ON A PEANO - TYPE AXIOMATICATION FOR FREE MONOIDS

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Sunt bine cunoscute Axiomele lui Peano pentru mulțimea numerelor naturale $\mathbb{N}$. Dacă X este o mulțime nevidă şi X^* este monoidul cuvintelor peste X; X^* poate fi caracterizat până la un izomorfism de o prioritate de universalitate şi de asemenea de proprietăți interne. Dacă $X = \{1\}$ are un singur element, atunci X^* este practic $\mathbb{N}$. In această lucrare se dă o exnsie a axiomelor lui Peano (de la $\mathbb{N}$ în X^*) şi de asemenea o generalizare la categorii mici libere.

It is well-known the Peano axiomatization for the set $\mathbb{N}$ of natural numbers. If X is a nonempty set and X^* is the monoid of the words over X, X^* can be characterized up to an isomorphism by an universality property and also by some internal properties. If $X = \{1\}$ hence a singleton, then X^* is practically $\mathbb{N}$. In this paper, we give an extension of the Peano axioms (from $\mathbb{N}$ to X^*) and also a generalization to free small categories.

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1. Introduction

A free monoid is a monoid together with a specified subset such that any function from this subset to another monoid extends uniquely to a morphism of monoids. The monoid X^* on an alphabet *X* is called the free monoid over *X*; this is indeed a free monoid (called an word monoid). Any two monoids which are free over the same set are isomorphic.

We propose to obtain an internal characterization of the free monoids. Such characterizations already exist, e.g. [2]. The idea is to show that the word monoid does satisfy the characterization and then one carries this, by isomorphism, to any other free monoid. Our approach is stronger, since we will

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not assume the existence of a special free monoid. Such an approach was applied in [3] to natural number objects in toposes, where the universality property is shown to be equivalent to the Peano's axioms. We intend to do the same for free monoids (in a set-theoretic framework) and this can be carried over to a topos.

2. The Axiomatic Characterization Theorem

Let W be a monoid $X \subset W$ a subset. By definition, W is free on X if any map $f : X \to M$ (M being any monoid) extends uniquely to a morphism $\overline{f} : W \to M$. In a categorical setting, we should have started with a map $j : X \to W$ instead of the inclusion but j is immediately proven injective. Our goal is to prove, in a set-theoretical frame which excludes consequences of the existence of a natural numbers sets, the following.

Theorem 1 of characterisation:

Fix a nonempty set X. A monoid W is free on X if and only if the following conditions hold:

(m_1) $e \notin X$ (e being the unit element in W);

(m_2) if $u \cdot v \in X$, then $u = e$ or $v = e$;

(m_3) if $u \cdot v = e$, then $u = e$ or $v = e$;

(m_4) if $u \cdot v = u'v'$; then there is w such that either $u = u'w$, $v' = wv$ or $u' = uw$, $v = wv'$;

(m_5) any submonoid of W which contains X is equal to W.

Note. The condition (m_4) is called "the subdivision property" and this has some significance in Automate theory (by [2]).

The condition (m_5) is equivalent to the following "principle of monoidal induction": Suppose that $p(w)$ is a predicate depending on $w \in W$ such that $p(x)$ holds for any $x \in X$ (induction basis) and moreover, $p(e)$ holds and whenever $p(u)$ and $p(v)$ hold, then $p(uv)$ holds. Then $p(w)$ holds for any $w \in W$.

The conditions (m_1) and (m_2) are concerned with the limited way in which the elements of can be computed, that conditions (m_3) and (m_4) describe the relation between the different ways of computing an element and that (m_5) is the induction characteristic to the monoid structure. Recall the Peano's axioms:

(p_1) 0 cannot be obtained by succession;

(p_2) if m is obtained by succession from n and p, then $n = p$.

(p_3) induction on natural numbers. It is exactly this correspondence in nature between (m_1) ÷ (m_5) and (p_1) ÷ (p_3) that made us to call (m_1) ÷ (m_5) a *Peano type axiomatization*. As we shall see, it is strongly related to a direct generalization of Peano's axioms that characterizes free X-dynamics.

In order to prove the above theorem, we need some lemmas.

Lemma 1. Let *W* (respectively *V*) be free monoids on *X* (resp. *Y*) and $h:W\to V$ be a morphism of monoids. The following assertions are equivalent:

a) $h(X)\subset Y$;

b) if $h(w)=e$, then $w=e$ and moreover, whenever $h(w)=st$, there are u,v such that $w=u\cdot v, h(u)=s$ and $h(v)=t$.

Proof. a)$\Rightarrow$b) Apply the monoidal induction by *w*. If $h(x)=st$, then $st\in Y$, so $s=e$ or $t=e$. In the first case, we can take $u=e, v=x$ and in the second, $u=x, v=e$. If $h(e)=st$, then $st=e$, so $s=e$, $t=e$ and take $u=e, v=e$. Suppose now that the assertion holds for w_1 and w_2. If $h(w_1,w_2)=st$, then $h(w_1)h(w_2)=st$, so there is *r* such that either $h(w_1)=sr$, $t=rh(w_1)$ or $s=h(w_1)\cdot r, h(w_2)=rt$. In the first case, $w_1=u_1v_1$, $h(u_1)=s, h(v_1)=r$, so we can take $u=u_1, v=v_1w_2$ and the assention holds also for w_1w_2. The second case is similar.

b) $\Rightarrow$ a) First we prove by monoid induction that for any $v\in V, v\neq e, v=ty, y\in Y$. If $x\in X$, then $h(x)\neq e$. So $h(x)=ty, y\in Y$. But $x=u\cdot v, h(u)=t, h(v)=y$. From $u\cdot v\in X$, one deduces $u=e$ or $v=e$. But $v=e$ implies $y=e$, so $u=e$ and $h(x)=h(v)=y\in Y$.

Recall that a X-dynamics ([2], [4]) means a pair (A,δ), where *A* is a set and $\delta:A\times X\to A$ a map. If (A',δ') is another X-dynamic, a *dynamorphism* between these X-dynamics is a map $h:A\to A'$ such that $h\circ\delta=\delta'\circ(h\times 1_X)$. A *subdynamics* of (A,δ) is a subset $S\subset A$ such that $\delta(S\times X)\subset S$; in this case, there is a structure of X-dynamics on *S* such that the inclusion map $i:S\to A$ becomes a dynamorfism. A X-dynamics (A,δ) *is free on* $a_0\in A$ if for any X-dynamics (A',δ') and any $a'\in A$, there is and is unique a dynamorphism $h:A\to A'$ such that $h(a_0)=a'$.

Lemma 2. The monoid *W* is free on *X* if and only if the X-dynamics (W,σ) is free on *e* (where $\sigma(w,x)=wx$ and *e* is the unit element in *W*).

Proof. Suppose that *W* is free on *X*; then any submonoid *S* such that $X\subset S\subset W$ equals *W* (indeed, the inclusion map $j:X\to S$ extends to a morphism $\overline{j}:W\to S$; the inclusion map $i:S\to W$ is also a morphism. Since

$i \circ \overline{j}$ and 1_W coincide on X, then they are equal. Particularly, i is surjective, hence $S = W$). Similarly, one can prove that whenever a X-dynamics (A, δ) is free on a_0, then any subdynamics containing a_0 is equal to A. We thus retain the following induction principle, that we call the successor induction: if p is a predicate, $p(a_0)$ and $\{(\forall) a \in A, p(a) \Rightarrow \quad (\forall) x \in X, p(\delta(a, x))\}$, then $(\forall) a \in A, p(a)$.

If W is free on X, (A, δ) is a X-dynamic and $a \in A$, then the map X-dynamics $\delta : A \times X \to A$ defines $\delta_1 : X \to A^A$. But A^A is a monoid, so δ_1 extends to a morphism $\overline{\delta}_1 : W \to A^A$ and define $h : W \to A$ $h(w) = \overline{\delta}_1(w)(a)$. Obviously, h is a dynamorphism and $h(e) = a$. (The uniqueness follows by making use of the successon induction).

Conversely, suppose that (W, σ) is foll on e and M be another with unit e' and $f : X \to M$. Then M is a X-dynamics by $\delta' : M \times X \to M$, $\delta'(m, x) = m f(x)$. Then one can prove by monoidal induction after v then $(\forall) v \in W, \overline{f}(ur) = \overline{f}(u)\overline{f}(r)$ for all $u \in W$. The uniqueness follow, from the fact that any morphism that extends f is a dynamorphism.

Note. The concept of X-dynamics generalizes the basic structure existing on $\mathbb{N}$, namely the successor structure. When $X = \{1\}$ is a singleton, then $\mathbb{N}$ is a X-dynamics which is free on a specified element, namely O.

Lemma 3. A X-dynamics (A, δ) is free on a_0 if and only if and only if the following conditions (generalized Peano axioms) hold

(d$_1$) $a_0 \notin \delta(A \times X)$

(d$_2$) δ is injective

(d$_3$) any subdynamics containing a_0 is equal to A.

Proof. Suppose that (A, δ) is free on a_0. The condition (d$_3$) follows from the proof of Lemma 2. For (d$_1$), consider the X-dynamics $\mathbb{B}$, with $\delta' : \mathbb{B} \times X \to \mathbb{B}$, $\delta'(b, x) = 1$. Then there is a dynomorphism $h : A \to \mathbb{B}$ such that $h(a_0) = 0$. If $a_0 = \delta(a, x)$, then $h(a_0) = \delta'(h(a), x) = 1$; contradiction. Prove now that δ is injective. For this, we not that $A \times X$ becomes a X-dynamics, with $\delta \times 1_X : (A \times X) \times X \to A \times X$, such that δ becomes a dynamorphism. Consider two dynamorphism h, h' defined on $A \times X$, coinciding on $\{a_0\} \times X$. One can prove by successor induction by *a*, that for all $a \in A, h(a,x) = h(a',x)$, for any $x \in X$. Hence $h = h'$. Now, let us add an extraelement $a_0^{'} \notin A \times X$; take $A' = A \times X \cup \{a_0^{'}\}$ and define $\delta' : A' \times X \to A'$, putting $\delta'(t,x) = (\delta(t), x)$ if $t \in A \times X$ and $\delta'(t,x) = a_0, x$ if $t = a_0^{'}$. Then the inclusion $e : A \times X \to A'$ is a dynamorphism and since (A, δ) is free on a_0, we will get a dynamorphisn $h : A \to A'$ such that $h(a_0) = a_0^{'}$. Since $h \circ \delta$ and *i* coincide on $\{a_0\} \times X$, they will be equal. Thus, δ is injective.

Conversely, assume that (d_1) ÷ (d_3) bold and consider another X-dynamics (A', δ') and $a' \in A$. Denote by *R* the set of all binary relations $\rho \subset A \times A'$ such that $(a_0, a') \in \rho$ and whenever $(a, a'') \in \rho$, then $(\delta(a,x), \delta'(a'',x)) \in \rho$ for all $x \in X$. Obviously, $A \times A' \in R$. Denote by *h* the intersection of all relations from *R*. By successor induction, one can prove that for any $a \in A$ the set $\{a'' \in A' | (a, a'') \in h\}$ has exactly one element only. Hence *h* is in fact a function and this is the checked dynamorphism. The uniqueness is immediate by successor induction.

Proof of the Theorem of Characterisation

Apply the lemmas 2 and 3 for the X-dynamics (W, σ) and $e \in W$.

We first note that the assertions (d_3) and (m_5) are equivalent. Indeed, suppose (m_5) and let *S* be a subdynamics of (W, σ). By monoid induction after *w*, one proves that for all $w \in W$ and $s \in S, sw \in S$. If *S* contains *e*, the $S = W$. Conversely, any submonoid which contains *X* is a subdynamics.

(d) $\Rightarrow$ (m) One first shows, by successor induction, that for all $w \in W$, if $w \neq e$, then there are $u \in W, x \in X$ such that $w = ux$; then (m_1), (m_2), (m_3) easily follow. To prove (m_4), fix u, u' and use the successor induction by *v* to check that for all $v \in W$, we have the subdivision property for all $v' \in W$.

(m) $\Rightarrow$ (d) (d_1) is direct; for (d_2), if $ux = vy$, then by (m_4), there is *w* such that $u = vw$, $y = wx$ or $v = uw, x = wy$. By (m_2), if $y = wx$, then $w = e$ (since $x \neq e$ by (m_1)) and if $x = wy$, then also $w = e$. Thus, $u = v, x = y$. What concerns (d_3), this is equivalent to (m_5), as we have seen. This completes the proof.

3. A Generalization to Free Categories

Let Z be a fixed set of vertices. A Z-graph means a set *A* of *arcs*, an initial vertex map $i_A : A \to Z$ and a terminal vertex map $t_A : A \to Z$. A morphism of graphs is a map $f : A \to B$ such that $i_B \circ f = i_A, t_B \circ f = t_A$. A *subgraph* of *A* is a subset of *A* (which is naturally a *Z*-graph). Define also the concatenation between $a \in A$ and $b \in B$ by $a * b$ if $t_A(a) = i_B(b)$.

A Z-category [5] consists of a *Z*-graph *C*, an associative partial *composition* defined only for concatenable arcs *a* and *b* such that $a * b$; denote this by $(a,b) \to ab$; moreover, for any $z \in Z$, there is an *identity* u_z, with $i(u_z) = t(u_z) = z$ and $cu_{t(c)} = u_{i(c)}c = c$, for any $c \in C$. A Z-function between two *Z*-categories is a morphism of graphs which commutes with composition and preserves the identities. A *subcategory* of *C* is a subgraph, which is closed under composition and contains all identities.

We introduce the following concepts. A *Z*-category *C* is *free* on a subgraph $X \subset C$ if for other *Z*-category C', any morphism of graphs $f : X \to C'$ extends uniquely to a *Z*-function $\overline{f} : C \to C'$. If *X* is a *Z*-graph, a X-dynamics is a *Z*-graph *A*, together a morphism of graphs $\delta : A \otimes X \to A$; here $A \otimes X = \{(a,x) \mid a * x\}$, $i(a,x) = i_A(a)$ and $t(a,x) = t_X(x)$. A *dynomorphism* $h : (A,\delta) \to (A',\delta')$ is a morphism of graphs $h : A \to A'$ such that $h(\delta(a,x)) = \delta'(h(a),x)$, for any $a \in A, x \in X$ such that $a * x$.

A *subdynamics* of (A,δ) is a subgraph $S \subset A$ such that wherever $s \in S$, then $\delta(s,x) \in S$ for any $x \in X$ such that $s * x$:

A *Z-family* of elements of a *Z*-graph *A* consists, for any $z \in X$, of an element $e_z \in A$, such that $i(e_z) = t(e_z) = z$. Finally, a *X*-dynamics (A,δ) is *free* on a *Z*-family of elements $\{e_z\}$ of *A* if for any other X-dynamics (A',δ') and any *Z*-family of elements $\{e_z^{'}\}$ of *A*', there is a unique dynamorphism $h : A \to A'$ such that $h(e_z) = e_z^{'}$, for any $z \in Z$.

Note. If *Z* is a singleton, then one obtains the corresponding notions used in §2.

In order to generalize the proof of the Lemma 3, one should take the disjoint union $Z \cup Z \times Z$ instead of $\mathbb{B}$. In generalizing for proof of Lemma 2, we need a construction of type A^A such that $\delta : A \otimes X \to A$ defines $\delta_1 : X \to A^A$ and A^A is a *Z*-category. Indeed, the arcs A^A will be triples $(z, z', \varphi); z \in Z$, $z' \in Z$, where $\varphi : t_A^{-1}(z) \to t_A^{-1}(z')$ with initial vertex *z* and terminal vertex *z*'. The map δ_1 is given by $\delta_1(x) = (z, z', \varphi)$, where $z = i_A(x), z' = t_A(x)$ and $\varphi(a) = \delta(a, x)$ (since $a \in t_A^{-1}(z), a * x$) A^A becomes a *Z*-category with $(z, z', \varphi)(z', z'', \varphi') = (z, z'', \varphi' \circ \varphi)$ and $u_z = (z, z, 1)$.

The *Z*-graphs and the morphism of graphs form a new category, that we denote by *Z-Graph*. The product $\otimes$ and the unity *Z*-graph $(Z, 1 : Z \to Z,$ $1 : Z \to Z)$ define an *Z-Graph* a structure of monoidal category. The construction A^A can be generalized such that for any *Z*-graph *Q*, there exists a function $A \to A^Q$; $(f : A \to B) \to \left(f^Q : A^Q \to B^Q\right)$, which is a right adjoint to the function $A \to Q \otimes A$; $(f : A \to B) \to \left(1_Q \otimes f : Q \otimes A \to Q \otimes B\right)$. Therefore, *Z-Graph* is an "almost closed" category (it lacks only symmetry, because generally $A \otimes B$ is not isomorphic to $B \otimes A$).

4. Conclusions

A monoid is free on a subset if it satisfies the well-known universality property. This papers refers to an internal characterization for the free monoids (theorem 1), which is similar in nature with the Peano's axiomatization of the natural numbers. This characterization is stated and proved independently on the existence of free monoids (that implying the independence of the existence of natural numbers). The proof requires the concept of dynamics, inspired from that of "transition of states" from Automata Theory and that of free dynamics. Lemma 2 describes the free monoids in terms of dynamics and lemma 3 explicits the generalized Peano axioms for free dynamics. In §3 we propose a generalization of the above results to free small categories.

R E F E R E N C E S

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