to give some backgrounds. Suppose that A and B are Banach algebras and also suppose that E and F are Banach A-bimodule and Banach B-bimodule, respectively. Via the following module actions, one can see that $E \otimes_p F$ becomes a Banach $A \otimes_p B$-bimodule:

$$(a \otimes b) \cdot (x \otimes y) = (a \cdot x) \otimes (b \cdot y), \quad (x \otimes y) \cdot (a \otimes b) = (x \cdot a) \otimes (y \cdot b),$$

for each $a \in A, x \in E, b \in B, y \in F$. One can see that $B(E, F)$ (the set of all bounded linear operator from E into F) is a Banach $A \otimes_p B$-bimodule via the following actions:

$$((a \otimes b) * T)(x) = b \cdot T(x \cdot a), \quad (T * (a \otimes b))(x) = T(a \cdot x) \cdot b,$$

for each $T \in B(E, F), a \in A, b \in B, x \in E$. We denote this Banach $A \otimes_p B$-bimodule by $\widetilde{B}(E, F)$. Also we can see that $B(F, E)$ becomes a Banach $A \otimes_p B$-bimodule via the following actions:

$$((a \otimes b) * T)(x) = a \cdot T(x \cdot b), \quad T * (a \otimes b))(x) = T(b \cdot x) \cdot a,$$

for each $T \in B(E, F), a \in A, b \in B, x \in E$. We denote this Banach $A \otimes_p B$-bimodule by $\widehat{B}(E, F)$. Note that for each $\lambda \in (E \otimes_p F)^*$ we can define $\widetilde{T_\lambda} \in B(E, F^*)$ and $\widehat{T_\lambda} \in B(F, E^*)$ by

$$< y, \widetilde{T_\lambda}(x) >=< x \otimes y, \lambda >, \quad < x, \widehat{T_\lambda}(y) >=< x \otimes y, \lambda >,$$

for each $x \in E, y \in F$. The map $\widetilde{\xi} : (E \otimes_p F)^* \to \widetilde{B}(E, F^*)$ given by $\widetilde{\xi}(\lambda) = \widetilde{T}_\lambda$ is an isometric $A \otimes_p B$-bimodule isomorphism. Also the map $\widehat{\xi} : (E \otimes_p F)^* \to \widehat{B}(F, E^*)$ given by $\widehat{\xi}(\lambda) = \widehat{T_\lambda}$ is an isometric $A \otimes_p B$-bimodule isomorphism. Then there exists a bounded isometric $A \otimes_p B$-bimodule isomorphism from $\widetilde{B}(E, F^*)$ into $\widehat{B}(F, E^*)$ which denoted by L. Also we remind that there exists an isometric $A \otimes_p B$-bimodule isomorphism from $(A \otimes_p A) \otimes_p (B \otimes_p B)$ into $(A \otimes_p B) \otimes_p (A \otimes_p B)$ defined by

$$\theta(a \otimes a' \otimes b \otimes b') = a \otimes b \otimes a' \otimes b' \quad (a, a' \in A, b, b' \in B).$$

It is clear that θ is a bounded $A \otimes_p B$-bimodule morphism.

Proposition 4.1. *Let A be a biflat Banach algebra and let B be approximate biflat Banach algebra. Then $A \otimes_p B$ is approximate biflat.*

Proof. Since A is approximate biflat, there exists a net $\rho_\alpha : (A \otimes_p A)^* \to A^*$ of bounded A-bimodule morphisms such that $\rho_\alpha \circ \pi_A^*(f) - f \xrightarrow{w^*} 0$, for every $f \in A^*$. Also since B is biflat, there exists a bounded B-bimodules from $(B \otimes_p B)^*$ into B, say ρ, such that $\rho \circ \pi_B^*(g) = g$ for each $g \in B^*$. Set

$$\begin{aligned}\bar{\rho}_\alpha : ((A \otimes_p B) \otimes_p (A \otimes_p B))^* &\xrightarrow{\theta^*} ((A \otimes_p A) \otimes_p (B \otimes_p B))^* \\ &\xrightarrow{\widetilde{\xi}} \widetilde{B}(A \otimes_p A, (B \otimes_p B)^*) \\ &\xrightarrow{T \mapsto \rho \circ T} \widetilde{B}(A \otimes_p A, B^*) \\ &\xrightarrow{L} \widehat{B}(B, (A \otimes_p A)^*) \\ &\xrightarrow{T \mapsto \rho_\alpha \circ T} \widehat{B}(B, A^*) \\ &\xrightarrow{\widehat{\xi^{-1}}} (A \otimes_p B)^*.\end{aligned} \quad (4)$$

Since $\bar{\rho}_\alpha$ is a composition of some $(A\otimes_p B)$-bimodule morphisms, then $\bar{\rho}_\alpha$ is a net of $(A\otimes_p B)$-bimodule morphism. Take $\lambda \in (A \otimes_p B)^*$. Using the following facts

$$\rho \circ \pi_B^* \circ \widetilde{T}_\lambda \circ \pi_A = \widetilde{T}_\lambda \circ \pi_A$$

and

$$L \circ \widetilde{T}_\lambda \circ \pi_A = \pi_A^* \circ \widehat{T}_\lambda,$$

we have

$$\overline{\rho}_\alpha \circ \pi^*_{A\otimes_p B}(\lambda) - \lambda = \widehat{\xi^{-1}} \circ \rho_\alpha \circ \pi_A^* \circ \widehat{T_\lambda} - \lambda \to \lambda - \lambda = 0.$$

This finishes the proof. □

We show the partial converse of above Proposition in the following theorem.

Theorem 4.1. *Let A and B be Banach algebras. Suppose that A has an identity and B has a non-zero idempotent. If $A \otimes_p B$ is approximately biflat, then A is approximately biflat, so A is pseudo-amenable.*

Proof. Suppose that $A \otimes_p B$ is approximately biflat. Then by Proposition 3.1 there exists a net (ρ_α) of A-bimodule morphism from $A \otimes_p B$ into $((A \otimes_p B) \otimes_p (A \otimes_p B))^{**}$ such that $\pi^{**}_{A\otimes_p B} \circ \rho_\alpha(x) \to x$ for each $x \in A \otimes_p B$. Take $e \in A$ the identity and $b_0 \in B$ the non-zero idempotent. Note that $A \otimes_p B$ becomes a Banach A-bimodule via the following actions:

$$a_1 \cdot (a_2 \otimes b) = a_1a_2 \otimes b, \quad (a_2 \otimes b) \cdot a_1 = a_2a_1 \otimes b \qquad (a_1, a_2 \in A, b \in B).$$

For each α, we have

$$\begin{aligned} \rho_\alpha(a_1a_2 \otimes b_0) &= \rho_\alpha((a_1 \otimes b_0)(a_2 \otimes b_0)) \\ &= (a_1 \otimes b_0) \cdot \rho_\alpha(a_2 \otimes b_0) \\ &= (a_1 \cdot (e \otimes b_0)) \cdot \rho_\alpha(a_2 \otimes b_0) \\ &= a_1 \cdot \rho_\alpha(ea_2 \otimes b_0b_0) \\ &= a_1 \cdot \rho_\alpha(a_2 \otimes b_0). \end{aligned} \tag{5}$$

For each α, we can also see that

$$\rho_\alpha((a_2 \otimes b_0) \cdot a_1) = \rho_\alpha(a_2 \otimes b_0) \cdot a_1.$$

For each α set $\overline{\rho}_\alpha(a) = \rho_\alpha(a \otimes b_0)$. It is easy to see that $(\overline{\rho}_\alpha)$ is a net of A-bimodule morphism. Since b_0 is a non-zero element in B, by Hahn-Banach theorem there exists a functional $f \in B^*$ such that $f(b_0) = 1$. Define $T : (A \otimes_p B) \otimes_p (A \otimes_p B) \to A \otimes_p A$ by

$$T(a \otimes b \otimes c \otimes d) = f(bd)a \otimes c$$

for each $a, c \in A$ and $b, d \in B$. Clearly T is a bounded linear map. One can see that $\pi_A^{**} \circ T^{**} = (id_A \otimes f)^{**} \circ \pi^{**}_{A\otimes_p B}$, where

$$id_A \otimes f(a \otimes b) = f(b)a \quad (a \in A, b \in B).$$

Set $\widetilde{\rho}_\alpha = T^{**} \circ \overline{\rho}_\alpha$. One can see that $(\widetilde{\rho}_\alpha)_\alpha$ is a net of bounded A-bimodule morphism. Since $f(b_0) = 1$, we have

$$\begin{aligned} \pi_A^{**} \circ \widetilde{\rho}_\alpha(a) = \pi_A^{**} \circ T^{**} \circ \overline{\rho}_\alpha(a) &= (\pi_A \circ T)^{**} \circ \rho_\alpha(a \otimes b_0) \\ &= (id_A \otimes f)^{**} \circ \pi^{**}_{A\otimes_p B} \cdot \rho_\alpha(a \otimes b_0) \\ &\to (id_A \otimes f)^{**}(a \otimes b_0) = a, \end{aligned} \tag{6}$$

for each $a \in A$. Using Proposition 3.1 A is approximately biflat. Since A has an identity by [14, Theorem 2.4], A is pseudo-amenable. □

Corollary 4.1. *Let A be a Banach algebra. If A is approximately biflat, then $\mathbb{M}_\Lambda(A)$ is approximately biflat. Converse is true, provided that A has a unit.*

Proof. Let A be approximately biflat. It is well-known that there exists an isometric isomorphism between $\mathbb{M}_\Lambda(A)$ and $A\otimes_p \mathbb{M}_\Lambda(\mathbb{C})$. Using the fact that $\mathbb{M}_\Lambda(\mathbb{C})$ is always biprojective [12, Proposition 2.7], then $\mathbb{M}_\Lambda(\mathbb{C})$ is biflat. Now by Proposition 4.1, $A\otimes_p \mathbb{M}_\Lambda(\mathbb{C})$ is approximately biflat.

For the converse, since A is unital and $\mathbb{M}_\Lambda(\mathbb{C})$ has a non-zero idempotent, by the previous Theorem, approximate biflatness of $\mathbb{M}_\Lambda(A)\cong A\otimes_p \mathbb{M}_\Lambda(\mathbb{C})$ implies that A is approximately biflat A, so the converse is clear. □

Theorem 4.2. *Let S be an inverse semigroup such that $E(S)$ is uniformly locally finite. Then $\ell^1(S)$ is approximately biflat if and only if $\ell^1(S)$ is biflat.*

Proof. Suppose that $\ell^1(S)$ is approximately biflat. Then by Proposition 3.1 there exists a net $(\rho_\alpha)_{\alpha\in I}$ of $\ell^1(S)$-bimodule morphism from $\ell^1(S)$ into $(\ell^1(S)\otimes_p \ell^1(S))^{**}$ such that $\pi^{**}_{\ell^1(S)}\circ\rho_\alpha(a)-a\to 0$ for each $a\in\ell^1(S)$. Since S is uniformly locally finite, by [12, Theorem 2.18] we have

$$\ell^1(S)\cong \ell^1-\bigoplus\{\mathbb{M}_{E(\mathfrak{D}_\lambda)}(\ell^1(G_{p_\lambda}))\},$$

where $\mathfrak{D}_\lambda$ is a $\mathfrak{D}$-class and G_{p_λ} is a maximal subgroup at p_λ. Then the map $P_{p_\lambda}:\ell^1(S)\to \mathbb{M}_{E(\mathfrak{D}_\lambda)}(\ell^1(G_{p_\lambda}))$ is a continuous homomorphism with a dense range. Define

$$\eta_\alpha:\mathbb{M}_{E(\mathfrak{D}_\lambda)}(\ell^1(G_{p_\lambda}))\to(\mathbb{M}_{E(\mathfrak{D}_\lambda)}(\ell^1(G_{p_\lambda}))\otimes_p\mathbb{M}_{E(\mathfrak{D}_\lambda)}(\ell^1(G_{p_\lambda})))^{**}$$

by $\eta_\alpha=(P\otimes P)^{**}\circ\rho_\alpha|_{\mathbb{M}_{E(\mathfrak{D}_\lambda)}(\ell^1(G_{p_\lambda}))}$. It is easy to see that (η_α) is a net of $\mathbb{M}_{E(\mathfrak{D}_\lambda)}(\ell^1(G_{p_\lambda}))$-bimodule morphism. For each $a\in\mathbb{M}_{E(\mathfrak{D}_\lambda)}(\ell^1(G_{p_\lambda}))$ we have

$$\begin{aligned}\pi^{**}_{\mathbb{M}_{E(\mathfrak{D}_\lambda)}(\ell^1(G_{p_\lambda}))}\circ\eta_\alpha(a)&=\pi^{**}_{\mathbb{M}_{E(\mathfrak{D}_\lambda)}(\ell^1(G_{p_\lambda}))}\circ(P_{p_\lambda}\otimes P_{p_\lambda})^{**}\circ\rho_\alpha|_{\mathbb{M}_{E(\mathfrak{D}_\lambda)}(\ell^1(G_{p_\lambda}))}(a)\\&=P^{**}_{p_\lambda}\circ\pi^{**}_{\ell^1(S)}\circ\rho_\alpha|_{\mathbb{M}_{E(\mathfrak{D}_\lambda)}(\ell^1(G_{p_\lambda}))}(a)\to a.\end{aligned}\tag{7}$$

Hence $\mathbb{M}_{E(\mathfrak{D}_\lambda)}(\ell^1(G_{p_\lambda}))$ is approximately biflat. By Theorem 4.1, $\ell^1(G_{p_\lambda})$ is approximately biflat. Since $\ell^1(G_{p_\lambda})$ is unital, by [14, Theorem 2.4] $\ell^1(G_{p_\lambda})$ is pseudo-amenable, hence by [5, Proposition 4.1] G_{p_λ} is amenable. Applying [3, Theorem 3.7] to finish the proof.

The converse is clear. □

Corollary 4.2. *Let $S=\cup_{e\in E(S)}G_e$ be a Clifford semigroup such that $E(S)$ is uniformly locally finite. If $\ell^1(S)^{**}$ is approximately biflat, then G_e is amenable for each $e\in E(S)$.*

Proof. Suppose that $\ell^1(S)^{**}$ is approximately biflat. It is well-known that $\ell^1(S)\cong\ell^1-\oplus_{e\in E(S)}\ell^1(G_e)$. Since $\ell^1(G_e)$ is unital, then $\ell^1(S)$ has an approximate identity. The previous Theorem, implies that $\ell^1(S)$ is pseudo-amenable. Then by [3, Theorem 3.7] G_e is amenable, for each $e\in E(S)$. □

5. **An application to Triangular Banach algebras**

In this section we give some examples of matrix algebras which are never approximately biflat. Similar results for ϕ-biprojectivity of some Matrix algebras have been investigated in [11].

Let A be a Banach algebra and let X be a Banach A-bimodule. Suppose that $\phi\in\Delta(A)$. We say that X has a left ϕ-character if there exists a non-zero map $\psi\in X^*$ such that

$$\psi(a\cdot x)=\phi(a)\psi(x),\quad (a\in A, x\in X).$$

Similarly we can define right case and two sided case. It is easy to see that for $\phi \in \Delta(A)$, $\phi \otimes \phi$ on $A \otimes_p A$ is a left ϕ-character. Also if A has a closed ideal $I \subseteq \ker \phi$, then $\overline{\phi}$ on $\frac{A}{I}$ is a left ϕ-character, where $\overline{\phi} : \frac{A}{L} \to \mathbb{C}$ given by $\overline{\phi}(a + I) = \phi(a)$ for all $a \in A$.

Let A and B be Banach algebras and let X be a Banach (A, B)-module, that is, X is a Banach space, a left A-module and a right B-module with the compatible module action that satisfies $(a \cdot x) \cdot b = a \cdot (x \cdot b)$ and $||a \cdot x \cdot b|| \leq ||a||||x||||b||$ for every $a \in A, x \in X, b \in B$. With the usual matrix operation and $||\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}|| = ||a|| + ||x|| + ||b||$, $T = \text{Tri}(A, X, B) = \begin{pmatrix} A & X \\ 0 & B \end{pmatrix}$ becomes a Banach algebra which is called Triangular Banach algebra. Let $\phi \in \Delta(B)$. We define a character $\psi_\phi \in \Delta(T)$ via $\psi_\phi \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} = \phi(b)$ for every $a \in A$, $b \in B$ and $x \in X$.

Theorem 5.1. *Let A and B be Banach algebras and let X be a Banach (A, B)-module such that $\overline{A^2} = A$ and $\overline{A \cdot X} = X$. Suppose that $\phi \in \Delta(B)$ with $\overline{B \ker \phi} = \ker \phi$. If one of the followings hold*

(i) *B is not left ϕ-amenable;*
(ii) *X has a right ϕ-character;*

then $T^{**} = \text{Tri}(A, X, B)^{**}$ is not approximately biflat.

Proof. We go toward a contradiction and suppose that T^{**} is approximately biflat. Let ψ_ϕ be same as above. It is clear that $\ker \psi_\phi = \text{Tri}(A, X, \ker \phi)$. Since $\overline{A^2} = A$, $\overline{A \cdot X} = X$ and $\overline{B \ker \phi} = \ker \phi$, then $\overline{T \ker \psi_\phi} = \psi_\phi$. Then by Theorem 3.3, T is left ψ_ϕ-amenable. Set $I = \text{Tri}(0, X, B)$. It is clear that I is closed ideal and $\psi_\phi|_I \neq 0$. Using [8, Lemma 3.1], one can see that I is left ψ_ϕ-amenable. Thus by [8, Theorem 1.4], there exists a bounded net (i_α) in I such that

$$ii_\alpha - \psi_\phi(i)i_\alpha \to 0, \quad \psi_\phi(i_\alpha) = 1, \quad (i \in I).$$

Take (x_α) in X and (b_α) in B such that $i_\alpha = \begin{pmatrix} 0 & x_\alpha \\ 0 & b_\alpha \end{pmatrix}$. Hence we have

$$\begin{pmatrix} 0 & x \\ 0 & b \end{pmatrix} \begin{pmatrix} 0 & x_\alpha \\ 0 & b_\alpha \end{pmatrix} - \psi_\phi(\begin{pmatrix} 0 & x \\ 0 & b \end{pmatrix}) \begin{pmatrix} 0 & x_\alpha \\ 0 & b_\alpha \end{pmatrix} \to 0$$

and $\quad \psi_\phi(\begin{pmatrix} 0 & x_\alpha \\ 0 & b_\alpha \end{pmatrix}) = \phi(b_\alpha) = 1$ for each $x \in X, b \in B$. Thus we have

$$xb_\alpha - \phi(b)x_\alpha \to 0, \quad bb_\alpha - \phi(b)b_\alpha \to 0, \quad \phi(b_\alpha) = 1, \quad (x \in X, b \in B).$$

If (i) holds the facts

$$bb_\alpha - \phi(b)b_\alpha \to 0, \quad \phi(b_\alpha) = 1, \quad (x \in X, b \in B)$$

give us a contradiction (left ϕ-amenability condition for B).

Suppose that (ii) happens. Take η as a right ϕ-character on X. Since $xb_\alpha - \phi(b)x_\alpha \to 0, \phi(b_\alpha) = 1$ for each $x \in X, b \in B$, then we have

$$\eta(xb_\alpha - \phi(b)x_\alpha) = \eta(xb_\alpha) - \phi(b)\eta(x_\alpha) = \phi(b_\alpha)\eta(x) - \phi(b)\eta(x_\alpha) \to 0$$

for each $x \in X, b \in B$. Thus we have $\lim \phi(b)\eta(x_\alpha) = \eta(x)$. Take $b \in \ker \phi$, then we have $\eta(x) = 0$ for each $x \in X$ which is a contradiction (η is a non-zero functional). $\square$

Corollary 5.1. *Let G be a locally compact group. Then* $\mathrm{Tri}(S(G), L^1(G), S(G))^{**}$ *is not approximately biflat.*

Proof. Suppose that $\phi \in \Delta(S(G))$. It is well-known that $S(G)$ has a left approximate identity and is a dense left ideal of $L^1(G)$ and also $L^1(G)$ has a bounded approximate identity. Then

$$\overline{S(G)^2} = S(G), \quad \overline{S(G)L^1(G)} = L^1(G), \quad \overline{S(G)\ker\phi} = \ker\phi.$$

Since $S(G)$ is a left ideal in $L^1(G)$, by [2, Lemma 2.2] ϕ can be extended to a character on $L^1(G)$, which is a right ϕ-character for $L^1(G)$. Now apply Theorem 5.1 to show that $\mathrm{Tri}(S(G), L^1(G), S(G))^{**}$ is not approximately biflat. □

Corollary 5.2. *Let G be a locally compact group. Then* $\mathrm{Tri}(L^1(G), S(G) \otimes_p S(G), S(G))^{**}$ *is not approximately biflat.*

Proof. Since $L^1(G)$ has a bounded approximate identity, we have $\overline{L^1(G)^2} = L^1(G)$. Also using Cohn factorization theorem we have $\overline{L^1(G) \cdot (S(G) \otimes_p S(G))} = S(G) \otimes_p S(G)$. Since $S(G)$ has a left approximate identity, then $\overline{S(G)\ker\phi} = \ker\phi$ for each $\phi \in \Delta(S(G))$. Note that for each $\phi \in \Delta(S(G))$, $\phi \otimes \phi$ which is defined by

$$\phi \otimes \phi(a \otimes b) = \phi(a)\phi(b) \quad (a, b \in S(G))$$

is a right ϕ-character on $S(G) \otimes_p S(G)$. Then apply Theorem 5.1 to show that $\mathrm{Tri}(L^1(G), S(G) \otimes_p S(G), S(G))^{**}$ is not approximately biflat. □

Similarly one can show the following result.

Corollary 5.3. *Let G be a locally compact group. Then* $\mathrm{Tri}(L^1(G), M(G), S(G))^{**}$ *is not approximately biflat.*

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