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TRANSITIVITY, ENTROPY AND LI-YORKE CHAOS OF MULTIPLE MAPPINGS

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In 2016, Hou and Wang[14] *introduced the concept of multiple mappings based on iterated function system. In this paper, we defined the transitivity of the multiple mappings from a set-valued perspective, which is completely different from the previous research perspective on iterated function systems. We show that multiple mappings and its continuous self-maps don't imply each other in terms of transitivity. While a sufficient condition for multiple mappings to be transitive is provided. And we show that transitivity plus fixed point implies Li-Yorke chaos for open multiple mappings. While positive entropy cant imply Li-Yorke chaos for multiple mappings, which is different from the corresponding conclusion of a single continuous map.*

Keywords: Multiple mappings, set-valued view, transitivity, Li-Yorke chaos, entropy.

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1. **Introduction**

Chaos is a highly significant research topic in the field of topological dynamical systems. The first rigorous mathematical definition of chaos can be traced back to 1975 by Li and Yorke[1]. Since then, scholars have embarked on vigorous research on chaos. Scholars from different fields integrated the study of chaos into their own research directions, describing the properties of chaos from different perspectives. Then, a series of different concepts of chaos have emerged, such as sensitivity and transitivity[4], Kato chaos[3], Devaney chaos[2], distribution chaos[5], shadowing properties[6] and others(see [7, 8, 9], for example). The study of the implication relationship between these different concepts has always held an essential position in the field of topological dynamical systems. Xiong [16] proved that transitivity plus a fixed point implies Li-Yorke chaos for continuous self-maps on compact metric space.

In 2016, Hou and Wang[14] defined multiple mappings derived from iterated function system. Their focus was primarily on studying the Hausdorff metric entropy and Friedland entropy of multiple mappings. Additionally, they

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introduced the notions of Hausdorff metric Li-Yorke chaos and Hausdorff metric distributional chaos from a set-valued perspective. It is worth noting that researchers studying iterated function systems often approach the topic from a group perspective rather than a set-valued perspective. This also establishes a close connection between multiple mappings and set-valued mappings, or we can consider multiple mappings as a special case of set-valued mappings.

It is important to acknowledge the valuable role of set-valued mappings in addressing complex problems involving uncertainty, ambiguity, or multiple criteria. Set-valued mappings offer versatility and flexibility, making them highly beneficial across various fields. One prominent application of set-valued mappings is in optimization problems, where the objective is to identify the optimal set of solutions. For instance, in multi-objective optimization, a set-valued mapping can represent the Pareto front, encompassing all non-dominated solutions.

Set-valued mappings also prove useful in decision-making processes that require considering multiple criteria or preferences. By representing feasible solutions as sets, decision-makers can thoroughly analyze and compare different options, enabling them to make well-informed decisions. Additionally, set-valued mappings find applications in data analysis tasks such as clustering and classification. Unlike assigning each data point to a single category, set-valued mappings can represent uncertainty or ambiguity by assigning data points to multiple categories. In fact, the applications of set-valued mappings are vast and diverse, encompassing numerous fields beyond those mentioned here.

In [10], we studied that if there exists at least one Hausdorff metric disdributionally chaotic pair of multiple mappings F, especially F is disdributionally chaotic, then there exists at least two strongly nonwandering points of F and we provide a condition that is sufficient for F to exhibit distributional chaos in a sequence and chaos in the strong sense of Li-Yorke. Zeng et al. [15] proved the existence of Hausdorff metric Li-Yorke chaos or Hausdorff distributional chaos in multiple mappings simultaneously for two topologically conjugate dynamical systems and the multiple mappings F and its 2-tuple of continuous self-maps f_1, f_2 are not mutually implied in terms of Hausdorff metric Li-Yorke chaos.

For a single continuous self-map, Blanchard et al. [11] claimed that positive entropy implies Li-Yorke chaos by ergodic methods. Then, a combined proof of this result is presented by Kerr and Li [12]. T. Downarowicz [13] have established that positive entropy implies type 2 distributional chaos. Then we may ask If the multiple mappings F has positive Hausdorff metric entropy, is it Hausdorff metric chaotic? In [14], Bingzhe Hou and Xu Wang studied the Hausdorff metric entropy to multiple mappings and presented two problems. In this paper, we will give a negative answer both to the following problems.

Problem 1.1. *Let (X, d) be a compact metric space and G be a semigroup generated by a finite set $F = \{f_1, f_2\}$. Does positive entropy imply Hausdorff metric Li-Yorke chaos for F?*

Problem 1.2. *Let (X, d) be a compact metric space and G be a semigroup generated by a finite set $F = \{f_1, f_2\}$. Does positive entropy imply Hausdorff metric distributional chaos of type 2 for F?*

The current paper aims to consider the set formed by the images of a single point under multiple mappings (as a compact set). We primarily consider the relationship between multiple mappings F and its 2-tuple of continuous self-maps f_1, f_2 in terms of transitivity, the relationship between transitivity and Li-Yorke chaos of multiple mappings and the implication between positive entropy and chaos under Hausdorff metric of multiple mappings.

The structure of the current paper is outlined as follows. In Section 2, we provide an introduction to the preliminaries and definitions. Then we study the relation between multiple mappings and its continuous self-maps in terms of transitivity in Section 3. We show that transitivity + fixed point ⇒ Li-Yorke chaos for open multiple mappings in Section 4 and give a negative answer both to Problem 1.1 and Problem 1.2 in Section 5.

2. **Preliminaries**

Throughout this paper, (X, f) is a dynamical system, in which X is metric space with a metric d, f is a continuous self-map on X. And $\mathbb{N} = \{0, 1, 2, \cdots\}$, $\mathbb{Z}^+ = \{1, 2, 3, \cdots\}$. A map f is said to be transitive, if for any nonempty open sets U, V of X there exists $n \in \mathbb{Z}^+$ such that $f^n(U) \bigcap V \neq \emptyset$.

Let $F = \{f_1, f_2\}$ be a multiple mappings with 2-tuple of continuous self-maps on X. Then for $\forall x \in X$, $F(x) = \{f_1(x), f_2(x)\} \subset X$ is compact. Let

$$\mathbb{K}(X) = \{K \subset X \mid K \text{ is compact and} K \neq \emptyset\}.$$

Then F is from X to $\mathbb{K}(X)$. The metric on $\mathbb{K}(X)$ is denoted by d_H, which is called Hausdorff metric. Specifically, it is defined as

$$d_H(A, B) = \max\{\sup_{a\in A} \inf_{b\in B} d(a, b), \sup_{b\in B} \inf_{a\in A} d(a, b)\}, \forall A, B \subset X.$$

It is clear that $(\mathbb{K}(X), d_H)$ is compact.

For any $n \in \mathbb{Z}^+$, $F^n : X \to \mathbb{K}(X)$ is defined by for any $x \in X$,

$$F^n(x) = \{f_{i_1} f_{i_2} \cdots f_{i_n}(x) \mid i_1, i_2, \cdots, i_n = 1 \text{ or } 2\}.$$

It is obvious that $F^n(x) \in \mathbb{K}(X)$. For $\forall A \subset X$, let

$$F^n(A) = \{f_{i_1} f_{i_2} \cdots f_{i_n}(a) \mid a \in A, i_1, i_2, \cdots, i_n = 1 \text{ or } 2\} = \bigcup_{a\in A} F^n(a).$$

For $\forall \mathcal{U} \subset \mathbb{K}(X)$,

$$F^{-n}(\mathcal{U}) = \{x \in X \mid F^n(x) \in \mathcal{U}\}.$$

Particularly, if $A \in \mathbb{K}(X)$, then $F^n(A) \in \mathbb{K}(X)$. Then, F naturally induces a continuous self-map on $\mathbb{K}(X)$, denoted by $\widetilde{F} : \mathbb{K}(X) \to \mathbb{K}(X)$.

Definition 2.1. *For any $x \in X$, we say*

$$\{F(x), F^2(X), F^3(X), \cdots\} := \overset{\circ}{orb}(x, F)$$

is the deleted orbit of x under F.

Since we are studying the dynamical systems of multiple mappings from a set-valued perspective, there is no natural way to extend the notion of transitivity. However, we have discovered a meaningful approach to extend it as follows. Put $Ran(F) = \{F^n(x) \mid n \geq 1, x \in X\}$.

Definition 2.2. *The multiple mappings $F = \{f_1, f_2\}$ is said to be transitive, if for any open sets $U \neq \emptyset$ of X and $\mathcal{U} \neq \emptyset$ of $Ran(F)$, we can find $n \in \mathbb{Z}^+$ satisfying*

$$\{F^n(u) \mid u \in U\} \bigcap \mathcal{U} \neq \emptyset.$$

Definition 2.3. *$x \in X$ is said to be a transitive point of the multiple mappings $F = \{f_1, f_2\}$, if its deleted orbit $\overset{\circ}{orb}(x, F)$ is dense in $Ran(F)$.*

It is easy to see that the Hausdorff metric transitivity of multiple mappings, in the case of degradation (where the multiple mappings consists of only one continuous self-map), is the same as the transitivity of a classical single continuous self-map. Next we provide some examples to illustrate the existence of the newly defined concept Definition 2.2.

Example 2.1. *Consider the multiple map defined on $[0, 1]$ as $F = \{f_1, f_2\}$, in which*

$$f_1(x) = 0, \forall x \in [0, 1], f_2(x) = 1, \forall x \in [0, 1].$$

Then $Ran(F) = \{\{0, 1\}\}$ and $F(0) = \{0, 1\} = F(1)$. So, F is transitive.

Example 2.2. *Consider the multiple map defined on $X = [0, 1]$ as $F = \{f_1, f_2\}$, in which $f_1(x) = 0, \forall x \in [0, 1]$ and*

$$f_2(x) = \begin{cases} 2x, & 0 \leq x \leq \frac{1}{2}, \\ 2 - 2x, & \frac{1}{2} < x \leq 1. \end{cases}$$

Then

$$Ran(F) = \{\{0, f_2^n(x)\} \mid n \geq 1, x \in [0, 1]\}.$$

Let $U \subset X$ and $\mathcal{U} \subset Ran(F)$ be nonempty open sets. Then there exists open set $V \neq \emptyset$ of $[0, 1]$ satisfying $\{\{0, v\} \mid v \in V\} \subset \mathcal{U}$. As we all know, f_2 is the tent map. It is transitive. Then there is $n \in \mathbb{Z}^+$ with $f_2^n(U) \bigcap V \neq \emptyset$. Thus, there is $u \in U$ with $f_2^n(u) \in V$. Then

$$F^n(u) = \{0, f_2^n(u)\} \in \mathcal{U}.$$

So, F is transitive.

Of course, there are also many multiple mappings that are not transitive. Now we give an example of multiple mappings which isn't transitive.

Example 2.3. *Consider the multiple mappings defined on $X = [0,1]$ as $F = \{f_1, f_2\}$, in which*

$$f_1(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2}, \\ 1, & \frac{1}{2} < x \le 1, \end{cases}$$

$$f_2(x) = \begin{cases} 1, & 0 \le x \le \frac{1}{2}, \\ 2-2x, & \frac{1}{2} < x \le 1. \end{cases}$$

Let

$$f(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2}, \\ 2-2x, & \frac{1}{2} < x \le 1. \end{cases}$$

Then

$$Ran(F) = \{\{1, f(x)\} \mid x \in [0,1]\} \bigcup \{\{0, 1, f^n(x)\} \mid n \ge 2, x \in X\}.$$

Let $U = [0, \frac{1}{64})$ and $\mathcal{U} = \{\{1, v\} \mid v \in (\frac{1}{16}, \frac{3}{16})\}$. Then

$$\{F^n(u) \mid n \ge 1, u \in U\} \bigcap \mathcal{U} = \emptyset.$$

So, F is not transitive.

In [14], the concepts of Hausdorff metric Li-Yorke chaos, (sequential) distributional chaos of multiple mappings were introduced. Let us revisit these definitions. It is important to note that all forms of chaos discussed in this paper are defined using the Hausdorff metric.

Definition 2.4. *The multiple mappings F is said to be chaotic in the sense of Li-Yorke, if there is an uncountable set $S \subset X$ satisfying for any $x \ne y \in S$,*

$$\liminf_{n\to\infty} d_H(F^n(x), F^n(y)) = 0, \ \limsup_{n\to\infty} d_H(F^n(x), F^n(y)) > 0.$$

Definition 2.5. *Define distributional function*

$$\phi^n_{xy}(F, \cdot) : R \to [0,1]$$

by

$$\phi^n_{xy}(F,t) = \frac{1}{n}\sharp\{0 \le i \le n-1 \mid d_H(F^i(x), F^i(y)) < t\}, \forall t \in R, x, y \in X.$$

It is clear that $\phi^n_{xy}(F,t) = 0, \forall t < 0$. Put

$$\phi_{xy}(F,t) = \liminf_{n\to\infty} \phi^n_{xy}(F,t), \ \phi^*_{xy}(F,t) = \limsup_{n\to\infty} \phi^n_{xy}(F,t).$$

F is said to be distributionally chaotic of type $k \in \{1,2,3\}$ (briefly referred to as HDC1, HDC2 and HDC3, respectively), if there is an uncountable subset $D \subset X$ such that any two points $x \ne y \in D$ satisfy condition (k) as following:

(1) $\phi^*_{xy}(F,t) \equiv 1$, $\forall t > 0$ *and* $\phi_{xy}(F,\varepsilon) = 0$ *for some* $\varepsilon \in \mathbb{Z}^+$.
(2) $\phi^*_{xy}(F,t) \equiv 1$, $\forall t > 0$ *and* $\phi^*_{xy} > \phi_{xy}$.
(3) $\phi^*_{xy} > \phi_{xy}$.

Definition 2.6. *Let* $x \neq y \in X$ *and the sequence* $\{p_k\} \subset \mathbb{Z}^+$. *For any* $\delta > 0$, *put*

$$\phi_{xy}(\delta, \{p_k\}) = \liminf_{n\to\infty} \frac{1}{n} \sharp\{1 \le k \le n \mid d_H(F^{p_k}(x), F^{p_k}(y)) < \delta\},$$

$$\phi^*_{xy}(\delta, \{p_k\}) = \limsup_{n\to\infty} \frac{1}{n} \sharp\{1 \le k \le n \mid d_H(F^{p_k}(x), F^{p_k}(y)) < \delta\}.$$

F is said to be distributionally chaotic in the sequence $\{p_k\}$, *if there is an uncountable subset* $D \subset X$ *such that for any two points* $x \neq y \in D$,
(1) $\phi^*_{xy}(t, \{p_k\}) = 1$ *for all* $t > 0$;
(2) $\phi_{xy}(\varepsilon, \{p_k\}) = 0$ *for some* $\varepsilon \in \mathbb{Z}^+$.

Obviously, if F is HDC1, then it is HDC2, HDC3 and sequentially distributionally chaotic. If F is HDC2, it is HDC3. If F is HDC1 or HDC2 or sequentially distributionally chaotic, it is Li-Yorke chaotic.

3. **Transitivity**

Firstly, we present two classical propositions on transitivity.

Proposition 3.1. *The multiple mapping F is transitive if and only if for any open sets* $\mathcal{U} \neq \emptyset$ *of* $Ran(F)$ *and* $V \neq \emptyset$ *of* X, *one can find* $n \in \mathbb{Z}^+$ *satisfying*

$$F^{-n}(\mathcal{U}) \bigcap V \neq \emptyset.$$

Proof. Necessity$\Rightarrow$: Let $\mathcal{U} \neq \emptyset \subset Ran(F)$ and $U \neq \emptyset \subset X$ be open. Then there is $n \in \mathbb{Z}^+$ such that

$$\{F^n(u) \mid u \in U\} \bigcap \mathcal{U} \neq \emptyset.$$

Thus there are $u \in U$ and $\mu \in \mathcal{U}$ satisfying $F^n(u) = \mu$. Then $F^{-n}(\mu) = u$. That is $F^{-n}(\mathcal{U}) \in U$. So,

$$F^{-n}(\mathcal{U}) \bigcap U \neq \emptyset.$$

Sufficiency$\Leftarrow$: Let $\mathcal{U} \neq \emptyset \subset Ran(F)$ and $U \neq \emptyset \subset X$ be open. Then one can find $n \in \mathbb{Z}^+$ with $F^{-n}(\mathcal{U}) \bigcap U \neq \emptyset$. Thus there are $u \in U$ and $\mu \in \mathcal{U}$ such that $F^{-n}(\mu) = u$. That is $F^n(u) = \mu \in \mathcal{U}$. Then

$$\{F^n(u) \mid u \in U\} \bigcap \mathcal{U} \neq \emptyset.$$

So, F is transitive.

□

Proposition 3.2. *If the multiple mapping F is transitive, then* $\{x \in X \mid \overline{\mathring{orb}(x,F)} = Ran(F)\}$ *is a* G_δ *set.*

Proof. X is compact, then it implies $\mathbb{K}(X)$ is also compact. Then there is a countable topological basis $\{\mathcal{U}_n\}_{n=1}^{\infty}$ of $\mathbb{K}(X)$. Thus

$$\{x \in X \mid \overline{\overset{\circ}{orb}(x, F)} = Ran(F)\} = \bigcap_{n=1}^{\infty} \bigcup_{m=0}^{\infty} F^{-m}(\mathcal{U}_n).$$

By Proposition 3.1,

$$\overline{\bigcup_{m=0}^{\infty} F^{-m}(\mathcal{U}_n)} = X.$$

So, $\{x \in X \mid \overline{\overset{\circ}{orb}(x, F)} = Ran(F)\}$ is a G_δ set. □

A natural question is what is the implication between the transitivity of multiple mappings $F = \{f_1, f_2\}$ and the transitivity of its 2-tuple of continuous self-maps f_1, f_2? The Example 2.1 shows that the transitivity of multiple mappings $F = \{f_1, f_2\}$ doesn't imply the transitivity of f_1 or f_2. Then combined with the following example, the transitivity of multiple mappings $F = \{f_1, f_2\}$ and its 2-tuple of continuous self-maps f_1, f_2 do not imply each other.

Example 3.1. *Consider the multiple mappings $F = \{f_1, f_2\}$ defined on $\{0, 1, 2\}$, in which*

$$f_1 : 0 \longmapsto 1 \longmapsto 2 \longmapsto 0$$

and

$$f_2 : 0 \longmapsto 2 \longmapsto 1 \longmapsto 0.$$

It is easy to see that both f_1 and f_2 is transitive and weakly mixing. Next we show F is not transitive.

$Ran(F) = \{\{1, 2\}, \{0, 2\}, \{0, 1\}, \{0, 1, 2\}\}$. Let $U = \{0\}$ and $\mathcal{U} = \{\{0, 2\}\}$. Then U is a nonempty open set of $\{0, 1, 2\}$ and $\mathcal{U}$ is a nonempty open set of $Ran(F)$. While

$$F : 0 \mapsto \{1, 2\} \mapsto \{0, 1, 2\} \mapsto \cdots \mapsto \{0, 1, 2\} \mapsto \cdots .$$

So, F is not transitive.

Although that both f_1 and f_2 are transitive can't imply $F = \{f_1, f_2\}$ is transitive, We present Theorem 3.1 as a sufficient condition for F to be transitive.

Theorem 3.1. *If $f_1(x) = c$ (c is a constant, $\forall x \in X$) and f_2 is transitive, then the multiple mapping $F = \{f_1, f_2\}$ is transitive.*

Proof. Clearly, $Ran(F) = \{\{c, f_2^n(x)\} \mid n \geq 1, x \in X\}$. Let $U \subset X$ and $\mathcal{U} \subset Ran(F)$ be two nonempty open sets. Then there exists nonempty open set $V \subset X$ such that $\{\{c, v\} \mid v \in V\} \subset \mathcal{U}$. Since f_2 is transitive, there exists $n \in \mathbb{Z}^+$ such that $f_2^n(U) \bigcap V \neq \emptyset$. Then there exists $u \in U$ such that $f_2^n(u) \in V$, that is, $F^n(u) \in \mathcal{U}$. So, F is transitive. □

4. **Transitivity and Li-Yorke chaos**

In this section, we will show that transitivity + fixed point $\Rightarrow$ Li-Yorke chaos for open multiple mappings. Prior to presenting this result, we provide a list of several lemmas that are instrumental in proving the result.

Lemma 4.1. [16] *Let* $h_i : X \to [0, +\infty]$ *be semicontinuous for any* $i \in \mathbb{Z}^+$ *and* $a \in [0, +\infty]$. *Define*

$$g(x) = \liminf_{i\to\infty} h_i(x), x \in X.$$

If $\overline{g^{-1}([0, a])} = X$, *then* $g^{-1}([0, a])$ *is a* G_δ *set.*

Lemma 4.2. *Suppose that the multiple mappings* $F = \{f_1, f_2\}$ *is an open mapping. If there are at least one transitive point of the multiple mappings* F *and one common fixed point* $v \in X$ *of* f_1 *and* f_2 *(that is* $f_1(v) = f_2(v) = v$*), then one can find a dense* G_δ *set* B *of* $X \times X$ *satisfying for* $\forall (x_1, x_2) \in B$,

$$\liminf_{n\to\infty} d_H(F^n(x_1), F^n(x_2)) = 0.$$

Proof. Define $\mathcal{F} : X \times X \to \mathbb{R}$ as for $\forall (x_1, x_2) \in X \times X$,

$$\mathcal{F}(x_1, x_2) = \liminf_{n\to\infty} d_H(F^n(x_1), F^n(x_2)).$$

Define $\overline{\mathcal{F}} : \mathbb{K}(X) \times \mathbb{K}(X) \to \mathbb{R}$ as for $\forall (A_1, A_2) \in \mathbb{K}(X) \times \mathbb{K}(X)$,

$$\overline{\mathcal{F}}(A_1, A_2) = \liminf_{n\to\infty} d_H(F^n(A_1), F^n(A_2)).$$

Let ω be a transitive point of the multiple mappings F, then for $\forall A = (A_1, A_2) \in \mathring{orb}(\omega, F) \times \mathring{orb}(\omega, F)$, there exist positive integers k_1, k_2 such that $F^{k_1}(\omega) = A_1$ and $F^{k_2}(\omega) = A_2$. Let $v \in X$ be the common fixed point of f_1 and f_2. Then $F(v) = \{v\} \in Ran(F)$ and there exists $\{n_i\}$ such that $\lim_{n_i\to\infty} F^{n_i}(\omega) = \{v\}$. Thus for each $j = 1, 2$,

$$\lim_{n_i\to\infty} F^{n_i}(A_j) = \lim_{n_i\to\infty} F^{n_i}(F^{k_j}(\omega)) = \lim_{n_i\to\infty} F^{k_j}(F^{n_i}(\omega)) = F^{k_j}(\omega) = \{v\}.$$

Then $\overline{\mathcal{F}}(A) = 0$. Since $\overline{\mathring{orb}(\omega, F) \times \mathring{orb}(\omega, F)} = Ran(F) \times Ran(F)$, $\overline{\overline{\mathcal{F}}^{-1}(0)} = Ran(F) \times Ran(F)$.

Now we show that $\mathcal{F}^{-1}(0)$ is dense in $X \times X$ by contradiction.

Suppose that $\overline{\mathcal{F}^{-1}(0)} \neq X \times X$. Then one can find a nonempty open set $U \times V \subset X \times X$ satisfying $\mathcal{F}^{-1}(0) \bigcap U \times V = \emptyset$. Thus for $\forall (a, b) \in U \times V$, $\mathcal{F}(a, b) = \liminf_{n\to\infty} d_H(F^n(a), F^n(b)) \neq 0$.

Let $A_1 = F(a)$ and $A_2 = F(b)$. If $(a, b) \in U \times V$, then

$$\liminf_{n\to\infty} d_H(F^n(A_1), F^n(A_2)) \neq 0.$$

Since F is an open mapping,

$$\{F(a) \mid a \in U\} \times \{F(b) \mid b \in V\} := \mathcal{U} \times \mathcal{V}$$

is a nonempty open set of $Ran(F) \times Ran(F)$. Then for any $(A_1, A_2) \in \mathcal{U} \times \mathcal{V}$, $\overline{\mathcal{F}}(A_1, A_2) \neq 0$, which is contradictory to $\overline{\mathcal{F}}^{-1}(0)$ being dense in $Ran(F) \times Ran(F)$. So, $\mathcal{F}^{-1}(0)$ is dense in $X \times X$.

By Lemma 4.1, $\mathcal{F}^{-1}(0)$ is a dense G_δ set in $X \times X$. □

Lemma 4.3. *If $Rec(F) := \{x \in X \mid \exists\{n_i\}$ with $\lim_{n_i \to \infty} F^{n_i}(x) = x\}$ is dense in X, then $Rec(F)$ is a dense G_δ set.*

Proof. Define $\mathcal{F} : X \to [0, +\infty)$ as for $\forall x \in X$,

$$\mathcal{F}(x) = \liminf_{n \to \infty} d_H(F^n(x), \{x\}).$$

Then $x \in Rec(F) \Leftrightarrow \mathcal{F}(x) = 0$. Thus $\overline{\mathcal{F}^{-1}(0)} = X$. By Lemma 4.1, $Rec(F) = \mathcal{F}^{-1}(0)$ is a dense G_δ set of X. □

Lemma 4.4. *If the multiple mappings F is transitive, then $Rec(F \times F)$ is a dense G_δ set of $X \times X$.*

Proof. By Proposition 3.2, one can find at least one transitive point $\omega \in X$ of F. Then $\mathring{orb}(\omega, F) \times \mathring{orb}(\omega, F)$ is dense in $Ran(F) \times Ran(F)$ and $\mathring{orb}(\omega, F) \times \mathring{orb}(\omega, F) \subset Rec(F \times F)$. Thus $Rec(F \times F)$ is dense in $X \times X$. By Lemma 4.3, $Rec(F \times F)$ is a dense G_δ set of $X \times X$. □

Theorem 4.1. *If the multiple mappings F is transitive and there is at least one common fixed point of f_1 and f_2, then F is Hausdorff metric Li-Yorke chaotic.*

Proof. Let $D = Rec(F \times F) \bigcap B$, where B is identical to the B referred to in lemma 4.2. By Proposition 3.2, one can find at least one transitive point of F. By Lemma 4.2 and Lemma 4.4, $D \subset X \times X$ and it is a residual set.

Due to the compactness of X, it is both complete and separable. As stated in [17], this implies the existence of a dense Mycielski set K in which any distinct pair of elements (x_1, x_2) satisfies $(x_1, x_2) \in D$. So, F Li-Yorke chaotic. □

5. **Entropy and Chaos**

In this section, we mainly show positive entropy doesn't imply neither Li-Yorke chaos nor HDC2 for multiple mappings by Example 4.1 of [14].

Example 5.1. *Denote X by the unit interval $[0, 1]$. Let $F = \{f_1, f_2\}$, where $f_1, f_2 : X \to X$ defined by*

$$f_1(x) = \begin{cases} x, & 0 \leq x \leq \frac{1}{3}, \\ 3x - \frac{2}{3}, & \frac{1}{3} < x \leq \frac{4}{9}, \\ \frac{3}{5}x + \frac{2}{5}, & \frac{4}{9} < x \leq 1, \end{cases}$$

$$f_2(x) = \begin{cases} \frac{3}{5}x, & 0 \le x \le \frac{5}{9}, \\ 3x - \frac{4}{3}, & \frac{5}{9} < x \le \frac{2}{3}, \\ x, & \frac{2}{3} < x \le 1, \end{cases}$$

Hou and Wang have already proven F has positive topological entropy. Now we use a case-by-case analysis to show F is not Li-Yorke chaotic.

(1) *For any $x \in [0, \frac{1}{3}]$ and any $n \in \mathbb{Z}^+$, $F^n(x) = \{x, \frac{3}{5}x, (\frac{3}{5})^2 x, \cdots, (\frac{3}{5})^n x\}$. Then $\lim_{n\to\infty} F^n(x) \in \mathbb{K}(X)$, $\forall x \in [0, \frac{1}{3}]$.*

(2) *For any $x \in [\frac{2}{3}, 1]$ and any $n \in \mathbb{Z}^+$, $F^n(x) = \{x, \frac{3}{5}(x-1)+1, (\frac{3}{5})^2(x-1)+1, \cdots, (\frac{3}{5})^n(x-1)+1\}$. Then $\lim_{n\to\infty} F^n(x) \in \mathbb{K}(X)$, $\forall x \in [\frac{2}{3}, 1]$.*

(3) *Let $G = \{f_{i_1} f_{i_2} \cdots f_{i_n} \mid n \in \mathbb{Z}^+, i_1, i_2, \cdots, i_n = 1 \text{ or } 2\}$. For any $x \in (\frac{1}{3}, \frac{2}{3})$ and any $g \in G$, there exists $n \in \mathbb{Z}^+$ such that $g(x) \in F^n(x)$. Then there exists a monotone increasing sequence $\{g_k(x)\}_{k=n}^{\infty}$ with $g_k(x) \in F^k(x)$, $k = n, n+1, \cdots$ and a monotone decreasing sequence $\{h_k(x)\}_{k=n}^{\infty}$ with $h_k(x) \in F^k(x)$, $k = n, n+1, \cdots$ such that*

$$g(x) \in \{g_k(x)\}_{k=n}^{\infty} \text{ and } g(x) \in \{h_k(x)\}_{k=n}^{\infty}.$$

Suppose that there exist $x \in (\frac{1}{3}, \frac{2}{3})$, $A \neq B \in \mathbb{K}(X)$ and $\{n_i\}$, $\{m_i\}$ such that $\lim_{n_i\to\infty} F^{n_i}(x) = A$ and $\lim_{m_i\to\infty} F^{n_i}(x) = B$. Select $a \in A - B$, then there exists $\{y_{n_i}\}_{i=1}^{\infty}$ with $y_{n_i} \in F^{n_i}(x)$, $i = 1, 2\cdots$ such that $\lim_{n_i\to\infty} y_{n_i} = a$. Thus there exists a monotone increasing (or decreasing) sequence $\{y_{n_{i_j}}\}_{j=1}^{\infty} \subset \{y_{n_i}\}_{i=1}^{\infty}$ such that $\lim_{j\to\infty} y_{n_{i_j}} = a$. Let $\{y_{n_{i_j}}\}_{j=1}^{\infty}$ be monotone increasing (When it is monotone decreasing, it is in a similar way.). There must be a monotone increasing sequence $\{y_{m_i}\}_{i=1}^{\infty}$ with $y_{m_i} \in F^{m_i}(x)$, $i = 1, 2\cdots$ such that $\{y_{n_{i_j}}\}_{j=1}^{\infty} \bigcup \{y_{m_i}\}_{i=1}^{\infty} = \{y_{t_i}\}_{i=1}^{\infty}$ is monotone increasing. While $\lim_{m_i\to\infty} y_{m_i} \neq a$. This is a contradiction. So, F is not Li-Yorke chaotic.

By Example 5.1, we have the following theorem, which is different from the corresponding conclusion of a single continuous map.

Theorem 5.1. *For multiple mappings, positive entropy cant imply Li-Yorke chaos or HDC1 or HDC2 or (sequentially) distributional chaos.*

6. Conclusions

We define and study transitivity of multiple mappings from the perspective of a set-valued view. This perspective is different from the view that has been previously studied in the context of dynamical systems of iterated function systems. We show that

(1) transitivity of multiple mappings $F = \{f_1, f_2\}$ and its 2-tuple of continuous self-maps f_1, f_2 do not imply each other. And, a sufficient condition for F to be transitive is provided.

(2) transitivity + fixed point $\Rightarrow$ Li-Yorke chaos for open multiple mappings.

(3) positive entropy doesn't imply neither Li-Yorke chaos nor HDC2 for multiple mappings.

The above conclusions not only deepen our understanding of continuous self-maps but also enable us to use relatively simple continuous self-maps to comprehend relatively complex multiple mappings. It is worth mentioning that we have defined the transitivity property of multiple mappings and have obtained some conclusions through our study. This has opened up new avenues for investigating the topological structure and properties of multiple mappings. And we give a negative answer to [problem 5.3 and problem 5.4] of [14].

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