SOME APPROXIMATION RESULTS FOR BERNSTEIN-KANTOROVICH OPERATORS BASED ON (p,q)-CALCULUS

M. Mursaleen[1], Khursheed J. Ansari[2], Asif Khan[3]

In this paper, a new analogue of Bernstein-Kantorovich operators as (p,q)-Bernstein-Kantorovich operators are introduced. We discuss approximation properties for these operators based on Korovkin's type approximation theorem and we compute the order of convergence using usual modulus of continuity and also the rate of convergence when the function f belongs to the class $Lip_M(\alpha)$. Moreover, the local approximation property of the sequence of positive linear operators $K_n^{(p,q)}$ has been studied. We show comparisons and some illustrative graphics for the convergence of operators to a function. In comparison to q-analogue of Bernstein-Kantorovich operators, our generalization gives more flexibility for the convergence of operators to a function.

1. Introduction and preliminaries

During the last two decades, the applications of q-calculus emerged as a new area in the field of approximation theory. The rapid development of q-calculus has led to the discovery of various generalizations of Bernstein polynomials involving q-integers. The aim of these generalizations is to provide appropriate and powerful tools to application areas such as numerical analysis, computer-aided geometric design and solutions of differential equations.

Using q-integers, Lupaş [15] introduced the first q-Bernstein operators [4] and investigated its approximating and shape-preserving properties. Another q-analogue of the Bernstein polynomials is due to Phillips [25]. Since then several generalizations of well-known positive linear operators based on q-integers have been introduced and studied their approximation properties. For instance, q-Bleimann, Butzer and Hahn operators [3]; q-parametric Szász-Mirakjan operators [16]; q-Bernstein-Durrmeyer operators [17]; q-analogue of Szász-Kantorovich operators [18].

---

[1]Professor, Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India, e-mail: `mursaleenm@gmail.com`

[2] Assistant Professor, Department of Mathematics, College of Science, King Khalid University, 61413, Abha, Saudi Arabia, e-mail:`ansari.jkhursheed@gmail.com`

[3] Assistant Professor, Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India, e-mail:`asifjnu07@gmail.com`

Recently, Mursaleen *et al* introduced (p, q)-calculus in approximation theory and constructed the (p, q)-analogue of Bernstein operators [21] and (p, q)-analogue of Bernstein-Stancu operators [22], (p, q)-analogue of Bleimann-Butzer-Hahm operators [23], Bernstein-Schurer operarors [24] and investigated their approximation properties. The (p, q)-analog of Szász-Mirakyan operators [1], Kantorovich type Bernstein-Stancu-Schurer operators [5] and Kantorovich variant of (p, q)-Szász-Mirakjan operators [19] have recently been studied too.

Motivated by above mentioned work, in this paper, authors introduce a new analogue of Bernstein-Kantorovich operators. Paper is organized as follows: In Section 2, we define (p, q)-Bernstein-Kantorovich operators and establish a basic lemma which is used in proving main results. In Section 3, we discuss approximation properties for these operators based on Korovkin's type approximation theorem and we compute the order of convergence using usual modulus of continuity and also the rate of convergence when the function f belongs to the class $\text{Lip}_M(\alpha)$. Moreover, we also study the local approximation property of the sequence of positive linear operators $K_n^{(p,q)}$. In Section 4, we give some examples to show comparisons and some illustrative graphics for the convergence of operators to a function.

Let us recall certain definitions and notations of (p, q)-calculus:

The (p, q)-integer was introduced in order to generalize or unify several forms of q-oscillator algebras well known in the earlier physics literature related to the representation theory of single parameter quantum algebras [6].

For any $p > 0$ and $q > 0$, the (p, q) integers $[n]_{p,q}$ are defined by

$$[n]_{p,q} = p^{n-1} + p^{n-2}q + p^{n-3}q^2 + \ldots + pq^{n-2} + q^{n-1} = \begin{cases} \frac{p^n - q^n}{p - q}, & \text{when } p \neq q \neq 1 \\[2mm] n\, p^{n-1}, & \text{when } p = q \neq 1 \\[2mm] [n]_q, & \text{when } p = 1 \\ n, & \text{when } p = q = 1 \end{cases}$$

where $[n]_q$ denotes the q-integers and $n = 0, 1, 2, \cdots$.

The (p, q)-Binomial expansion is

$$(x + y)_{p,q}^n := (x + y)(px + qy)(p^2x + q^2y) \cdots (p^{n-1}x + q^{n-1}y)$$

and the (p, q)-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} := \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}.$$

In [21], we studied Korovkin's type approximation properties for (p, q)-Bernstein Operators to approximate any continuous functions under the condition $0 < q < p \leq 1$.

The (p,q)-Bernstein Operators are as follow:

$$B_{n,p,q}(f;x) = \frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}} x^k \prod_{s=0}^{n-k-1} (p^s - q^s x) f\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\right), \; x \in [0,1] \tag{1}$$

Also, we have

$$(1-x)_{p,q}^n = \prod_{s=0}^{n-1}(p^s - q^s x) = (1-x)(p-qx)(p^2 - q^2 x)...(p^{n-1} - q^{n-1}x)$$

$$= \sum_{k=0}^{n}(-1)^k p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^k$$

Note when $p = 1$, (p,q)-Bernstein Operators given by (1) turns out to be q-Bernstein Operators.

The definite integrals of the function f are defined by

$$\int_0^a f(x)d_{p,q}x = (q-p)a \sum_{k=0}^{\infty} \frac{p^k}{q^{k+1}} f\left(\frac{p^k}{q^{k+1}}a\right), \text{ when } \left|\frac{p}{q}\right| < 1,$$

and

$$\int_0^a f(x)d_{p,q}x = (p-q)a \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} f\left(\frac{q^k}{p^{k+1}}a\right), \text{ when } \left|\frac{p}{q}\right| > 1.$$

Details on (p,q)-calculus can be found in [9, 13, 14, 26, 27].
For $p = 1$, all the notions of (p,q)-calculus are reduced to q-calculus [10].

2. Construction of Operators

Dalmanoglu [7] defined the Bernstein-Kantorovich [11] operators using q-calculus as follows:

$$K_{n,q}(f;x) = [n+1]_q \sum_{k=0}^{n} p_{n,k}(q;x) \int_{[k]_q/[n+1]_q}^{[k+1]_q/[n+1]_q} f(t)d_q t, \; x \in [0,1], \tag{2}$$

$$p_{n,k}(q;x) := \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-k-1}(1-q^s x).$$

where $K_{n,q} : C[0,1] \to C[0,1]$ are defined for any $n \in \mathbb{N}$ and for any function $f \in C[0,1]$.

Now, we introduce (p,q)-analogue of Bernstein-Kantorovich operators as

$$K_n^{(p,q)}(f;x) = \frac{[n]_{p,q}}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^{n} \frac{b_{n,k}^{(p,q)}(x)}{p^{n-k}q^k} \int_{\frac{[k]_{p,q}}{p^{k-n-1}[n]_{p,q}}}^{\frac{[k+1]_{p,q}}{p^{k-n}[n]_{p,q}}} f(t)d_{p,q}t, \; x \in [0,1] \tag{3}$$

where

$$b_{n,k}^{(p,q)}(x) = \left[\begin{array}{c} n \\ k \end{array} \right]_{p,q} (x)_{p,q}^k (1-x)_{p,q}^{n-k}$$

$$= \left[\begin{array}{c} n \\ k \end{array} \right]_{p,q} p^{\frac{k(k-1)}{2}} x^k (1-x)_{p,q}^{n-k}$$

$$= \left[\begin{array}{c} n \\ k \end{array} \right]_{p,q} p^{\frac{k(k-1)}{2}} x^k \prod_{s=0}^{n-k-1} (p^s - q^s x)$$

and $(x)_{p,q}^k := x(px)(p^2 x) \cdots (p^{k-1} x) = p^{\frac{k(k-1)}{2}} x^k$.

For $p = 1$, operators (3) turns out to be the classical q-Bernstein-Kantorovich operators (2).

First, we prove the following basic lemmas:

Lemma 2.1. For $x \in [0,1]$, $0 < q < p \leq 1$

(i) $K_n^{(p,q)}(1; x) = 1$,

(ii) $K_n^{(p,q)}(t; x) = x + \frac{p^n}{[2]_{p,q}[n]_{p,q}}$,

(iii) $K_n^{(p,q)}(t^2; x) = \frac{q}{p} \frac{[n-1]_{p,q}}{[n]_{p,q}} x^2 + \left(\frac{p^n(2q+p)}{[3]_{p,q}[n]_{p,q}} + \frac{p^{n-1}}{[n]_{p,q}} \right) x + \frac{p^{2n}}{[3]_{p,q}[n]_{p,q}^2}$,

(iv) $K_n^{(p,q)}\left((t-x)^2; x\right) = \left(\frac{q}{p} \frac{[n-1]_{p,q}}{[n]_{p,q}} - 1 \right) x^2 + \left(\frac{p^n(2q+p)}{[3]_{p,q}[n]_{p,q}} + \frac{p^{n-1}}{[n]_{p,q}} - \frac{2p^n}{[2]_{p,q}[n]_{p,q}} \right) x + \frac{p^{2n}}{[3]_{p,q}[n]_{p,q}^2}$.

Proof. (i)

$$K_n^{(p,q)}(1; x) = \frac{[n]_{p,q}}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^{n} \frac{b_{n,k}^{(p,q)}(x)}{p^{n-k} q^k} \int_{\frac{[k]_{p,q}}{p^{k-n-1} [n]_{p,q}}}^{\frac{[k+1]_{p,q}}{p^{k-n} [n]_{p,q}}} d_{p,q} t = 1.$$

(ii)

$$K_n^{(p,q)}(t; x) = \frac{[n]_{p,q}}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^{n} \frac{b_{n,k}^{(p,q)}(x)}{p^{n-k} q^k} \int_{\frac{[k]_{p,q}}{p^{k-n-1} [n]_{p,q}}}^{\frac{[k+1]_{p,q}}{p^{k-n} [n]_{p,q}}} t \, d_{p,q} t$$

$$= \frac{1}{[2]_{p,q}[n]_{p,q}} \frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^{n} \frac{b_{n,k}^{(p,q)}(x)}{p^{n-k} q^k} \left(\frac{[k+1]_{p,q}^2 - p^2 [k]_{p,q}^2}{p^{2k-2n}} \right).$$

Using $[k+1]_{p,q} = q^k + p[k]_{p,q}$, we have

$$
\begin{aligned}
K_n^{(p,q)}(t;x) &= \frac{1}{[2]_{p,q}[n]_{p,q}} \frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^n \frac{b_{n,k}^{(p,q)}(x)(p^k + [2]_{p,q}[k]_{p,q})}{p^{n-k}q^k} \\
&= \frac{1}{[2]_{p,q}[n]_{p,q}} \frac{p^n}{p^{\frac{n(n-1)}{2}}} \left(\sum_{k=0}^n b_{n,k}^{(p,q)}(x) + [2]_{p,q} \sum_{k=0}^n b_{n,k}^{(p,q)}(x) \frac{[k]_{p,q}}{p^k} \right) \\
&= \frac{1}{[2]_{p,q}[n]_{p,q}} \frac{p^n}{p^{\frac{n(n-1)}{2}}} \left(p^{\frac{n(n-1)}{2}} + [2]_{p,q} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} (x)_{p,q}^k (1-x)_{p,q}^{n-k} \frac{[k]_{p,q}}{p^k} \right) \\
&= \frac{p^n}{[2]_{p,q}[n]_{p,q}} + \frac{1}{p^{\frac{n(n-3)}{2}}} \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} (x)_{p,q}^{k+1} (1-x)_{p,q}^{n-k-1} \frac{1}{p^{k+1}} \\
&= \frac{p^n}{[2]_{p,q}[n]_{p,q}} + \frac{1}{p^{\frac{(n-1)(n-2)}{2}}} \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} (p^k x)(x)_{p,q}^k (1-x)_{p,q}^{n-k-1} \frac{1}{p^k} \\
&= \frac{p^n}{[2]_{p,q}[n]_{p,q}} + \frac{x}{p^{\frac{(n-1)(n-2)}{2}}} \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} (x)_{p,q}^k (1-x)_{p,q}^{n-k-1} \\
&= \frac{p^n}{[2]_{p,q}[n]_{p,q}} + x.
\end{aligned}
$$

(iii)

$$
\begin{aligned}
K_n^{(p,q)}(t^2;x) &= \frac{[n]_{p,q}}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^n \frac{b_{n,k}^{(p,q)}(x)}{p^{n-k}q^k} \int_{\frac{[k]_{p,q}}{p^{k-n-1}[n]_{p,q}}}^{\frac{[k+1]_{p,q}}{p^{k-n}[n]_{p,q}}} t^2 \, d_{p,q}t \\
&= \frac{1}{[3]_{p,q}[n]_{p,q}^2} \frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^n \frac{b_{n,k}^{(p,q)}(x)}{p^{n-k}q^k} \left(\frac{[k+1]_{p,q}^3 - p^3[k]_{p,q}^3}{p^{3k-3n}} \right) \\
&= \frac{p^{2n}}{[3]_{p,q}[n]_{p,q}^2} \frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^n b_{n,k}^{(p,q)}(x) \left(1 + (2q+p)\frac{[k]_{p,q}}{p^k} + \frac{[3]_{p,q}[k]_{p,q}^2}{p^{2k}} \right) \\
&= \frac{p^{2n}}{[3]_{p,q}[n]_{p,q}^2} \left(1 + \frac{(2q+p)}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^n b_{n,k}^{(p,q)}(x)\frac{[k]_{p,q}}{p^k} + \frac{[3]_{p,q}}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^n b_{n,k}^{(p,q)}(x)\frac{[k]_{p,q}^2}{p^{2k}} \right).
\end{aligned}
$$

With the help of the previous calculations, we have

$$
\frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^n b_{n,k}^{(p,q)}(x)\frac{[k]_{p,q}}{p^k} = \frac{[n]_{p,q}}{p^n}x.
$$

And

$$
\begin{aligned}
\frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^n b_{n,k}^{(p,q)}(x)\frac{[k]_{p,q}^2}{p^{2k}} &= \frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} (x)_{p,q}^k (1-x)_{p,q}^{n-k}\frac{[k]_{p,q}^2}{p^{2k}} \\
&= \frac{[n]_{p,q}}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} (x)_{p,q}^{k+1} (1-x)_{p,q}^{n-k-1}\frac{[k+1]_{p,q}}{p^{2k+2}}.
\end{aligned}
$$

Using $[k+1]_{p,q} = p^k + q[k]_{p,q}$, we have

$$\frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^{n} b_{n,k}^{(p,q)}(x) \frac{[k]_{p,q}^2}{p^{2k}}$$

$$= \frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} (x)_{p,q}^{k+1}(1-x)_{p,q}^{n-k-1}\left(\frac{1}{p^{k+2}} + \frac{q[k]_{p,q}}{p^{2k+2}}\right)$$

$$= \frac{1}{p^{\frac{n(n-1)}{2}}} \left\{ \frac{p^{\frac{(n-1)(n-2)}{2}}x}{p^2} + q[n-1]_{p,q} \sum_{k=0}^{n-2} \begin{bmatrix} n-2 \\ k \end{bmatrix}_{p,q} (x)_{p,q}^{k+2}(1-x)_{p,q}^{n-k-2}\frac{1}{p^{2k+4}} \right\}$$

$$= \frac{1}{p^{\frac{n(n-1)}{2}}} \left\{ \frac{p^{\frac{(n-1)(n-2)}{2}}x}{p^2} + q[n-1]_{p,q} \sum_{k=0}^{n-2} \begin{bmatrix} n-2 \\ k \end{bmatrix}_{p,q} (p^{k+1}x)(p^k x)(x)_{p,q}^{k}(1-x)_{p,q}^{n-k-2}\frac{1}{p^{2k+4}} \right\}$$

$$= \frac{1}{p^{\frac{n(n-1)}{2}}} \left\{ \frac{p^{\frac{(n-1)(n-2)}{2}}x}{p^2} + q[n-1]_{p,q} \frac{p^{\frac{(n-2)(n-3)}{2}}x^2}{p^4} \right\}$$

$$= \frac{1}{p^{n+1}}x + \frac{q[n-1]_{p,q}}{p^{2n+1}}x^2.$$

Using the above equalities, we have

$$K_n^{(p,q)}(t^2; x) = \frac{q}{p}\frac{[n-1]_{p,q}}{[n]_{p,q}}x^2 + \left(\frac{p^n(2q+p)}{[3]_{p,q}[n]_{p,q}} + \frac{p^{n-1}}{[n]_{p,q}}\right)x + \frac{p^{2n}}{[3]_{p,q}[n]_{p,q}^2}.$$

(iv) Using the linearity of the operators $K_n^{(p,q)}$, we have

$$K_n^{(p,q)}\big((t-x)^2; x\big)$$

$$= K_n^{(p,q)}(t^2; x) - 2x K_n^{(p,q)}(t; x) + x^2 K_n^{(p,q)}(1; x)$$

$$= \frac{q}{p}\frac{[n-1]_{p,q}}{[n]_{p,q}}x^2 + \left(\frac{p^n(2q+p)}{[3]_{p,q}[n]_{p,q}} + \frac{p^{n-1}}{[n]_{p,q}}\right)x + \frac{p^{2n}}{[3]_{p,q}[n]_{p,q}^2} - 2x\left(x + \frac{p^n}{[2]_{p,q}[n]_{p,q}}\right) + x^2$$

$$= \left(\frac{q}{p}\frac{[n-1]_{p,q}}{[n]_{p,q}} - 1\right)x^2 + \left(\frac{p^n(2q+p)}{[3]_{p,q}[n]_{p,q}} + \frac{p^{n-1}}{[n]_{p,q}} - \frac{2p^n}{[2]_{p,q}[n]_{p,q}}\right)x + \frac{p^{2n}}{[3]_{p,q}[n]_{p,q}^2}.$$

3. Main Results

Let $C[a,b]$ be the linear space of all real valued continuous functions f on $[a,b]$ and let T be a linear operator which maps $C[a,b]$ into itself. We say that T is *positive* if for every non-negative $f \in C[a,b]$, we have $T(f,x) \geq 0$ for all $x \in [a,b]$.

The classical Korovkin approximation theorem [2, 12, 28] states as follows:

Let (T_n) be a sequence of positive linear operators from $C[a,b]$ into $C[a,b]$. Then $\lim_n \|T_n(f,x) - f(x)\|_{C[a,b]} = 0$, for all $f \in C[a,b]$ if and only if $\lim_n \|T_n(f_i,x) - f_i(x)\|_{C[a,b]} = 0$, for $i = 0,1,2$, where $f_0(x) = 1$, $f_1(x) = x$ and $f_2(x) = x^2$.

Remark 3.1. For $q \in (0,1)$ and $p \in (q,1]$ it is obvious that $\lim_{n\to\infty}[n]_{p,q} = 0$ or $\frac{1}{p-q}$. In order to reach to convergence results of the operator $K_n^{(p,q)}(f;x)$, we take

a sequence $q_n \in (0,1)$ and $p_n \in (q_n, 1]$ such that $\lim\limits_{n\to\infty} p_n = 1$, $\lim\limits_{n\to\infty} q_n = 1$ and $\lim\limits_{n\to\infty} p_n^n = a$, $\lim\limits_{n\to\infty} q_n^n = b$ with $0 < a, b \leq 1$. So we get $\lim\limits_{n\to\infty} [n]_{p_n,q_n} = \infty$.

Theorem 3.1. Let $0 < q_n < p_n \leq 1$ such that $\lim\limits_{n\to\infty} p_n = 1$ and $\lim\limits_{n\to\infty} q_n = 1$ satisfying Remark 3.1. Then for each $f \in C[0,1]$, $K_n^{(p_n,q_n)}(f;x)$ converges uniformly to f on $[0,1]$.

Proof. By the Korovkin Theorem it is sufficient to show that

$$\lim_{n\to\infty} \|K_n^{(p_n,q_n)}(t^m;x) - x^m\|_{C[0,1]} = 0, \ m = 0, 1, 2.$$

By Lemma 2.1 (i), it is clear that

$$\lim_{n\to\infty} \|K_n^{(p_n,q_n)}(1;x) - 1\|_{C[0,1]} = 0.$$

Now, by Lemma 2.1 (ii)

$$|K_n^{(p_n,q_n)}(t;x) - x| = \frac{p_n^n}{[2]_{p_n,q_n}[n]_{p_n,q_n}}$$

which yields

$$\lim_{n\to\infty} \|K_n^{(p_n,q_n)}(t;x) - x\|_{C[0,1]} = 0.$$

Similarly,

$$|K_n^{(p_n,q_n)}(t^2;x) - x^2|$$
$$= \left| \left(\frac{q_n}{p_n} \frac{[n-1]_{p_n,q_n}}{[n]_{p_n,q_n}} - 1 \right) x^2 + \left(\frac{p_n^n(2q_n + p_n)}{[3]_{p_n,q_n}[n]_{p_n,q_n}} + \frac{p_n^{n-1}}{[n]_{p_n,q_n}} \right) x + \frac{p_n^{2n}}{[3]_{p_n,q_n}[n]_{p_n,q_n}^2} \right|$$
$$\leq \left(\frac{q_n}{p_n} \frac{[n-1]_{p_n,q_n}}{[n]_{p_n,q_n}} - 1 \right) x^2 + \left(\frac{p_n^n(2q_n + p_n)}{[3]_{p_n,q_n}[n]_{p_n,q_n}} + \frac{p_n^{n-1}}{[n]_{p_n,q_n}} \right) x + \frac{p_n^{2n}}{[3]_{p_n,q_n}[n]_{p_n,q_n}^2}.$$

Taking maximum of both sides of the above inequality, we get

$$\|K_n^{(p_n,q_n)}(t^2;x) - x^2\| \leq \frac{q_n}{p_n} \frac{[n-1]_{p_n,q_n}}{[n]_{p_n,q_n}} - 1 + \frac{p_n^n(2q_n + p_n)}{[3]_{p_n,q_n}[n]_{p_n,q_n}} + \frac{p_n^{n-1}}{[n]_{p_n,q_n}} + \frac{p_n^{2n}}{[3]_{p_n,q_n}[n]_{p_n,q_n}^2}$$

which concludes

$$\lim_{n\to\infty} \|K_n^{(p_n,q_n)}(t^2;x) - x^2\|_{C[0,1]} = 0.$$

Thus the proof is completed.

Now we will compute the rate of convergence in terms of modulus of continuity.

Let $f \in C[0,1]$. The modulus of continuity of f denoted by $\omega(f,\delta)$ gives the maximum oscillation of f in any interval of length not exceeding $\delta > 0$ and it is given by the relation

$$\omega(f,\delta) = \sup_{|x-y|\leq\delta} |f(x) - f(y)|, \ x, y \in [0,b].$$

It is known that $\lim\limits_{\delta\to 0^+} \omega(f,\delta) = 0$ for $f \in C[0,b]$ and for any $\delta > 0$ one has

$$|f(y) - f(x)| \leq \omega(f,\delta)\left(\frac{(y-x)^2}{\delta^2} + 1 \right). \tag{3.1}$$

Theorem 3.2. If $f \in C[0,1]$, then

$$\left|K_n^{(p,q)}(f;x) - f(x)\right| \leq 2\omega\big(f, \delta_n(x)\big)$$

takes place, where $\delta_n(x) = \sqrt{K_n^{(p,q)}\big((t-x)^2\big)}$.

Proof. Since $K_n^{(p,q)}(1;x) = 1$, we have

$$
\begin{aligned}
\left|K_n^{(p,q)}(f;x) - f(x)\right| & \leq K_n^{(p,q)}\big(|f(t) - f(x)|;x\big) \\
& \leq \frac{[n]_{p,q}}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^{n} \frac{b_{n,k}^{(p,q)}(x)}{p^{n-k}q^k} \int_{\frac{[k]_{p,q}}{p^{k-n-1}[n]_{p,q}}}^{\frac{[k+1]_{p,q}}{p^{k-n}[n]_{p,q}}} |f(t) - f(x)|d_{p,q}t.
\end{aligned}
$$

In view of (3.1), we get

$$
\begin{aligned}
\left|K_n^{(p,q)}(f;x) - f(x)\right| & \leq \left\{ \frac{[n]_{p,q}}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^{n} \frac{b_{n,k}^{(p,q)}(x)}{p^{n-k}q^k} \int_{\frac{[k]_{p,q}}{p^{k-n-1}[n]_{p,q}}}^{\frac{[k+1]_{p,q}}{p^{k-n}[n]_{p,q}}} \left(\frac{|t-x|^2}{\delta^2} + 1 \right) d_{p,q}t \right\} \omega(f,\delta) \\
& = \left\{ \frac{1}{\delta^2} K_n^{(p,q)}\big((t-x)^2;x\big) + 1 \right\} \omega(f,\delta).
\end{aligned}
$$

Choosing $\delta = \delta_n(x) = \sqrt{K_n^{(p,q)}\big((t-x)^2\big)}$, we have

$$\left|K_n^{(p,q)}(f;x) - f(x)\right| \leq 2\omega\big(f, \delta_n(x)\big).$$

This completes the proof of the theorem.

Now we give the rate of convergence of the operators $K_n^{(p,q)}$ in terms of the elements of the usual Lipschitz class $\mathrm{Lip}_M(\alpha)$.
Let $f \in C[0,1]$, $M > 0$ and $0 < \alpha \leq 1$. We recall that f belongs to the class $\mathrm{Lip}_M(\alpha)$ if the inequality

$$|f(t) - f(x)| \leq M|t-x|^\alpha, \ (t,x \in [0,1])$$

is satisfied.

Theorem 3.3. Let $0 < q < p \leq 1$. Then for each $f \in \mathrm{Lip}_M(\alpha)$ we have

$$|K_n^{(p,q)}(f;x) - f(x)| \leq M\delta_n^\alpha(x),$$

where $\delta_n(x) = \sqrt{K_n^{(p,q)}\big((t-x)^2;x\big)}$.

Proof. By the monotonicity of the operators $K_n^{(p,q)}$, we can write

$$
\begin{aligned}
\left|K_n^{(p,q)}(f;x) - f(x)\right| & \leq K_n^{(p,q)}\big(|f(t) - f(x)|;x\big) \\
& \leq \frac{[n]_{p,q}}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^{n} \frac{b_{n,k}^{(p,q)}(x)}{p^{n-k}q^k} \int_{\frac{[k]_{p,q}}{p^{k-n-1}[n]_{p,q}}}^{\frac{[k+1]_{p,q}}{p^{k-n}[n]_{p,q}}} |f(t) - f(x)|d_{p,q}t \\
& \leq M \frac{[n]_{p,q}}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^{n} \frac{b_{n,k}^{(p,q)}(x)}{p^{n-k}q^k} \int_{\frac{[k]_{p,q}}{p^{k-n-1}[n]_{p,q}}}^{\frac{[k+1]_{p,q}}{p^{k-n}[n]_{p,q}}} |t-x|^\alpha \ d_{p,q}t.
\end{aligned}
$$

Now applying the Hölder's inequality for the sum with $p_1 = \frac{2}{\alpha}$ and $p_2 = \frac{2}{2-\alpha}$ and taking into consideration Lemma 2.1(i) and Lemma 2.2(ii), we have

$$
\begin{aligned}
|K_n^{(p,q)}(f;x) - f(x)| \;\leq\; & M \sum_{k=0}^{n} \left\{ \frac{[n]_{p,q}}{p^{\frac{n(n-1)}{2}}} \frac{b_{n,k}^{(p,q)}(x)}{p^{n-k}q^k} \int_{\frac{[k]_{p,q}}{p^{k-n-1}\,[n]_{p,q}}}^{\frac{[k+1]_{p,q}}{p^{k-n}\,[n]_{p,q}}} (t-x)^2 d_{p,q}t \right\}^{\frac{\alpha}{2}} \\
& \times \left\{ \frac{[n]_{p,q}}{p^{\frac{n(n-1)}{2}}} \frac{b_{n,k}^{(p,q)}(x)}{p^{n-k}q^k} \int_{\frac{[k]_{p,q}}{p^{k-n-1}\,[n]_{p,q}}}^{\frac{[k+1]_{p,q}}{p^{k-n}\,[n]_{p,q}}} 1 d_{p,q}t \right\}^{\frac{2-\alpha}{2}} \\
\leq\; & M \left\{ \frac{[n]_{p,q}}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^{n} \frac{b_{n,k}^{(p,q)}(x)}{p^{n-k}q^k} \int_{\frac{[k]_{p,q}}{p^{k-n-1}\,[n]_{p,q}}}^{\frac{[k+1]_{p,q}}{p^{k-n}\,[n]_{p,q}}} (t-x)^2 d_{p,q}t \right\}^{\frac{\alpha}{2}} \\
& \times \left\{ \frac{[n]_{p,q}}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^{n} \frac{b_{n,k}^{(p,q)}(x)}{p^{n-k}q^k} \int_{\frac{[k]_{p,q}}{p^{k-n-1}\,[n]_{p,q}}}^{\frac{[k+1]_{p,q}}{p^{k-n}\,[n]_{p,q}}} 1 d_{p,q}t \right\}^{\frac{2-\alpha}{2}} \\
=\; & M \left\{ K_n^{(p,q)}\big((t-x)^2;x\big) \right\}^{\frac{\alpha}{2}}.
\end{aligned}
$$

Choosing $\delta^2(x) = \delta_n^2(x) = K_n^{(p,q)}\big((t-x)^2;x\big)$, we arrive at our desired result.

Next, we prove the local approximation property for the operators $K_n^{(p,q)}$. The Peetre's K-functional is defined by

$$
K_2(f,\delta) = \inf[\{\|f - g\| + \delta\|g''\|\} : g \in W^2],
$$

where

$$
W^2 = \{g \in C[0,1] : g', g'' \in C[0,1]\}.
$$

By [8], there exists a positive constant $C > 0$ such that $K_2(f,\delta) \leq C\omega_2(f, \sqrt{\delta})$, $\delta > 0$, where the second order modulus of continuity is given by

$$
\omega_2(f, \sqrt{\delta}) = \sup_{0 < h \leq \sqrt{\delta}} \sup_{x \in [0,1]} \mid f(x+2h) - 2f(x+h) + f(x) \mid .
$$

Theorem 3.4. Let $f \in C[0,1]$ and $0 < q < p \leq 1$. Then for all $n \in \mathbb{N}$, there exists an absolute constant $C > 0$ such that

$$
\left| K_n^{(p,q)}(f;x) - f(x) \right| \leq C\omega_2\big(f, \delta_n(x)\big) + \omega\big(f, \alpha_n(x)\big)
$$

where

$$
\delta_n(x) = \sqrt{K_n^{(p,q)}\big((t-x)^2;x\big) + \frac{p^{2n}}{[2]_{p,q}^2[n]_{p,q}^2}} \text{ and } \alpha_n = \frac{p^n}{[2_{p,q}][n]_{p,q}}.
$$

Proof. For $x \in [0,1]$, we consider the auxiliary operators K_n^* defined by

$$
K_n^*(f;x) = K_n^{(p,q)}(f;x) + f(x) - f\left(x + \frac{p^n}{[2_{p,q}][n]_{p,q}} \right).
$$

From Lemma 2.1, we observe that the operators $K_n^*(f;x)$ are linear and reproduce the linear functions. Hence

$$
\begin{aligned}
K_n^*(1;x) &= K_n^{(p,q)}(1;x) + 1 - 1 = 1, \\
K_n^*(t;x) &= K_n^{(p,q)}(t;x) + x - \left(x + \frac{p^n}{[2_{p,q}][n]_{p,q}} \right) = x, \\
\text{so } K_n^*(t-x;x) &= K_n^*(t;x) - xK_n^*(1;x) = 0.
\end{aligned}
$$

Let $x \in [0,1]$ and $g \in C_B^2[0,1]$. Using the Taylor's formula

$$
g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)\, g''(u)\, du.
$$

Applying K_n^* to both sides of the above equation, we have

$$
\begin{aligned}
K_n^*(g;x) - g(x) &= K_n^*\big((t-x)g'(x);x\big) + K_n^*\left(\int_x^t (t-u)g''(u)du; x \right) \\
&= g'(x)K_n^*\big((t-x);x\big) + K_n^{(p,q)}\left(\int_x^t (t-u)g''(u)du; x \right) \\
&\quad - \int_x^{x+\frac{p^n}{[2_{p,q}][n]_{p,q}}} \left(x + \frac{p^n}{[2_{p,q}][n]_{p,q}} - u \right) g''(u)du \\
&= K_n^{(p,q)}\left(\int_x^t (t-u)g''(u)du; x \right) \\
&\quad - \int_x^{x+\frac{p^n}{[2_{p,q}][n]_{p,q}}} \left(x + \frac{p^n}{[2_{p,q}][n]_{p,q}} - u \right) g''(u)du.
\end{aligned}
$$

On the other hand, since

$$
\left| \int_x^t (t-u)g''(u)du \right| \leq \int_x^t |t-u||g''(u)|du \leq \|g''\| \int_x^t |t-u|du \leq (t-x)^2\|g''\|
$$

and

$$
\left| \int_x^{x+\frac{p^n}{[2_{p,q}][n]_{p,q}}} \left(x + \frac{p^n}{[2_{p,q}][n]_{p,q}} - u \right) g''(u)du \right| \leq \frac{p^{2n}}{[2]_{p,q}^2[n]_{p,q}^2}\|g''\|.
$$

We conclude that

$$
\begin{aligned}
\left| K_n^*(g;x) - g(x) \right| &= \left| K_n^{(p,q)}\left(\int_x^t (t-u)g''(u)du; x \right) \right. \\
&\quad \left. - \int_x^{x+\frac{p^n}{[2_{p,q}][n]_{p,q}}} \left(x + \frac{p^n}{[2_{p,q}][n]_{p,q}} - u \right) g''(u)du \right| \\
&\leq \|g''\| K_n^{(p,q)}\big((t-x)^2;x\big) + \frac{p^{2n}}{[2]_{p,q}^2[n]_{p,q}^2}\|g''\| \\
&= \delta_n^2(x)\|g''\|.
\end{aligned}
$$

Now, taking into account boundedness of K_n^*, we have

$$
\left| K_n^*(f;x) \right| \leq \left| K_n^{(p,q)}(f;x) \right| + 2\|f\| \leq 3\|f\|.
$$

Therefore

$$
\begin{aligned}
\left| K_n^{(p,q)}(f;x) - f(x) \right| &\leq \left| K_n^*(f;x) - f(x) \right| + \left| f(x) - f\left(x + \frac{p^n}{[2]_{p,q}[n]_{p,q}} \right) \right| \\
&\leq \left| K_n^*(f-g;x) - (f-g)(x) \right| \\
&\quad + \left| f(x) - f\left(x + \frac{p^n}{[2]_{p,q}[n]_{p,q}} \right) \right| + \left| K_n^*(g;x) - g(x) \right| \\
&\leq \left| K_n^*(f-g;x) \right| + \left| (f-g)(x) \right| \\
&\quad + \left| f(x) - f\left(x + \frac{p^n}{[2]_{p,q}[n]_{p,q}} \right) \right| + \left| K_n^*(g;x) - g(x) \right| \\
&\leq 4\|f-g\| + \omega\left(f, \frac{p^n}{[2]_{p,q}[n]_{p,q}} \right) + \delta_n^2(x)\|g''\|.
\end{aligned}
$$

Hence, taking the infimum on the right-hand side over all $g \in W^2$, we have the following result

$$
\left| K_n^{(p,q)}(f;x) - f(x) \right| \leq 4K_2\big(f, \delta_n^2(x)\big) + \omega\big(f, \alpha_n(x)\big).
$$

In view of the property of K-functional, we get

$$
\left| K_n^{(p,q)}(f;x) - f(x) \right| \leq C\omega_2\big(f, \delta_n(x)\big) + \omega\big(f, \alpha_n(x)\big).
$$

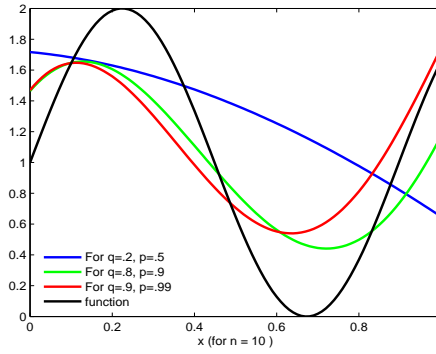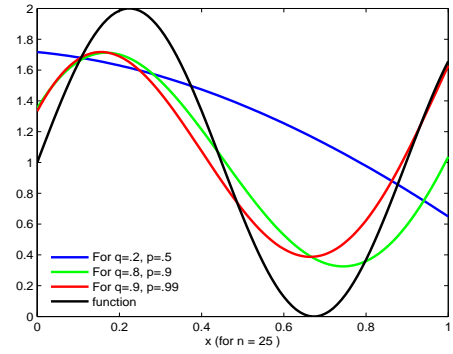This completes the proof of the theorem.
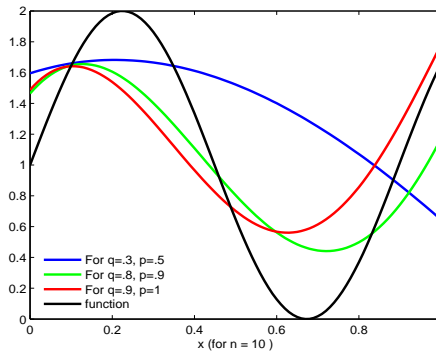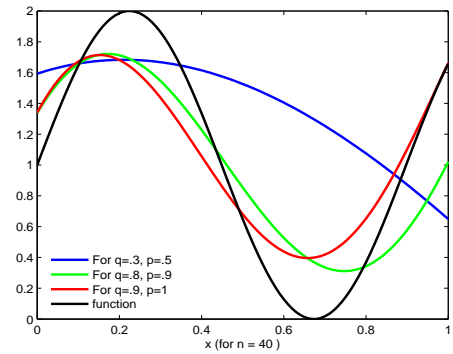
FIGURE 1

FIGURE 2

FIGURE 3

FIGURE 4

4. Graphical Examples

With the help of Matlab, we show comparisons and some illustrative graphics [20] for the convergence of operators to the function $f(x) = 1 + sin(7x)$.

From Fig. 1 it can be observed that as the value of q and p approaches towards 1 provided $0 < q < p \leq 1$, (p, q)-Bernstein-Kantorovich operators converge towards the function. The parameters q and p adds flexibility in approximation of functions by positive linear operators.

In comparison to Fig. 1 as the value the n increases, operators given by (3) converge towards the function which is shown in figure (2).

Similarly for different values of parameters p, q, n convergence of operators to the function is shown in Fig. 3 and Fig. 4.

Thus in comparison to q-analogoue of Bernstein-Kantorovich operators, our generalization gives us more flexibility for the convergence of operators to a function.

R E F E R E N C E S

[1] *T. Acar,* (p,q)-Generalization of Szsz-Mirakyan operators, Math. Meth. Appl. Sci., 39(10)(2016)2685-2695.

[2] *F. Altomare, M. Campiti,* Korovkin type approximation theory and its applications, de Gruyter Stud. Math. 17, Berlin, 1994.

[3] *A. Aral, O. Doğru,* Bleimann Butzer and Hahn operators based on q-integers, J. Inequal. Appl., (2007) 1-12. Art. ID 79410.

[4] *S.N. Bernstein,* Démostration du théorème de Weierstrass fondée sur le calcul de probabilités, Comm. Soc. Math. Kharkow (2), 13 (1912-1913) 1-2.

[5] *Q.-B. Cai and G. Zhou,* On (p,q)-analogue of Kantorovich type Bernstein–Stancu–Schurer operators, Appl. Math. Comput., 276 (2016) 12-20.

[6] *R. Chakrabarti and R. Jagannathan,* A (p,q)-oscillator realization of two parameter quantum algebras, J. Phys. A: Math. Gen., 24 (1991) 711-718.

[7] *O. Dalmanoglu,* Approximation by Kantorovich type q-Bernstein operators, in Proceedings of the 12th WSEAS International Conference on Applied Mathematics, Cairo, Egypt (2007), pp. 113-117.

[8] *R.A. Devore, G.G. Lorentz,* Constructive Approximation, Springer, Berlin, 1993.

[9] *M.N. Hounkonnou, J. Désiré, B. Kyemba,* $\mathcal{R}(p,q)$-calculus: differentiation and integration, SUT Journal of Mathematics, Vol. 49, No. 2 (2013), 145-167.

[10] *V. Kac, P. Cheung,* Quantum Calculus, Springer-Verlag New York, 2002.

[11] *L.V. Kantorovich,* Sur certains développements suivant les polynômes de la forme de S. Bernstein, I, II, C. R. Acad. URSS, (1930), 563-568, 595-600.

[12] *P.P. Korovkin,* Linear operators and approximation theory, Hindustan Publishing Corporation, Delhi, 1960.

[13] *J. Katriel, M. Kibler,* Normal ordering for deformed boson operators and operator-valued deformed Stirling numbers, J. Phys. A: Math. Gen. 25 (1992) 2683-2691.

[14] *R. Jagannathan, K. S. Rao,* Two-parameter quantum algebras, twin-basic numbers, and associated generalized hypergeometric series, Proceedings of the International Conference on Number Theory and Mathematical Physics, 20-21 December 2005.

[15] *A. Lupaş,* A q-analogue of the Bernstein operator, Seminar on Numerical and Statistical Calculus, University of Cluj-Napoca, 9(1987) 85-92.

[16] *N.I. Mahmudov,* On q-parametric Szász-Mirakjan operators, Mediterr. J. Math. 7(3), 297-311 (2010).

[17] *N.I. Mahmudov, P. Sabancigil,* On genuine q-Bernstein-Durrmeyer operators. Publ. Math. Debrecen 76(4) (2010).

[18] *N.I. Mahmudov, V. Gupta,* On certain q-analogue of Szász Kantorovich operators, J. Appl. Math. Comput., 37 (2011) 407-419.

[19] *M. Mursaleen, A. Alotaibi and K.J. Ansari,* On a Kantorovich variant of (p,q)-Szász-Mirakjan operators, Jour. Func. Spaces, Volume 2016, Article ID 1035253, 9 pages.

[20] *M. Mursaleen, Asif Khan,* Generalized q-Bernstein-Schurer Operators and Some Approximation Theorems, Jour. Func. Spaces, Volume 2013, Article ID 719834, 7 pages.

[21] *M. Mursaleen, K.J. Ansari and A. Khan,* On (p,q)-analogue of Bernstein Operators, Applied Mathematics and Computation, 266 (2015) 874-882, [Erratum: Appl. Math. Comput. 266 (2015) 874-882].

[22] *M. Mursaleen, K.J. Ansari and A. Khan,* Some approximation results by (p,q)-analogue of Bernstein-Stancu operators, Appl. Math. Comput. , 264 (2015) 392–402 [Corrigendum: Appl. Math. Comput, 269 (2015) 744-746].

[23] *M. Mursaleen, Md. Nasiruzzaman, Asif Khan and K. J. Ansari,* Some approximation results on Bleimann-Butzer-Hahn operators defined by (p, q)-integers, Filomat 30:3 (2016), 639-648.

[24] *M. Mursaleen, Md. Nasiuzzaman and A. Nurgali,* Some approximation results on Bernstein-Schurer operators defined by (p, q)-integers, J. Ineq. Appl., 2015 (2015): 249.

[25] *G.M. Phillips,* Bernstein polynomials based on the q-integers, The heritage of P.L.Chebyshev, Ann. Numer. Math., 4 (1997) 511-518.

[26] *V. Sahai, S. Yadav,* Representations of two parameter quantum algebras and p,q-special functions, J. Math. Anal. Appl. 335 (2007) 268-279.

[27] *P.N. Sadjang,* On the fundamental theorem of (p, q)-calculus and some (p, q)-Taylor formulas, arXiv:1309.3934v1 [math.QA], 2013.

[28] *F. Schurer,* Linear positive operators in approximation theory, Math. Inst. Techn. Univ. Delft Report, 1962.