CONSTRUCTION OF MULTITIME RAYLEIGH SOLITONS

Laura Gabriela MATEI[1], Constantin UDRIŞTE[2]

In the present paper, some multitime Rayleigh evolution PDEs are built, as extensions of the single-time corresponding PDEs, using elements of differential geometry. Introducing some additional assumptions, multitime soliton solutions for the new PDEs are generated. To obtain families of multitime Rayleigh solitons, special partial differential equations are integrated. The original results refer to the technique used to create multitime versions for Rayleigh type classical PDEs and to the construction of specific multitime solitons, as well as directions to approach the stability of these solitons.

Keywords: Rayleigh PDEs, multitime Rayleigh PDEs, multitime Rayleigh solitons, stability analysis.
MSC2000: 74J35, 74J30, 78A40.

1. Classical single-time Rayleigh PDEs and solitons

In the physical-mathematical literature [5], [6], [7], we find the *Rayleigh wave equation*

$$u_{tt} - u_{xx} = \epsilon(u_t - u_t^3) \quad (1)$$

related to *Rayleigh wave equation of Van der Pol type*

$$u_{tt} - u_{xx} = \epsilon(1 - u^2)u_t. \quad (2)$$

Each of these has been used to model physical phenomena. Now, the PDE (1) serves as a model for the large amplitude vibrations of wind-blown, ice-laden power transmission lines, in time that, the PDE (2) describes plane electromagnetic waves propagating between two parallel planes in a region where the conductivity varies quadratically with the electric field.

Just as their counterparts from ordinary differential equations, the PDEs (1) and (2) can be transformed one to another. Their solutions can be obtained by simple operations performed on the solution of a certain first order, nonlinear wave equation.

An initial-boundary value problem for Rayleigh nonlinear wave equation can be considered to be a simple model to describe the galloping oscillations of overhead power transmission lines in a wind field. One end of the transmission line is assumed to be fixed, whereas the other end of the line is assumed to be attached to a dashpot system.

[1]Assistant, Department of Mathematics and Informatics, Faculty of Applied Sciences, University "POLITEHNICA" of Bucharest, Romania, E-mail: mateiglaura@yahoo.com

[2]Professor, Department of Mathematics and Informatics, Faculty of Applied Sciences, University "POLITEHNICA" of Bucharest, Romania, E-mail: udriste@mathem.pub.ro

A *single time Rayleigh soliton* is a self-reinforcing wave solution $u(x,t) = \phi(x-\lambda t)$ of the PDE (1) or (2), that maintains its shape while it travels at constant speed λ.

Some papers (see [13], [7], [11], [12], [14]) describe as Rayleigh wave a type of seismic surface wave that moves with a rolling motion that consists of a combination of particle motion perpendicular and parallel to the main direction of wave propagation. The amplitude of this motion decreases with depth. Like primary waves, Rayleigh waves are alternatively compressional and extensional (they cause changes in the volume of the rocks they pass through).

The Rayleigh waves are of particular importance in seismology, acoustic, geophysics and electronics applications ([5]-[7], [10]-[14]).

Section 2 analyzes the geometric objects (fundamental tensor, linear connection, vector fields, tensor fields) capable of transforming single-time Rayleigh PDEs into multitime PDEs, showing the existence of an infinity of geometrical structures such that the multitime Rayleigh PDEs are prolongations of single-time Rayleigh PDEs. These ingredients permit to define an original ultra-parabolic-hyperbolic differential operators defining the multitime Rayleigh wave equations. Sections 3 underlines the technique which produces multitime Rayleigh solitons. Sections 4 and 5 praise explicit formulas for the multitime Rayleigh solitons. Section 6 comments the stability of multitime Rayleigh solitons.

Remark The *Ricci solitons* ([1]-[4], or the *Riemann solitons* ([22]-[23]) and *type wave solitons* ([6]) have in common only that they are special solutions of evolutionary PDEs in differential geometry, respectively in physics.

2. Multitime extensions of Rayleigh PDEs via geometrical elements

Generally, the passing from systems of PDEs with a single-time variable t to related PDE systems with $m \geq 2$ evolution variables $t = (t^\alpha)$, $\alpha = 1, ..., m$, is substantially complicated due to necessity of praising some reasons and some techniques of such change. The most natural way of changing is to use geometrical ingredients (derivation, trace etc) that extend the initial PDE system. The theory and a systematic procedure for the construction of such new PDE systems is presented here in the context of Rayleigh nonlinear waves.

This paper provides new results regarding the multitime solitons in two and more temporal dimensions that can be of interest in physics. We overpass the complexity and, furthermore, the difficulty of performing hand computations for Rayleigh PDE systems involving many temporal variables by using the symbolic software package in MAPLE. Our sources of inspiration for introducing and studying multitime soliton PDEs are the papers [8], [9], [20]. Also the papers [15]-[19] and [21] contains a lot of ideas in this direction, including the multitime optimal control.

Let us introduce and study some *multitime geometrical prolongations of the Rayleigh PDEs*, using related connection, fundamental tensor field, vector fields, tensor fields which leave on the jet bundle as ingredients in the Differential Geometry of the manifold $\mathbb{R}^m$.

Suppose the *multitime* $t = (t^1, ..., t^m) \in \mathbb{R}^m$ is a parameter of evolution. We endow the manifold (jet bundle of order one) $J^1(\mathbb{R} \times \mathbb{R}^m, \mathbb{R} \times \mathbb{R}^m)$ with a *distinguished symmetric linear connection* $\Gamma^{\gamma}_{\alpha\beta} = \Gamma^{\gamma}_{\alpha\beta}\left(x, t, u, \frac{\partial u}{\partial t}\right)$, and with a *distinguished fundamental symmetric contravariant tensor field* $h = \left(h^{\alpha\beta}(x, t, u, \frac{\partial u}{\partial t})\right)$ of constant signature (r, z, s), $r + z + s = m$. Using a C^2 function $u : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$, we build the *Hessian operator*

$$(Hess_{\Gamma} u)_{\alpha\beta} = \frac{\partial^2 u}{\partial t^{\alpha} \partial t^{\beta}} - \Gamma^{\gamma}_{\alpha\beta} \frac{\partial u}{\partial t^{\gamma}}, \quad \alpha, \beta, \gamma \in \{1, ..., m\}$$

and its trace, called *ultra-parabolic-hyperbolic operator*,

$$\Box_{\Gamma,h} u = h^{\alpha\beta} (Hess_{\Gamma} u)_{\alpha\beta}.$$

We define a *multitime PDE* as

$$\Box_{\Gamma,h} u - \frac{\partial^2 u}{\partial x^2} = 0, \tag{3}$$

where $x \in R$ and $t = (t^1, ..., t^m) \in R^m$.

- Let

$$C^{\gamma}(x, t, \eta, \xi),\ \gamma = 1, ..., m$$

be a *distinguished vector field* and

$$B^{\alpha\beta\gamma}(x, t, \eta, \xi),\ \alpha, \beta, \gamma \in \{1, ..., m\}$$

be a *distinguished tensor field.* If we adopt the hypothesis

$$h^{\alpha\beta}(x, t, \eta, \xi) \Gamma^{\gamma}_{\alpha\beta}(x, t, \eta, \xi) \xi_{\gamma} = C^{\gamma}(x, t, \eta, \xi) \xi_{\gamma} - B^{\alpha\beta\gamma}(x, t, \eta, \xi) \xi_{\alpha} \xi_{\beta} \xi_{\gamma}, \tag{4}$$

then we obtain the *multitime Rayleigh PDE*

$$h^{\alpha\beta} \frac{\partial^2 u}{\partial t^{\alpha} \partial t^{\beta}} - C^{\gamma} \frac{\partial u}{\partial t^{\gamma}} + B^{\alpha\beta\gamma} \frac{\partial u}{\partial t^{\alpha}} \frac{\partial u}{\partial t^{\beta}} \frac{\partial u}{\partial t^{\gamma}} - \frac{\partial^2 u}{\partial x^2} = 0. \tag{5}$$

- If $C^{\gamma}(x, t, \eta, \xi)$, $\gamma = 1, ..., m$ and $D^{\gamma}(x, t, \eta, \xi)$, $\gamma = 1, ..., m$ are *distinguished vector fields* and the constraint relation is

$$h^{\alpha\beta}(x, t, \eta, \xi) \Gamma^{\gamma}_{\alpha\beta}(x, t, \eta, \xi) \xi_{\gamma} = C^{\gamma}(x, t, \eta, \xi) \xi_{\gamma} - D^{\gamma}(x, t, \eta, \xi) \eta^2 \xi_{\gamma}, \tag{6}$$

then we get a *multitime Rayleigh wave equation of Van der Pol type*

$$h^{\alpha\beta} \frac{\partial^2 u}{\partial t^{\alpha} \partial t^{\beta}} - C^{\gamma} \frac{\partial u}{\partial t^{\gamma}} + u^2 D^{\gamma} \frac{\partial u}{\partial t^{\gamma}} - \frac{\partial^2 u}{\partial x^2} = 0. \tag{7}$$

The PDE (5) has two important properties: (i) It is *multitime-reversible* if and only if $h^{\alpha\beta}(x, t) = h^{\alpha\beta}(x, -t)$, $C^{\alpha}(x, -t) = -C^{\alpha}(x, t)$, $B^{\alpha\beta\gamma}(x, -t) = -B^{\alpha\beta\gamma}(x, t)$. In this case, the functions $u(x, t)$ and $u(x, -t)$ are solutions of this PDE; (ii) It has a stationary solution $u(x)$ if and only if $u(x)$ is solution of the equation $u''(x) = 0$. In other words, the stationary solution is $u(x) = Ax + B$. Geometrically, its graph $(x, t, u(x))$ is a hyperplane in $\mathbb{R}^{1+m+1}$.

The PDE (7) has similar properties: (i) It is *multitime-reversible* if and only if $h^{\alpha\beta}(x, t) = h^{\alpha\beta}(x, -t)$, $C^{\alpha}(x, -t) = -C^{\alpha}(x, t)$, $D^{\alpha}(x, -t) = -D^{\alpha}(x, t)$. In this case, the functions $u(x, t)$ and $u(x, -t)$ are solutions of the PDE (7); (ii) It has a stationary solution $u(x) = Ax + B$, whose graph $(x, t, u(x))$ is a hyperplane in $\mathbb{R}^{1+m+1}$.

Theorem 2.1. *(i) There exists an infinity of geometrical structures* $\Gamma^{\gamma}_{\alpha\beta}$, $h^{\alpha\beta}$, C^{γ}, $B^{\alpha\beta\gamma}$ *on* $\mathbb{R}^m$ *such that a solution of the Rayleigh PDE (1) is also a solution of the multitime Rayleigh PDE (5).*

(ii) There exists an infinity of geometrical structures $\Gamma^{\gamma}_{\alpha\beta}$, $h^{\alpha\beta}$, C^{γ}, D^{γ} *on* $\mathbb{R}^m$ *such that a solution of the Rayleigh PDE (2) is also a solution of the multitime Rayleigh PDE of Van der Pol type (7).*

Proof Let $t^1 = t$ and $u = u(x, t^1)$.

(i) Suppose $u = u(x, t^1)$ is a solution of single-time Rayleigh PDE (1). The function $v(x, t^1, ..., t^m) = u(x, t^1)$ is a solution of the multitime Rayleigh PDE (5) if the family of geometrical structures $\Gamma^{\gamma}_{\alpha\beta}$, $h^{\alpha\beta}$, C^{γ}, $B^{\alpha\beta\gamma}$ is fixed by

$$h^{\alpha\beta}\Gamma^{1}_{\alpha\beta}\xi_1 = C^1\xi_1 - B^{111}\xi_1\xi_1\xi_1.$$

It is obvious that we have an infinity of geometrical structures that satisfy this algebraic equation.

(ii) Similar.

The foregoing Theorem justifies the term *multitime geometrical prolongations of the Rayleigh PDEs.*

Conversely, if we want to obtain a solution of a single-time Rayleigh PDE from a solution of the multitime Rayleigh PDE, we can use: (1) a suitable curve $\tau \to \phi(\tau)$, $t^{\alpha} = \phi^{\alpha}(\tau)$, $\alpha = 1, ..., m$, which imposes some conditions on the coefficients; particularly, we can look for a solution of type $u(x, (\tau, ..., \tau))$; (2) solutions $u(x, t)$ depending only one variable t^{α}, $\alpha =$ fixed; for example $u = u(x, t^1)$.

3. Construction of multitime Rayleigh solitons

The first aim of this Section is to find some multitime solitons solutions for the multitime Rayleigh PDE. In spite of the mathematical beauty, the distance between theoretical multitime models and real situations where they apply is still too high. It is not so far from our understanding, but it is still hidden for unsuspecting researchers.

Let $\phi : I \subset \mathbb{R} \to \mathbb{R}$ be a function of class C^4. We seek for solutions of the PDEs (5) and (7) in the form of *multitime solitons*

$$u(x, t) = \phi(x - \lambda_{\alpha}t^{\alpha}) = \phi(z),$$

where (λ_{α}), $\alpha = 1, ..., m$, is a constant vector and $z = x - \lambda_{\alpha}t^{\alpha}$. Then, the partial derivatives of the unknown function $u(x, t)$ are

$$\frac{\partial^2 u}{\partial x^2} = \phi''(z),\ \frac{\partial u}{\partial t^{\alpha}} = \phi'(z)(-\lambda_{\alpha}),\ \frac{\partial^2 u}{\partial t^{\alpha}\partial t^{\beta}} = \phi''(z)\lambda_{\alpha}\lambda_{\beta}.$$

Substituting these derivatives in the PDEs (5) and (7), we obtain the second order ODEs,

$$[h^{\alpha\beta}\lambda_{\alpha}\lambda_{\beta} - 1]\,\phi''(z) - B^{\alpha\beta\gamma}\lambda_{\alpha}\lambda_{\beta}\lambda_{\gamma}\,\phi'(z)^3 + C^{\gamma}\lambda_{\gamma}\,\phi'(z) = 0 \quad (8)$$

and respectively

$$[h^{\alpha\beta}\lambda_{\alpha}\lambda_{\beta} - 1]\,\phi''(z) - D^{\gamma}\lambda_{\gamma}\,\phi^2(z)\phi'(z) + C^{\gamma}\lambda_{\gamma}\,\phi'(z) = 0. \quad (9)$$

Summarizing, we have

Theorem 3.1. *If $\phi(z)$ is a solution of second order ODE (8) or (9) , then $u(x,t) = \phi(x - \lambda_\alpha t^\alpha)$ is a multitime soliton solution of the multitime Rayleigh PDEs (5) or (7) respectively.*

In order to find some multitime Rayleigh solitons, we make some particular choices of the elements that appear in the construction of the two multitime PDEs.

First choise: For the ODE (8), we relate the metric tensor $h^{\alpha\beta}(x,t,\eta,\xi)$, the tensor field $B^{\alpha\beta\gamma}(x,t,\eta,\xi)$, the vector field $C^\gamma(x,t,\eta,\xi)$ and the constant vector λ_α by the conditions

$$h^{\alpha\beta}(x,t,\eta,\xi)\lambda_\alpha\lambda_\beta - 1 = a(x - \lambda_\alpha t^\alpha) = a(z) \neq 0,$$

$$B^{\alpha\beta\gamma}(x,t,\eta,\xi)\lambda_\alpha\lambda_\beta\lambda_\gamma = b(x - \lambda_\alpha t^\alpha) = b(z),\ C^\gamma(x,t,\eta,\xi)\lambda_\gamma = c(x - \lambda_\alpha t^\alpha) = c(z).$$

With these conditions, the ODE (8) becomes

$$a(z)\phi''(z) - b(z)(\phi')^3(z) + c(z)\phi'(z) = 0. \tag{10}$$

Second choise: For the ODE (9), we relate the metric tensor $h^{\alpha\beta}(x,t,\eta,\xi)$, the vector fields $C^\gamma(x,t,\eta,\xi)$ and $D^\gamma(x,t,\eta,\xi)$ and the constant vector λ_α by the conditions

$$h^{\alpha\beta}(x,t,\eta,\xi)\lambda_\alpha\lambda_\beta - 1 = a(x - \lambda_\alpha t^\alpha) = a(z) \neq 0,$$

$$D^\gamma(x,t,\eta,\xi)\lambda_\gamma = b(x - \lambda_\alpha t^\alpha) = d(z),\ C^\gamma(x,t,\eta,\xi)\lambda_\gamma = c(x - \lambda_\alpha t^\alpha) = c(z).$$

With this choice, the ODE (9) transforms in

$$a(z)\phi''(z) - d(z)\phi^2(z)\phi'(z) + c(z)\phi'(z) = 0. \tag{11}$$

We are looking for some solutions of the ODE (10) and (11), with a view to finding multitime Rayleigh solitons.

4. Multitime Rayleigh soliton families

We start with the multitime Rayleigh solitons based on the ODE (10).

4.1. Case of coefficients depending on z

Denoting $\phi' \overset{\text{not}}{=} \psi$, the second order ODE (10) becomes a first order ODE,

$$a(z)\,\psi'(z) - b(z)\,\psi^3(z) + c(z)\,\psi(z) = 0,$$

called *Bernoulli ODE.* The general form of this ODE is called *Abel ODE* of the *first kind*, and it arose in the context of the studies of Niels Henrik Abel on the theory of *elliptic functions*, and represents a natural generalization of the Riccati equation.

Since $a(z) \neq 0$, we write the differential equation in the form

$$\psi' = -\frac{c(z)}{a(z)}\,\psi + \frac{b(z)}{a(z)}\,\psi^3.$$

By a change of the unknown function, $\xi = \psi^{-2}$, the Bernoulli ODE becomes a linear ODE,

$$\xi' - 2\,\frac{c(z)}{a(z)}\,\xi = -2\,\frac{b(z)}{a(z)},$$

with the solutions

$$\xi(z) = \exp\left(2\int \frac{c(z)}{a(z)}dz\right)\left[K - 2\int \frac{b(z)}{a(z)}\,\exp\left(-2\int \frac{c(z)}{a(z)}dz\right)dz\right],\ K \in \mathbb{R}.$$

Since $\psi = \xi^{-\frac{1}{2}}$, we obtain

$$\phi'(z) = \frac{\exp\left(-\int \frac{c(z)}{a(z)}dz\right)}{\sqrt{K - 2\int \frac{b(z)}{a(z)} \exp\left(-2\int \frac{c(z)}{a(z)}dz\right) dz}}, \ K \in \mathbb{R}.$$

Therefore, we have found solutions of the multitime PDE Rayleigh (5).

Theorem 4.1. *If we fix the above coefficients by the conditions*

$$h^{\alpha\beta}(x,t,\eta,\xi)\lambda_\alpha\lambda_\beta - 1 = a(x - \lambda_\alpha t^\alpha) \neq 0,$$

$$B^{\alpha\beta\gamma}(x,t,\eta,\xi)\lambda_\alpha\lambda_\beta\lambda_\gamma = b(x - \lambda_\alpha t^\alpha), C^\gamma(x,t,\eta,\xi)\lambda_\gamma = c(x - \lambda_\alpha t^\alpha),$$

then the function

$$u(x,t) = \phi(x - \lambda_\alpha t^\alpha)$$

represents a multitime soliton-solution for the multitime PDE Rayleigh (3) for every ϕ given by

$$\phi(z) = \int \frac{\exp\left(-\int \frac{c(z)}{a(z)}\, dz\right)}{\sqrt{K - 2\int \frac{b(z)}{a(z)} \exp\left(-2\int \frac{c(z)}{a(z)}\, dz\right) dz}}\, dz, \ K \in \mathbb{R}.$$

4.2. Case of constant coefficients

Let $\lambda = (\lambda_\alpha), \ \alpha \in \{1, ..., m\}$ be a parameter. If we fix the elements h, B, C by the constants, and using the notations $h^{\alpha\beta}\lambda_\alpha\lambda_\beta - 1 = a \neq 0, \ B^{\alpha\beta\gamma}\lambda_\alpha\lambda_\beta\lambda_\gamma = b, \ C^\gamma\lambda_\gamma = c$, then the ODE (10) takes the form

$$a\phi''(z) - b(\phi')^3(z) + c\phi'(z) = 0.$$

Denoting $\phi' \overset{\text{not}}{=} \psi$, this second order ODE with constant coefficients becomes a first order *Bernoulli ODE*,

$$a\,\psi' - b\,\psi^3 + c\,\psi = 0.$$

Having separable variables, this ODE can be written

$$a\,\frac{d\psi}{b\psi^3 - c\psi} = dz \ \Leftrightarrow \ \frac{a}{c}\left(-\int \frac{d\psi}{\psi} + \int \frac{\psi d\psi}{\psi^2 - \frac{c}{b}}\right) = \int dz$$

and we get

$$\psi(z) = \pm\sqrt{\frac{c}{b\left(1 - p\exp\left(\frac{2c}{a}z\right)\right)}}, \ \ p \in \mathbb{R}^*$$

that is

$$\phi(z) = \pm\int\sqrt{\frac{c}{b\left(1 - p\exp\left(\frac{2c}{a}z\right)\right)}}\, dz, \ \ p \in \mathbb{R}^*,$$

with $\dfrac{c}{b\left(1 - p\,\exp\left(\frac{2c}{a}z\right)\right)} \geq 0$. To calculate the primitive from the right hand side, we amplify the fraction by $\exp\left(\dfrac{2c}{a}z\right)$ and then make the change of variables $\exp\left(\dfrac{c}{a}z\right) = t$. We obtain the integral

$$I = \pm\frac{a}{c}\int\frac{1}{t}\sqrt{\frac{c}{b(1-pt^2)}}\,dt$$

and a new change of variables, $\dfrac{1}{t} = s$, gives a new integral,

$$J = \pm\frac{a}{c}\int\sqrt{\frac{c}{b(s^2-p)}}\,ds.$$

There are two separate possibilities:

a) If we take $\dfrac{c}{b} > 0$, then

$$J = \pm\frac{a}{c}\sqrt{\frac{c}{b}}\int\frac{1}{\sqrt{s^2-p}}\,ds.$$

- For $p > 0$, the integral is

$$J = \pm\frac{a}{c}\sqrt{\frac{c}{b}}\,\mathrm{ch}^{-1}\left|\frac{s}{\sqrt{p}}\right| + r,\ \ r \in \mathbb{R},$$

that is

$$\phi(z) = \pm\frac{a}{c}\sqrt{\frac{c}{b}}\,\mathrm{ch}^{-1}\left|\frac{1}{\sqrt{p}\,\exp\left(\frac{c}{a}z\right)}\right| + r,\ \ r \in \mathbb{R},\ \ p \in \mathbb{R}_+^*$$

and we can write

$$\phi(z) = \pm\frac{a}{c}\sqrt{\frac{c}{b}}\,\mathrm{ch}^{-1}\left(K\,\exp\left(-\frac{c}{a}z\right)\right) + r,\ \ r \in \mathbb{R},\ \ K \in \mathbb{R}_+^*.$$

- For $p < 0$, we have

$$J = \pm\frac{a}{c}\sqrt{\frac{c}{b}}\int\sqrt{\frac{1}{s^2+p'}}\,ds,\ \ p' = -p,\ \ p' > 0.$$

It follows

$$J = \pm\frac{a}{c}\sqrt{\frac{c}{b}}\,\mathrm{sh}^{-1}\left|\frac{s}{\sqrt{p'}}\right| + r,\ \ r \in \mathbb{R},$$

that is

$$\phi(z) = \pm\frac{a}{c}\sqrt{\frac{c}{b}}\,\mathrm{sh}^{-1}\left|\frac{1}{\sqrt{p'}\,\exp\left(\frac{c}{a}z\right)}\right| + r,\ \ r \in \mathbb{R},\ \ p' \in \mathbb{R}_+^*$$

and then

$$\phi(z) = \pm\frac{a}{c}\sqrt{\frac{c}{b}}\,\mathrm{sh}^{-1}\left(K\exp\left(-\frac{c}{a}z\right)\right) + r,\ \ r \in \mathbb{R},\ \ K \in \mathbb{R}_+^*.$$

b) If we suppose $\frac{c}{b} < 0$, then, via $\frac{c}{b\left(1 - p\,\exp\left(\frac{2c}{a}z\right)\right)} \geq 0$, it follows $p > 0$ and the integral J becomes

$$J = \pm\frac{a}{c}\sqrt{\frac{-c}{b}}\,\arcsin\left(\frac{s}{\sqrt{p}}\right) + r, \quad r \in \mathbb{R}, \quad p \in \mathbb{R}_+^*.$$

It follows

$$\phi(z) = \pm\frac{a}{c}\sqrt{\frac{-c}{b}}\,\arcsin\left(\frac{1}{\sqrt{p}\,\exp\left(\frac{c}{a}z\right)}\right) + r, \quad r \in \mathbb{R}, \quad p \in \mathbb{R}_+^*,$$

that is

$$\phi(z) = \pm\frac{a}{c}\sqrt{\frac{-c}{b}}\,\arcsin\Big(K\exp\Big(-\frac{c}{a}z\Big)\Big) + r, \quad r \in \mathbb{R}, \quad K \in \mathbb{R}_+^*.$$

Therefore, we have found three families of solutions of the multitime Rayleigh PDE (5).

Theorem 4.2. *a) If we fix the foregoing coefficients by the condition* $\frac{c}{b} > 0$*, then we get two families of soliton solutions*

$$u(x,t) = \pm\frac{a}{c}\sqrt{\frac{c}{b}}\,ch^{-1}\Big(K\exp\Big(-\frac{c}{a}(x - \lambda_\alpha t^\alpha)\Big)\Big) + r,$$

$$u(x,t) = \pm\frac{a}{c}\sqrt{\frac{c}{b}}\,sh^{-1}\Big(K\exp\Big(-\frac{c}{a}(x - \lambda_\alpha t^\alpha)\Big)\Big) + r,$$

where $r \in \mathbb{R}, \quad K \in \mathbb{R}_+^*$*;*

b) if we take $\frac{c}{b} < 0$*, then we have another family of soliton solutions,*

$$u(x,t) = \pm\frac{a}{c}\sqrt{-\frac{c}{b}}\,\arcsin\Big(K\exp\Big(-\frac{c}{a}(x - \lambda_\alpha t^\alpha)\Big)\Big) + r, \quad r \in \mathbb{R}, \quad K \in \mathbb{R}_+^*.$$

4.3. Maclaurin series soliton of multitime Rayleigh PDE

In a previous section, we have obtained a second order ODE in ϕ, namely,

$$a(z)\phi''(z) - b(z)(\phi')^3(z) + c(z)\phi'(z) = 0.$$

One assume that it has a solution which is analytic on an interval around $z = 0$ and we search a Maclaurin series solution. Then we express ϕ as a power series in the form

$$\phi(z) = \sum_{n=0}^{\infty} \alpha_n\, z^n \tag{12}$$

and we try to determine what the α_n's need to be. The resulting power series need to converge on an interval around origin. We compute $\phi'(z)$, $\phi'^3(z)$ and $\phi''(z)$:

$$\phi'(z) = \sum_{n=1}^{\infty} n\alpha_n\, z^{n-1} = \sum_{n=0}^{\infty}(n+1)\alpha_{n+1}\, z^n = \sum_{n=0}^{\infty}\beta_n\, z^n,$$

$$(\phi')^2(z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\beta_k\beta_{n-k}\right) z^n$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \alpha_{k+1}\, \alpha_{n-k+1} (k+1)(n-k+1) \right) z^n = \sum_{n=0}^{\infty} \gamma_n\, z^n,$$

$$(\phi')^3(z) = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} \gamma_i\, \beta_{n-i} \right) z^n$$

$$= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} \sum_{k=0}^{i} \alpha_{k+1}\, \alpha_{i-k+1}\, \alpha_{n-i+1} (k+1)(i-k+1)(n-i+1) \right) z^n,$$

$$\phi''(z) = \sum_{n=1}^{\infty} n(n+1)\alpha_{n+1}\, z^{n-1} = \sum_{n=0}^{\infty} (n+1)(n+2)\alpha_{n+2}\, z^n.$$

Consider the particular case

$$a(z) = mz + a,\ \ b(z) = pz + b,\ \ c(z) = qz + c,\ \ \ m, p, q, a, b, c \in \mathbb{R},$$

that is the coefficients $a(z), b(z), c(z)$ of the ODE (10) are affine functions in z. Consequently, the foregoing ODE gives the identity

$$(mz + a) \sum_{n=0}^{\infty} (n+2)(n+1)\alpha_{n+2}\, z^n + (qz + c) \sum_{n=0}^{\infty} \alpha_{n+1}(n+1)\, z^n - (pz + b)$$

$$\times \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} \sum_{k=0}^{i} \alpha_{k+1}\, \alpha_{i-k+1}\, \alpha_{n-i+1} (k+1)(i-k+1)(n-i+1) \right) z^n = 0.$$

This identity can be written

$$m \sum_{n=1}^{\infty} n(n+1)\alpha_{n+1}\, z^n + 2a\alpha_2 + a \sum_{n=1}^{\infty} (n+2)(n+1)\alpha_{n+2}\, z^n$$

$$-p \sum_{n=1}^{\infty} \left(\sum_{i=0}^{n-1} \sum_{k=0}^{i} \alpha_{k+1}\, \alpha_{i-k+1}\, \alpha_{n-i} (k+1)(i-k+1)(n-i) \right) z^n - b\alpha_1^3$$

$$-b \sum_{n=1}^{\infty} \left(\sum_{i=0}^{n} \sum_{k=0}^{i} \alpha_{k+1}\, \alpha_{i-k+1}\, \alpha_{n-i+1} (k+1)(i-k+1)(n-i+1) \right) z^n$$

$$+q \sum_{n=1}^{\infty} n\alpha_n\, z^n + c\alpha_1 + c \sum_{n=1}^{\infty} \alpha_{n+1}(n+1)\, z^n = 0,$$

or, equivalent,

$$(2a\alpha_2 + c\alpha_1 - b\alpha_1^3) + \sum_{n=1}^{\infty} [mn(n+1)\alpha_{n+1} + a(n+2)(n+1)\alpha_{n+2}$$

$$-p \sum_{i=0}^{n-1} \sum_{k=0}^{i} \alpha_{k+1}\, \alpha_{i-k+1}\, \alpha_{n-i} (k+1)(i-k+1)(n-i)$$

$$-b \sum_{i=0}^{n} \sum_{k=0}^{i} \alpha_{k+1}\, \alpha_{i-k+1}\, \alpha_{n-i+1} (k+1)(i-k+1)(n-i+1)$$

$$+qn\alpha_n + c(n+1)\alpha_{n+1}] z^n = 0.$$

By identifying the coefficients of the powers of z with 0, we find the condition

$$2a\alpha_2 + c\alpha_1 - b\alpha_1^3 = 0$$

and the recurrence

$$mn(n+1)\alpha_{n+1} + a(n+2)(n+1)\alpha_{n+2}$$
$$-p\sum_{i=0}^{n-1}\sum_{k=0}^{i}\alpha_{k+1}\,\alpha_{i-k+1}\,\alpha_{n-i}(k+1)(i-k+1)(n-i)$$
$$-b\sum_{i=0}^{n}\sum_{k=0}^{i}\alpha_{k+1}\,\alpha_{i-k+1}\,\alpha_{n-i+1}(k+1)(i-k+1)(n-i+1)$$
$$+qn\alpha_n + c(n+1)\alpha_{n+1} = 0, \quad n \geq 1. \tag{13}$$

By the initial conditions $\phi(0) = \alpha_0$, $\phi'(0) = \alpha_1$ and $2a\alpha_2 + c\alpha_1 - b\alpha_1^3 = 0$, this recurrence gives us all the coefficients of the power series (12), but the difficult part is just solving the recurrence for the unknown α_n.

Theorem 4.3. *The multitime series soliton solution of the multitime Rayleigh PDE is*

$$u(x,t) = \sum_{n=0}^{\infty}\alpha_n(x - \lambda_\alpha t^\alpha)^n,$$

with α_0, α_1 *fixed,* $2a\alpha_2 + c\alpha_1 - b\alpha_1^3 = 0$ *and* α_n, $n \geq 2$ *given by the recurrence (13).*

5. Families of multitime Rayleigh solitons of Van der Pol type

Now we continue with the multitime Rayleigh solitons of Van der Pol type based on the ODE (11).

5.1. Case of coefficients depending on z

As we have seen in Section 3, if we can choose the constant vector λ_α by some conditions which relate the elements h, C, D, λ, then we get the equation

$$a(z)\phi''(z) - d(z)\phi^2(z)\phi'(z) + c(z)\phi'(z) = 0.$$

We can write this equation in the form

$$\phi^2(z)\phi'(z) = \frac{a(z)}{d(z)}\phi''(z) + \frac{c(z)}{d(z)}\phi'(z).$$

In order to find some solutions of this equation, we make a particular choice for λ_α, by a new condition: the coefficients $a(z) \overset{\text{not}}{=} h^{\alpha\beta}(x,t,\eta,\xi)\lambda_\alpha\lambda_\beta - 1$, $d(z) \overset{\text{not}}{=} D^\gamma(x,t,\eta,\xi)\lambda_\gamma$ and $c(z) \overset{\text{not}}{=} C^\gamma(x,t,\eta,\xi)\lambda_\gamma$ will be related by the equality

$$\left(\frac{a(z)}{d(z)}\right)' = \frac{c(z)}{d(z)},$$

or equivalent

$$a'(z)d(z) - a(z)d'(z) = d(z)c(z).$$

With such a selection of λ_α, the equation becomes

$$\phi^2(z)\phi'(z) = \left(\frac{a(z)}{d(z)}\phi'(z)\right)',$$

that is

$$\frac{\phi^3(z)}{3} = \frac{a(z)}{d(z)}\phi'(z) + k, \ \ k \in \mathbb{R}.$$

This Bernoulli ODE has separable variables. By integration, we get

$$\int \frac{3d\phi}{\phi^3 - 3k} = \int \frac{d(z)}{a(z)}\, dz, \ \ k \in \mathbb{R}. \qquad (14)$$

For simplicity, we can take $k = 0$ and we find a particular family of solutions, defined by the relation

$$\phi^2(z) = -\frac{2}{3}\int \frac{d(z)}{a(z)}\, dz.$$

If we keep k variable and non-zero, then making the substitution $3k$ by k_1^3, the equality (14) becomes

$$\int \frac{3d\phi}{\phi^3 - k_1^3} = \int \frac{d(z)}{a(z)}\, dz, \ \ k_1 \in \mathbb{R}^*.$$

The general solution of the equation (14) is expressed implicitly by the equality

$$\frac{1}{k_1^2}\ln\frac{|\phi - k_1|}{\sqrt{\phi^2 + \phi k_1 + k_1^2}} - \frac{\sqrt{3}}{k_1^2}\mathrm{arctg}\left(\frac{2\phi + k_1}{k_1\sqrt{3}}\right) = \int \frac{d(z)}{a(z)}\, dz, \ \ k_1 \in \mathbb{R}^*.$$

Summarizing, we can formulate the next result:

Theorem 5.1. *If we can take the constant vector λ_α so as $h^{\alpha\beta}(x,t,\eta,\xi)$, $B^\gamma(x,t,\eta,\xi)$, $C^\gamma(x,t,\eta,\xi)$ and λ_α, $\alpha,\beta,\gamma = 1,...,m$, to be related by the conditions*

$$h^{\alpha\beta}(x,t,\eta,\xi)\lambda_\alpha\lambda_\beta - 1 = a(x - \lambda_\alpha t^\alpha) \neq 0, \ B^\gamma(x,t,\eta,\xi)\lambda_\gamma = b(x - \lambda_\alpha t^\alpha),$$

$$C^\gamma(x,t,\eta,\xi)\lambda_\gamma = c(x - \lambda_\alpha t^\alpha), \ \ a'(z)d(z) - a(z)b'(z) = d(z)c(z).$$

then the function $u(x,t) = \phi(x - \lambda_\alpha t^\alpha)$ represents a multitime soliton-solution for the multitime PDE Van der Pol (7), for every ϕ defined implicitly by one of the equalities

$$\phi^2(z) = -\frac{2}{3}\int \frac{d(z)}{a(z)}\, dz$$

or

$$\frac{1}{k_1^2}\ln\frac{|\phi(z) - k_1|}{\sqrt{\phi^2(z) + \phi(z)k_1 + k_1^2}} - \frac{\sqrt{3}}{k_1^2}\mathit{arctg}\left(\frac{2\phi(z) + k_1}{k_1\sqrt{3}}\right) = \int \frac{d(z)}{a(z)}\, dz, \ k_1 \in \mathbb{R}^*.$$

5.**2. Case of constant coefficients**

Let $\lambda = (\lambda_\alpha)$, $\alpha \in \{1,...,m\}$ be a parameter. If we fix the elements h, C, D by constants, and we denote

$$h^{\alpha\beta}\lambda_\alpha\lambda_\beta - 1 = a \neq 0, \ D^\gamma\lambda_\gamma = d, \ C^\gamma\lambda_\gamma = c, \qquad (15)$$

then the ODE (11) takes the form

$$a\,\phi''(z) - d\,\phi^2(z)\phi'(z) + c\,\phi'(z) = 0.$$

Integrating, we find a first order ODE, with separable variables

$$a\,\phi' - \frac{d}{3}\,\phi^3 + c\phi = k, \ \ k \in \mathbb{R}.$$

In order to find some solutions of this equation, we take $k = 0$. By this choice, the equation becomes a *Bernoulli* ODE. We make a change of the unknown function, $\psi = \phi^{-2}$, and the Bernoulli ODE becomes a linear ODE $\psi' - \frac{2c}{a}\psi = -\frac{2d}{3a}$, with the solutions

$$\psi(z) = \exp\left(\int \frac{2c}{a} dz\right) \left[K - \int \frac{2d}{3a} \exp\left(-\int \frac{2c}{a} dz\right) dz\right], \quad K \in \mathbb{R},$$

that is

$$\psi(z) = K \exp\left(\frac{2c}{a} z\right) + \frac{d}{3c}, \quad K \in \mathbb{R}.$$

Since $\phi = \psi^{-\frac{1}{2}}$, we obtain

$$\phi(z) = \frac{1}{\sqrt{K \exp\left(\frac{2c}{a} z\right) + \frac{d}{3c}}}, \quad K \in \mathbb{R}.$$

Thus we obtain a family of multitime soliton-solutions of the multitime Van der Pol PDE (7) and we can formulate the next theorem:

Theorem 5.2. *If we take λ_α so as to fix h, D, C by the notations (15), then we get a family of multitime soliton-solutions*

$$u(x,t) = \frac{1}{\sqrt{K \exp\left(\frac{2c}{a}(x - \lambda_\alpha t^\alpha)\right) + \frac{d}{3c}}}, \quad K \in \mathbb{R}.$$

of the multitime Rayleigh wave equation of Van der Pol type (7).

6. Stability of multitime Rayleigh solitons

The multitime Rayleigh PDEs are evolution equations. To show what happen in *future multitime* t, we endow the set $\mathbb{R}^m_+$ with the product order. Also, suppose $\lambda_\alpha > 0$, for each index α. The constant vector $\lambda = (\lambda_\alpha)$ controls the speed, amplitude, and width of a multitime soliton.

The multitime Rayleigh PDEs have important properties of stability, among which there is the following: (i) if we specify the initial position $u(x, 0) = u_0(x)$, $u_t(x, 0) = u_1(x)$ of a multitime soliton $u(x, t)$ at multitime $t = 0$, the equation has a unique solution with that initial data for $t > 0$; (ii) if we impose suitable conditions for the coefficients in a multitime soliton $u(x, t)$, then $\lim_{||t||\to\infty} u(x, t) = M$; this underline the idea that a soliton rests in bounded region of the space; (iii) let $u(x, t)$ be a multitime soliton with the initial condition $u(x, 0) = u_0(x), u_t(x, 0) = u_1(x)$ and $u(x) = Ax + B$ be a stationary solution of the Rayleigh PDE of Van der Pol type; $\forall\delta > 0$, $\exists\varepsilon > 0$ such that

$$||u_0(x) - u(x)||^2_{L^2(x)} \leq \varepsilon, ||u_1(x)||^2_{L^2(x)} \leq \varepsilon \Longrightarrow \sup_{||t||} ||u(x,t) - u(x)||^2_{L^2(x)} \leq \delta,$$

under reasonable conditions; the restrictions on the initial conditions require a small speed λ (hint: from the first relation we determine the parameters A, B by the least squares method).

Acknowledgements Many thanks to Professor P. Malischewsky for reporting the difference between Rayleigh waves described by PDEs and seismic Rayleigh surface waves. We wish also to thank certain referees for their useful comments. Partially supported by University Politehnica of Bucharest and by Academy of Romanian Scientists.

REFERENCES

[1] *G. Bercu, C. Corcodel, M. Postolache*, Iterative gemetric structures, Int. J. Geom. Methods Mod. Physics, **7** (2010), No. 7, 1103-1114.

[2] *G. Bercu, M. Postolache*, Classes of gradient Ricci solitons, Int. J. Geom. Methods Mod. Physics, **8** (2011), No. 4, 783-796.

[3] *G. Bercu, M. Postolache*, Classification of steady Ricci solitons on two-manifolds, Int. J. Geom. Methods Mod. Physics, **9** (2012), No. 5, 1250049.

[4] *G. Bercu, M. Postolache*, Classes of gradient Ricci solitons on generalized Poincare manifolds, Int. J. Geom. Methods Mod. Physics, **9** (2012), No. 4, 1250027.

[5] *Darmawijoyo, W. T. van Horssen, P. Clement*, On a Rayleigh wave equation with boundary damping, Nonlinear Dynamics, **33** (2003), No. 4, 399-429.

[6] *W. S. Hall*, The Rayleigh and Van der Pol wave equations, some generalizations, Lecture Notes in Mathematics, Springer-Verlag, Berlin, **703** (1979), 103-138.

[7] *P. G. Malischewsky*, Comment to "A new formula for the velocity of Rayleigh waves" by D. Nkemzi, Wave Motion, **26** (1997), 199-205; Wave Motion, **31** (2000), 93-96.

[8] *L. Matei, C. Udrişte*, Multitime sine-Gordon solitons via geometric characteristics, Balkan J. Geom. Appl., **16** (2011), No. 2, 81-89.

[9] *L. Matei, C. Udrişte, C. Ghiu*, Multitime Boussinesque solitons, Int. J. Geom. Methods M., **9** (2012), No. 4, 19 p, ID 1250031.

[10] *H. Mechkour*, The exact expressions for the roots of Rayleigh wave equation, BSG Proceedings, Geometry Balkan Press, Bucharest, **8** (2003), 96 - 104.

[11] *D. Nkemzi*, A new formula for the velocity of Rayleigh waves, Wave Motion, **26** (1997), 199-205.

[12] *M. Rahman, J. R. Barber*, Exact expressions for the roots of the secular equation for Rayleigh waves, ASME, J. Appl. Mech., **62** (1995), 250-252.

[13] *J. W. S. Rayleigh*, On waves propagating along the plane surface of an elastic solid, Proc. London Math. Soc., **17** (1887), 4-11.

[14] *D. Royer*, A study of the secular equation for Rayleigh waves using the root locus method, Ultrasonics, **39** (2001), 223-225.

[15] *C. Udrişte*, Equivalence of multitime optimal control problems, Balkan J. Geom. Appl., **15** (2010), No. 1, 155-162.

[16] *C. Udrişte*, Minimal submanifolds and harmonic maps through multitime maximum principle, Balkan J. Geom. Appl., **18** (2013), No. 2, 69-82.

[17] *C. Udrişte, V. Damian, L. Matei, I. Ţevy*, Multitime differentiable stochastic processes, diffusion PDEs, Tzitzeica hypersurfaces, U.P.B. Scientific Bulletin, Series A, **74** (2012), No. 1, 3-10.

[18] *C. Udrişte, S. Dinu, I. Ţevy*, Multitime optimal control for linear PDEs with curvilinear cost functional, Balkan J. Geom. Appl., **18** (2013), No. 1, 87-100.

[19] *C. Udrişte, L. Matei, I. Duca*, Multitime Hamilton-Jacobi theory, Proceedings of 8-th WSEAS International Conference on Applied Computational Science (ACACOS-09), Hangzhou, China, May 20-22, 2009, 509-513.

[20] *C. Udrişte, L. Petrescu, L. Matei*, Multitime reaction-diffusion solitons, Balkan J. Geom. Appl., **17** (2012), No. 2, 115-128.

[21] *C. Udrişte, I. Ţevy*, Multitime dynamic programming for curvilinear integral actions, J. Optim. Theory Appl., **146** (2010), 189-207.

[22] *C. Udrişte*, Riemann flow and Riemann wave, An. Univ. Vest, Timişoara, Ser. Mat.-Inf., 48, 1-2 (2010), 265-274.

[23] *C. Udrişte*, Riemann flow and Riemann wave via bialternate product Riemannian metric, arXiv:1112.4279v1 [math.AP] 19 Dec 2011.