GENERAL HARMONIC-LIKE VARIATIONAL INEQUALITIES

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We introduce and study some new classes of general harmonic-like variational inequalities. Auxiliary principle technique is applied to discuss the existence of the solution and to propose several iterative schemes for computing the approximate solution. Convergence analysis is investigated under certain mild conditions. Several new problems such as harmonic-like complementarity problems, representation theorems and related optimization problems are discussed. The techniques and ideas developed in this paper can be used to develop new techniques for these problems.

Keywords: Variational inequalities, harmonic-like convex functions, auxiliary principle, algorithms, convergence criteria

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1. Introduction

Convexity theory contains a wealth of novel ideas and innovative techniques, which have played the significant role in the development of almost all the branches of pure and applied sciences such as fixed point, variational inequalities and optimizations.

It is well known fact that the minimum of the differentiable convex function on the convex set in a normed spaces can be characterized by an inequality, which is called the variational inequality. Lions and Stampacchia [12] considered and studied the variational inequalities. They also emphasized that the Riesz-Frechet representation theorem and Lax-Milgram lemma are special cases of the variational inequalities. Noor [15] introduced and studied some new classes of variational inequalities involving two monotone operators with applications in Bingham fluid, elasticity and optimizations. System of the absolute value equations, complementarity problems and hemivariational inequalities are special cases of the nonlinear variational inequalities involving two operators. Variational inequality theory can be viewed as a novel extension and generalization of the variational principles. By variational principles, we mean maximum and minimum problems arising in game theory, mechanics, geometrical optics, general relativity theory, economics, transportation, differential geometry, mathematical finance, machine learning, artificial intelligence and related areas. It is well known facts that variational theory provides us with a simple, natural, unified, novel and general framework to study an extensive range of unilateral, obstacle, free, moving and equilibrium problems arising in fluid flow through porous media, elasticity, circuit analysis, transportation, oceanography, operations research, finance, economics, and optimization. It is amazing that variational inequalities have influenced various areas of pure and applied sciences and are still continue to influence the recent research, see [3, 4, 8, 9, 10, 12, 14, 15, 16, 17, 18, 19, 21, 22, 23, 24, 25, 27, 28, 29, 30, 31, 32, 33, 34, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46]. Anderson et al. [2] have investigated several aspects of the harmonic convex sets and harmonic convex functions, which can viewed as important generalizations of the convex functions and convex sets. The harmonic means have

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novel applications in electrical circuits theory. Al-Azemi et al. [1] studied the Asian options with harmonic average, which can be viewed a new direction in the study of the risk analysis stock exchange and financial mathematics. The harmonic mean are being used to suggest some iterative methods for solving nonlinear equations. Noor et al. [22] have shown that the minimum of the differentiable harmonic convex functions on the harmonic convex set can be characterized by a class of harmonic variational inequalities. For more details and applications of the harmonic functions, see [2, 13, 21, 22, 23, 24, 26, 27, 28, 32] and the references therein. Noor et al. [26] introduced the new concepts of harmonic-like convex sets and harmonic-like convex functions. Motivated and inspire by this new direction, we introduce and consider some new classes of harmonic variational inequalities involving arbitrary two operators. Several new problems such as harmonic-like complementarity problems, harmonic-like Riesz-Frechet representation theorems and Lax-Milgram lemma are highlighted. Due to the structure and nonlinearity of the harmonic-like variational inequalities, the projection, resolvent operators and their variants can not be applied to study the existence of the solution and approximate the solution. To overcome these drawback, one uses the auxiliary principle technique, which is mainly due to Lions and Stampachia [12] and Glowinski et al. [8], to discuss various aspects of the such nonlinear problems. Noor [19], Noor et al. [24, 25, 27, 30, 31, 32] and Patrikson [33] have shown that the auxiliary principle technique can be used effectively to suggest some iterative methods for solving the boundary value problems and various classes of variational inequalities. In Section 2, we introduce the harmonic-like variational inequalities and discuss some important special cases. The auxiliary principle technique is used to discuss the existence of a unique solution as well as to suggest some iterative methods for the boundary value problems. Convergence analysis of the proposed method is also considered under some mild conditions. We have only considered theoretical aspects of the suggested methods. It is an interesting problem to implement these methods and to illustrate the their efficiency. Comparison with other methods need further research efforts. The ideas and techniques of this paper may can be extended for other classes of variational inequalities and related optimization problems.

2. **Formulations and basic facts**

Let Ω be a nonempty closed set in a real Hilbert space $\mathcal{H}$. We denote by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ be the inner product and norm, respectively.

First of all, we now show that the minimum of a differentiable harmoni-like convex function on a harmonic-like convex set Ω in $\mathcal{H}$ can be characterized by the harmonic-like variational inequalities. For this purpose, we recall the following well known concepts and results [5, 13, 27, 28].

Definition 2.1. [27, 28] *A set Ω is said to be a general harmonic-like convex set, if*

$$(\frac{2w\mu}{w+\mu}) + t(g(\nu) - g(\mu) \in \Omega, \quad \forall \mu, w, \nu \in \Omega,$$
$$\mu \leq w \leq \nu, \quad \mu \leq w \leq \nu, \quad t \in [0,1].$$

Definition 2.2. [27, 28] *A function $\phi : \Omega \longrightarrow \mathcal{H}$ is said to be a general harmonic-like convex with respect to an arbitrary operator g, if*

$$\phi((\frac{2w\mu}{w+\mu}) + t(g(\nu) - g(\mu))$$
$$\leq (1-t)\phi(g(\mu)) + t\phi(g(\nu)), \quad \forall w, \mu, \nu \in \Omega, \quad \mu \leq w \leq \nu, \quad t \in [0,1].$$

Remark 2.1. Here $w \in [\mu, \nu]$ can be viewed as the weight function.

(a). If $w = \mu, \quad g = I$, then the defintions 2.1 and 2.2 are exactly the convex set and convexfunctions.

(b). For $w = \nu$, the harmonic-like convex set and harmonic-like functions reduce to general harmonic mean set and general harmonic mean function respectively.

We remark that, if $\mu = w$, then the general harmonic-like convex set becomes the general convex set and the general harmonic-like convex functions reduces to the general convex function. For the properties and applications of convex sets and convex functions, see [5, 13] and the references therein Using the technique of Noor et al. [27, 28], one can easily prove that the minimum of the differentiable general harmonic-like convex function can be characterized by an inequality of the following type:

Theorem 2.1. *Let $\phi : \Omega \longrightarrow \mathcal{H}$ be a differentiable general harmonic-like convex function. Then $\mu \in \Omega$ is the minimum of general harmonic-like convex function ϕ on Ω, if and only if, $\mu \in \Omega$ satisfies the inequality*

$$\langle \phi'(\frac{2w\mu}{w+\mu}), g(\nu) - g(\mu) \rangle \geq 0, \quad \forall \nu, w \in \Omega, \tag{1}$$

where $\phi'(\frac{2w\mu}{w+\mu})$ is the differential of ϕ.

The inequality of the type (1) is called the general harmonic-like variational inequality. It is known that the problem (1) may not arise as the optimality conditions of the differentiable general harmonic-like convex functions. This motivated us to consider a more general problem of which the problem (1) is a special case.

To be more precise, for given operators $\mathcal{T}, g : \mathcal{H} \longrightarrow \mathcal{H}$, consider the problem of finding $\mu \in \Omega \subseteq \mathcal{H}$, such that

$$\langle \mathcal{T}(\frac{2w\mu}{w+\mu}), g(\nu) - g(\mu) \rangle \geq 0, \quad \forall w, \nu \in \Omega, \tag{2}$$

which is called the general harmonic-like variational inequality.

We now point out some very important and interesting problems, which can be obtained as special cases of the problem (2).

(1) For $w = \mu$, then problem (2) reduces to finding $\mu \in \Omega$, such that

$$\langle \mathcal{T}\mu, g(\nu) - g(\mu) \rangle \geq 0, \quad \forall \nu \in \Omega, \tag{3}$$

is called the general variational inequality introduced and studied by Noor [21] in 1988. For the applications and generalizations of the general variational inequalities, see[16, 17, 19, 30, 31] and the references therein.

(2) If $\mathcal{T} = I$, the identity operator, then problem (2)reduces to finding $\mu \in \Omega$ such that

$$\langle (\frac{2w\mu}{w+\mu}), \nu - \mu \rangle \geq 0, \quad \forall w, \nu \in \Omega, \tag{4}$$

This inequality is called the inverse harmonic-like variational inequality.

(3) If $g = I$, $\mu = w$, then the problem (2) collapses to finding $\mu \in \Omega$ such that

$$\langle \mathcal{T}\mu, \nu - \mu \rangle \geq 0, \quad \forall \nu \in \Omega, \tag{5}$$

which is called variational inequality, introduced by Lions and Stamapcchia [24] in the impulse control theory. For the numerical analysis, sensitivity analysis, dynamical systems and other aspects of variational inequalities and related optimization programming problems. see [1, 8, 12, 15, 16, 17, 18, 19, 20, 21, 23, 24, 25, 26, 27, 28, 31, 32, 33, 35, 36, 37, 38, 40, 42, 43, 44, 45] and the references therein.

(4) For the polar cone $\Omega^* = \{\mu \in \mathcal{H} : \langle \mu, \nu \rangle \geq 0, \quad \forall \nu \in \Omega\}$, the problem (2) is equivalent to finding $\mu \in \mathcal{H}$ such that

$$\mu \in \Omega, \quad \mathcal{T}(\frac{2w\mu}{w+\mu}) \in \Omega^* \quad \text{and} \quad \langle \mathcal{T}(\frac{2w\mu}{w+\mu}), g(\mu) \rangle = 0, \tag{6}$$

is called the general harmonic-like complementarity problem.

(5) If $\mu = w, \quad g = I$, then the problem (6) is called the general complementarity problem introduced and studied by Noor [16, 18]. For the applications, motivations, generalization, numerical methods and other aspects of the complementarity problems in engineering and applied sciences, see [4, 16, 18, 19, 20, 29, 31, 32] and the references therein.

(6) If $\Omega = \mathcal{H}$, then the problem (2) collapses to finding $\mu \in \mathcal{H}$ such that

$$\langle \mathcal{T}(\frac{2w\mu}{w+\mu}), g(\nu) - g(\mu)\rangle = 0, \quad \forall w, \nu \in \Omega, \tag{7}$$

which is called the general harmonic-like Lax-Milgram Lemma, introduced and studied in [27]. For the applications, generalizations and extensions of the Lax-Milgram Lemma, see [6, 10, 11, 12, 14, 15, 18, 19, 20, 22, 30, 31, 32]. and the references therein.

(7) If $\mathcal{T} = I, g = I$, then the problem (7) reduces to finding $\mu \in \mathcal{H}$ such that

$$\langle (\frac{2w\mu}{w+\mu}), \nu - \mu\rangle = 0, \quad \forall w, \nu \in \mathcal{H}, \tag{8}$$

which is the harmonic-like Riesz-Frechet representation theorem.

(8) For $w = \mu$, we obtain the problem of finding $\mu \in \mathcal{H}$, such that

$$\langle \mu, \nu\rangle = \langle f, v\rangle, \quad \forall \nu \in \mathcal{H},$$

which is known as the celebrated the Riesz-Frechet representation theorem, introduced and studded by Frechet [7] and Riesz [37]independently in 1907. Also, we see [7, 12, 14, 15, 20, 22, 27, 30, 31, 32, 35, 36] for the novel generalization and applications of the Reisz-Frechet representation theorems and its variant forms. For a different and appropriate choice of the operators and spaces, one can obtain several known and new classes of variational inequalities and related problems. This clearly shows that the problem (2) considered in this paper is more general and unifying one.

Remark 2.2. It is worth mentioning that for appropriate and suitable choices of the operators $\mathcal{T}, g$, closed convex set Ω and the spaces, one can obtain several classes of variational inequalities, complementarity problems and optimization problems as special cases of the nonlinear harmonic-like variational inequalities (2). This shows that the problem (2) is quite general and unifying one. It is interesting problem to develop efficient and implementable numerical methods for solving the general harmonic-like variational inequalities and their variants.

We need the following definitions and results in obtaining our results.

Definition 2.3. [27, 28] *An operator $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ with respect to an arbitrary operator g is said to be:*

(1) *Strongly general harmonic-like monotone, if there exist a constant $\alpha > 0$, such that*

$$\langle \mathcal{T}(\frac{2w\mu}{w+\mu}) - \mathcal{T}(\frac{2w\nu}{w+\nu}), g(\mu) - g(\nu)\rangle \geq \alpha\|\mu - \nu\|^2, \quad \forall w, \mu, \nu \in \mathcal{H}.$$

(2) *general harmonic-like monotone, if*

$$\langle \mathcal{T}(\frac{2w\mu}{w+\mu}) - \mathcal{T}(\frac{2w\nu}{w+\nu}), g(\mu) - g(\nu)\rangle \geq \alpha\|\mu - \nu\|^2, \quad \forall w, \mu, \nu \in \mathcal{H}.$$

(3) *harmonic-like Lipschitz continuous, if there exist a constant $\beta > 0$, such that*

$$\|\mathcal{T}(\frac{2w\mu}{w+\mu}) - \mathcal{T}(\frac{2w\nu}{w+\nu})\nu\| \leq \beta\|\mu - \nu\|, \quad \forall w, \mu, \nu \in \mathcal{H}.$$

Remark 2.3. Every strongly harmonic-like monotone operator is a harmonic-like monotone operator. Fur $w = \mu$, $g = I$, the definition 2.3 reduces to the classical strongly general monotonicity and Lipchitz continuity of the operator.

For $w = \mu$, Definition 2.3 reduces to the strongly mononivity, Lipschitz continuity and psedu-monotonicity of the operator.

3. **Main Results**

In this section, we use the auxiliary principle technique, the origin of which can be traced back to Lions and Stampacchia [12] and Glowinski et. al. [8]as developed in [3, 17, 24, 25, 27, 28, 30, 31, 33] to discuss the existence of the solution and consider some approximate schemes for the harmonic-like variational inequalities. The main of idea of this technique is to consider an arbitrary auxiliary problem related to the original problem. This way, one defines a mapping connecting the both problems. To prove the existence of a solution of the original problem, it is enough to show that this connecting mapping is a contraction which yields the solution of the original problem. Another novel feature of this approach is that this technique enables us to suggest some iterative methods for solving the harmonic-like variational inequalities. It has been shown that the projection, resolvent and descent methods can be derived for the auxilialry principle technique.

We now consider the auxiliary principle approach to discuss the existence of the solution of the problem (2).

Theorem 3.1. *Let the operators $\mathfrak{T}$ and g be Lipschitz continuous with constants $\beta > 0$ and $\sigma > 0$, respectively. If there exists a constant $\rho > 0$, such that*

$$\rho \leq \frac{1-\sigma}{\beta}, \tag{9}$$

then there exists a unique solution of the problem (2).

Proof. We now use the auxiliary principle technique to prove the existence of a solution of (2). To be more precise, for a given $\mu \in \Omega$ satisfying (2), consider the problem of finding $\eta \in \Omega$ such that

$$\langle \rho \mathfrak{T}(\frac{2w\mu}{w+\mu}), g(\nu) - g(\eta)\rangle + \langle g(\eta) - g(\mu), \rangle g(\nu) - g(\eta)\rangle \geq 0, \quad \forall w, \nu \in \Omega, \tag{10}$$

which is called the auxiliary problem, where $\rho > 0$ is a constant. It is clear that (10) defines a mapping η connecting the both problems (2) and (10). To prove the existence of a solution of (2), it is enough to show that the mapping η defined by (10) is a contraction mapping.

Let $\eta_1 \neq \eta_2$ (corresponding to $\mu_1 \neq \mu_2 \in \Omega$) be solutions of (10). Then

$$\begin{aligned} &\langle \rho \mathfrak{T}(\frac{2w\mu_1}{w+\mu_1}), g(\nu) - g(\eta_1)\rangle \\ &\quad + \langle g(\eta_1) - g(\mu_1), \rangle g(\nu) - g(\eta_1)\rangle \geq 0, \quad \forall w, \nu \in \Omega, \end{aligned} \tag{11}$$

$$\begin{aligned} &\langle \rho \mathfrak{T}(\frac{2w\mu_2}{w+\mu_2}), g(\nu) - g(\eta_2)\rangle \\ &\quad + \langle g(\eta_2) - g(\mu_2), \rangle g(\nu) - g(\eta_2)\rangle \geq 0, \quad \forall w, \nu \in \Omega, \end{aligned} \tag{12}$$

Taking $\nu = \eta_2$ in (11) and $\nu = \eta_1$ in (12) and adding the resultants, we have

$$\begin{aligned} & \|g(\eta_1) - g(\eta_2\|^2 = \langle g(\eta_1) - g(\eta_2), g(\eta_1) - g(\eta_2)\rangle \\ = \ & \langle g(\mu_1) - g(\mu_2, g(\eta_1) - g(\eta_2)\rangle + \rho\langle[\mathfrak{T}(\frac{2w\mu_2}{w+\mu_2}) - \mathfrak{T}(\frac{2w\mu_1}{w+\mu_1})], g(\eta_1) - g(\eta_2)\rangle \\ \leq \ & \|g(\mu_1) - g(\mu_2\|\|g(\eta_1) - g(\eta_2)\| + \rho\|\mathfrak{T}(\frac{2w\mu_2}{w+\mu_2}) - \mathfrak{T}(\frac{2w\mu_1}{w+\mu_1})\|\|g(\eta_1) - g(\eta_2)\| \\ \leq \ & \sigma\|\mu_1 - \mu_2\|g(\eta_1) - g(\eta_2)\| + \rho\|\mathfrak{T}(\frac{2w\mu_2}{w+\mu_2}) - \mathfrak{T}(\frac{2w\mu_1}{w+\mu_1})\|\|g(\eta_1) - g(\eta_2)\| \\ \leq \ & \sigma\|\mu_1 - \mu_2\|\|g(\eta_1) - g(\eta_2)\| + \rho\beta\|\mu_1 - \mu_2\|\|g(\eta_1) - g(\eta_2)\|, \end{aligned} \tag{13}$$

since the operators g and $\mathfrak{T}$ are Lipschitz continuous with constaants σ, β, respectively. From (13), we have

$$\|g(\eta_1) - g(\eta_2\| \leq (\sigma + \rho\beta)\|\mu_1 - \mu_2\| = \theta\|\mu_1 - \mu_2\|, \tag{14}$$

where $\theta = (\sigma + \rho\beta)$.

From (9), It follows that $\theta < 1$ for $0 < \rho < \frac{1-\sigma}{\beta}$. Since g^{-1} exists, so it has a fixed point $\eta(\mu) = \mu \in \Omega$ satisfying the problem (2). □

It is worth mentioning that $\eta = \mu$ is a solution of (2) This implies that the auxiliary principle technique enables us to suggest the following iterative method for solving the problem (2).

Algorithm 3.1. *For a given initial value μ_o,, compute the approximate solution μ_{n+1} by the iterative scheme*

$$\begin{aligned} & \langle \rho\mathfrak{T}(\frac{2w\mu_n}{w+\mu_n}), g(\mu_{n+1}) - g(\mu_n)\rangle \\ & \quad + \langle g(\mu_{n+1}) - g(\mu_n), g(\nu) - g(\mu_{n+1})\rangle \geq 0, \quad \forall w, \nu \in \Omega, \end{aligned}$$

We again use the auxiliary principle technique to suggest an implicit method for solving the problem (2). For a given $\mu \in \Omega$ satisfying (2), consider the problem of finding $\eta \in \Omega$ such that

$$\langle \rho\mathfrak{T}(\frac{2w\eta}{w+\eta}), g(\nu) - g(\eta)\rangle + \langle g(\eta) - g(\mu), \rangle g(\nu) - g(\eta)\rangle \geq 0, \quad \forall w, \nu \in \Omega, \tag{15}$$

which is called the auxiliary problem, where $\rho > 0$ is a constant. We note that the auxiliary problems (10) and (15) are quite different. Clearly $\eta = \mu \in \Omega$ is a solution of (2). This observation allows us to suggest the following iterative method for solving the problem (2).

Algorithm 3.2. *For a given initial value μ_o,, compute the approximate solution μ_{n+1} by the iterative scheme*

$$\begin{aligned} & \langle \rho\mathfrak{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}), g(\mu_{n+1}) - g(\mu_n)\rangle \\ & \quad + \langle g(\mu_{n+1}) - g(\mu_n), g(\nu) - g(\mu_{n+1})\rangle \geq 0, \quad \forall w, \nu \in \Omega, \end{aligned} \tag{16}$$

This is an implicit method. Using the predictor-corrector technique, we obtain the two-step method for solving the problem (2).

Algorithm 3.3. *For a given initial value μ_0, compute the approximate solution μ_{n+1} by the iterative schemes*

$$\begin{aligned} & \langle \rho\mathfrak{T}(\frac{2w\mu_n}{w+\mu_n}) + g(y_n) - g(\mu_n), g(\nu) - g(y_n)\rangle \geq 0, \quad \forall w, \nu \in \Omega \\ & \langle \rho\mathfrak{T}(\frac{2wy_n}{w+y_n}) + g(y\mu_{n+1}) - g(\mu_n), g(\nu) - g(\mu_{n+1})\rangle \geq 0, \quad \forall w, \nu \in \Omega, \end{aligned}$$

which is known the two step iterative method for solving the problem (2).

For the convergence criteria, we need the following concept.

Definition 3.1. *An operator* $\mathcal{T}$ *is siad to be pseudo general harmonic-like with respect to the operator* g, *if*

$$\langle \mathcal{T}(\frac{2w\mu}{w+\mu}) + g(\mu) - g(\nu), g(\nu) - g(\mu)\rangle \geq 0, \quad \forall w, \nu \in \Omega$$

$$\Longrightarrow$$

$$\langle \mathcal{T}(\frac{2w\nu}{w+\nu}) + g(\nu) - g(\mu), g(\mu) - g(\nu)\rangle \geq 0, \quad \forall w, \nu \in \Omega.$$

We now consider the convergence analysis of Algorithm 3.3 and this is the main motivation of mext result.

Theorem 3.2. *Let* $\mu \in \Omega$ *be the solution of* (2) *and* μ_{n+1} *be the approximate solution obtained from Algorithm* 3.3. *If the operator* $\mathcal{T}$ *is psuedo harmic-like operator with respect to* g, *then*

$$\|g(\mu_{n+1} - g(\mu)\|^2 \leq \|g(\mu) - g(\mu_n)\|^2 - \|g(u_{n+1}) - g(\mu_n)\|^2. \tag{17}$$

Proof. Let $\mu \in \Omega$ be a solution of (2). Then

$$\langle \mathcal{T}(\frac{2w\nu}{w+\nu}), g(\mu) - g(\nu)\rangle \geq 0, \quad \forall w, \nu \in \Omega, \tag{18}$$

since $\mathcal{T}$ is psuedo harmonic-like operator.

Taking $v = \mu_{n+1}$ in (18) and $v = \mu$ in (16), we have

$$\langle \mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}), g(\mu) - g(\mu_{n+1})\rangle \geq 0, \quad \forall w \in \Omega, \tag{19}$$

and

$$\langle \mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}) + g(\mu_{n+1}) - g(\mu), g(\mu) - g(\mu_{n+1})\rangle \geq 0, \quad \forall w \in \Omega. \tag{20}$$

From (19) and (20), we obtain

$$\langle g(\mu_{n+1}) - g(\mu), g(\mu) - g(\mu_{n+1})\rangle \geq 0.$$

From which, using

$$a, b \in \mathcal{H}, \quad 2\langle a, b\rangle = \|a\|^2 + \|b\|^2 - \|a - b\|^2,$$

we obtain

$$\|g(\mu_{n+1} - g(\mu)\|^2 \leq \|g(\mu) - g(\mu_n)\|^2 - \|g(u_{n+1}) - g(\mu_n)\|^2,$$

the required result. □

Theorem 3.3. *Let* $\mu \in \Omega$ *be the solution of* (2) *and* μ_{n+1} *be the approximate solution obtained from Algorithm* 3.3. *If all the assumptioms of Theorem* 3.2 *hold and* g^{-1} *exists, then*

$$\lim_{n\to\infty} \mu_n = \mu.$$

Proof. Let $\mu \in \Omega$ be a solution of (2). From the above arguments, it follows that the sequence $\{\|\mu - \mu_n\|\}$ is nonincreasing and consequently the sequence $\{\mu_n\}$ is bounded. Also, we have

$$\sum_{n\to\infty}^{\infty} \|\mu_{n+1} - \mu_n\|^2 \leq \|\mu - \mu_n\|^2$$

which implies that

$$\lim_{n\to\infty} \|\mu_{n+1} - \mu_n\| = 0,$$

since g^{-1} exists.

Let $\hat{\mu}$ be the cluster point of the sequence $\{\mu_n\}$ and the subsequence of $\{\mu_{n_j}\}$ of the sequence $\{\mu_n\}$ converges to $\hat{\mu}$. Replacing $\{\mu_n\}$ by $\{\mu_{n_j}\}$ in (15), we have

$$\langle \mathfrak{T}(\frac{2w\hat{\mu}}{w+\hat{\mu}}), g(\nu) - g(\hat{\mu}) \geq 0, \quad \forall w, \nu \in \Omega.$$

which shows that $\hat{\mu} \in \Omega$ is a solution of (2). □

We can apply the modified auxiliary principle technique involving an arbitrary operator M, which is mainly due to Noor [24], to suggest some hybrid itertaive methods for solving the general harmonic-like variational inequalities (2).

For given $\mu \in \Omega$ satisfying the problem (2), find $\eta \in \Omega$ such that

$$\begin{aligned} &\langle \rho\mathfrak{T}(\frac{2w\eta}{w+\eta}), g(\nu) - g(\eta)\rangle \\ &+\langle M(\eta) - M(\mu) + \xi(g(\mu) - g(\mu)), g(\nu) - g(\eta)\rangle \geq 0, \quad \forall w, \nu \in \Omega, \end{aligned}$$

where $\rho > 0$, ξ are constants.

For $M = I$, the identity operator and $\xi = 0$, the auxiliary problem is exactly the auxiliary problem [38]. For suitable choice of the operator $M$and the parameters, one can obtain some new auxiliary problems associated with problem (2) It is obvious that $\eta = \mu \in \Omega$ is a solution of the problem (2). This observation allows us to suggest the some new inertial iterative method for solving the problem (2).

Algorithm 3.4. *For given μ_0, μ_1, compute the approximate solution μ_{n+1} by the iterative schemes*

$$\begin{aligned} &\langle \rho\mathfrak{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}), g(\nu) - g(\mu_{n+1})\rangle \\ &\quad +\langle M(\mu_{n+1}) - M(\mu_n) + \xi(g(\mu_n) - g(\mu_{n-1})), g(\nu) - g(\mu_{n+1})\rangle \\ &\quad \geq 0, \quad \forall w, \nu \in \Omega, \end{aligned}$$

Algorithm 3.4 is called the hybrid inertial implicit iterative method, which contains Algorithm 3.3 and the following inertial iterative method for $M = 0$.

Algorithm 3.5. *For given μ_0, μ_1, compute the approximate solution μ_{n+1} by the iterative schemes*

$$\begin{aligned} &\langle \rho\mathfrak{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}), g(\nu) - g(\mu_{n+1})\rangle \\ &\quad +\langle \xi(g(\mu_n) - g(\mu_{n-1})), g(\nu) - g(\mu_{n+1})\rangle \geq 0, \quad \forall w, \nu \in \Omega. \end{aligned}$$

Algorithm 3.5 appears to be a new one. Using the predictor-corrector technique and updating the technique [23], one can propose and suggest some three step iterative schemes for solving the general harmonic-like variational inequalities and their variant forms. Using the technique of Noor and Noor [27], one can study the convergence criteria of Algorithm 3.5. For the applications and convergence analysis of the inertial type methods, see [3, 19, 27, 33] and the references therein.

Conclusion

In this paper, we have introduced and analyzed some new classes of general harmonic-like variational inequalities. The auxiliary principle approach is applied to discuss the existence and approximate solutions of the general harmonic-like variational inequalities. Convergence analysis is considered under some weaker conditions, which do not require the strongly monotonicity and Lischitz continuity of the involved operator. These results continued to hold for special cases such as complementarity problems, Lax-Milgram lemma and representation theorems. Our methods of investigation are very simple compared with other methods. It is interesting open problem to compare these techniques with other methods.

Contributions of the authors:

All the authors contributed equally in writing, editing, reviewing and agreed for the final version for publication.

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