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KINEMATICS AND DYNAMICS OF A 2-DOF ORIENTING GEAR TRAIN

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Articolul stabileşte relaţii matriceale pentru cinematica şi analiza dinamică a unui tren de roţi dinţate pentru orientare cu două grade de libertate. Mecanismul este un sistem paralel cu cinci elemente mobile şi trei cuplaje prin roţi dinţate controlate de două motoare electrice. Cunoscând mişcarea de rotaţie a efectorului, problema de dinamică inversă este rezolvată printr-un procedeu bazat pe principiul lucrului mecanic virtual, dar rezultatele au fost verificate în sistemul de lucru al ecuaţiilor lui Lagrange de speţa a doua. În final, se obţin câteva grafice pentru unghiurile de rotaţie la intrare, forţele active şi puterile mecanice ale celor trei sisteme de acţionare.

Recursive matrix relations for kinematics and dynamics analysis of a 2-DOF orienting gear train are established in this paper. The mechanism is a parallel system with five moving parts and three bevel gear pairs controlled by two electric motors. Knowing the rotation motion of the end-effector, the inverse dynamic problem is solved using an approach based on the principle of virtual work, but the results have been verified in the framework of the Lagrange equations of second kind. Finally, some graphs for the input angles of rotation, the forces and the powers of the actuators are obtained.

Keywords: Dynamics; Kinematics; Gear train; Virtual work

List of symbols

$p_{k,k-1}$: orthogonal relative transformation matrices from the frame $x_{k-1}y_{k-1}z_{k-1}$ to following frame $x_k y_k z_k$

$\vec{u}_1, \vec{u}_2, \vec{u}_3$: three orthogonal unit vectors

ϕ_1, ϕ_2 : angles giving the orientation of the end-effector

$\varphi_{k,k-1}$: relative rotation angle of $T_k, (k = 1, 2, 3)$ rigid body

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$\vec{\omega}_{k,k-1}$: relative angular velocity of T_k
$\vec{\omega}_{k0}$: absolute angular velocity of T_k
$\tilde{\omega}_{k,k-1}$: skew symmetric matrix associated to the angular velocity $\vec{\omega}_{k,k-1}$
$\vec{\varepsilon}_{k,k-1}$: relative angular acceleration of T_k
$\tilde{\varepsilon}_{k0}$: absolute angular acceleration of T_k
$\tilde{\varepsilon}_{k,k-1}$: skew symmetric matrix associated to the angular acceleration $\vec{\varepsilon}_{k,k-1}$
$\vec{r}_k^{\,C}$: position vector of the mass centre of T_k rigid body
m_k : mass of T_k rigid body
$\hat{J}_k$: symmetric matrix of tensor of inertia of T_k about the link-frame $x_k y_k z_k$
$m_{10}^A, m_{10}^C, P_{10}^A, P_{10}^C$: torques and powers of the two actuators

1 Introduction

The orienting mechanisms are incorporated in the structure of industrial robots and have two or three output rotations. Generally, these mechanical systems have conical and cylindrical toothed elements in their structure, while the input axes are parallel and the output axes are orthogonal. The three rotary orientation movements are usually performed around the axes of a Cartesian orthogonal frame, having its axes linked to the last arm of the robot's positioning mechanism.

The industrial robots with orienting gear trains can perform several operations such as welding, flame cutting, spray painting, milling or assembling. Being comparatively simple and compact in size, the bevel-gear wrist mechanisms can be sealed in a metallic box that keeps the device of contamination. Furthermore, using bevel gear trains for power transmission, the actuators can be mounted remotely on the forearm, thereby reducing the weight and inertia of a robot manipulator.

Planetary gear trains with three degrees of freedom are adopted as the design concept for robotic wrist ([1-5]).

2 Inverse kinematics model

Recursive matrix relations for kinematics and dynamics of a 2-DOF orienting gear train, which has a non-symmetrical kinematical schema, are developed in the paper (Fig. 1). The robot wrist must rotate around two orthogonal axes so that the mechanism has two degrees of freedom. A matrix methodology for the kinematics analysis based on the concept of fundamental circuit of an open-loop chain is presented. This method involves the identification of all open-loop chains and the derivation of the geometric relationships between the orientation of the end-effector and the joint angles of the chains, including the input actuator displacements ([6], [7], [8]).

Let $Oxyz$ be a fixed frame, about which the mechanism moves. The mechanism topology consists of five moving links, five revolute pairs and three bevel gear pairs (Fig. 2). Therefore, the wrist is a 2-DOF spherical mechanism, which has a limited rotation range about the vertical joint axis.

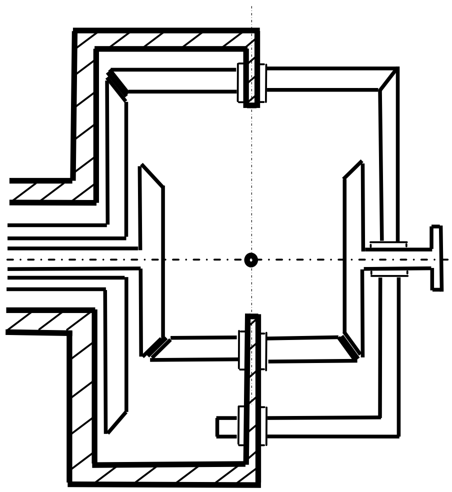

Fig. 1 The 2-DOF orienting gear train

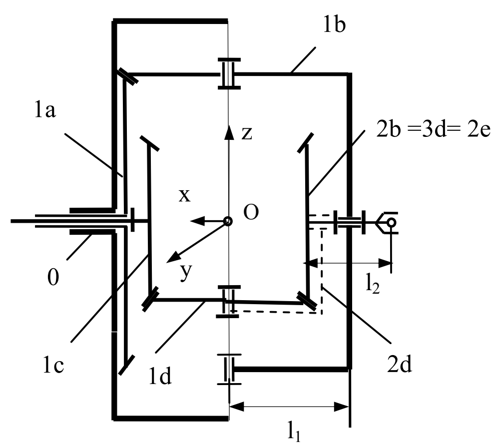

Fig. 2 Kinematical scheme of the mechanism

There are two active gears $1a$, $1c$ and two kinematic chains $0-1b-2b$ and $0-1d-2d-3d$ starting from the forearm and ending to a common element as end-effector $2e=2b=3d$.

In the wrist mechanism, the gear $1a$, of radius r_1, mass m_1 and tensor of inertia $\hat{J}_1$ is adjoining to the link $1b$ of length l_1, which could serve as carrier for the $1d$ - $3d$ bevel gear pair. This element includes a bevel gear of radius r_2, mass m_2 and tensor of inertia $\hat{J}_2$. Four gears $1a$, $1b$, $1c$, $1d$ are sun gears while $2e=2b=3d$ is a bevel planet gear fixed to the end-effector and adjacent to a fictitious carrier $2d$.

The gear $1c$ of radius r_3, mass m_3 and tensor of inertia $\hat{J}_3$ is connected to a second gear $1d$ of radius r_4, mass m_4 and tensor of inertia $\hat{J}_4$. Including the end-effector of length l_2 at the last gear $2e=2b=3d$ of radius r_5, mass m_5, tensor of inertia $\hat{J}_5$, we obtain an assembly that is free to arbitrarily undergo two concurrent rotations with respect to the common center O. We remark that the active gears $1a$ and $1c$ share one fixed common joint on axis x and that the links $1b$ and $1d$ share another common joint on axis z.

In the following, we apply the method of successive displacements to geometric analysis of closed-loop chains and we note that a joint variable is the displacement required to move a link from the initial location to the actual position. If every link is connected to at least two other links, the chain forms one or more independent closed-loops. The variable angles $\varphi_{k,k-1}$ of rotation about the joint axis z_k are the parameters needed to bring the next link from a reference configuration to the next configuration. We call the matrix $a^{\varphi}_{k,k-1}$, for example, the orthogonal transformation 3×3 matrix of relative rotation with the angle $\varphi^{A}_{k,k-1}$ of link T^{A}_{k} around z^{A}_{k} axis. In the study of the kinematics on constrained systems, we are interested in deriving a matrix equation relating the location of an arbitrary T_k body to the joint variables. When the change of coordinates is successively considered, the corresponding matrices are multiplied.

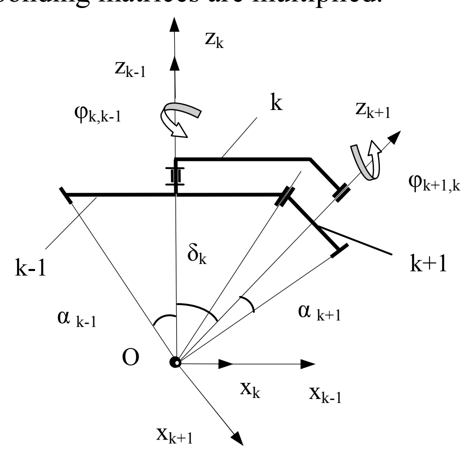

Fig. 3 Gear fundamental circuit

In what follows, we introduce a matrix approach that utilizes the *theory of fundamental circuits* [6]. There exists a *real* or *fictitious* carrier for every gear pair in a planetary gear train and a fundamental matrix equation for each loop can be written as

$$a_{k+1,k-1} = a^{\varphi}_{k+1,k}\theta^{\delta_k}_{y}a_{k,k-1},\ \varphi_{k,k-1} = n_{k+1,k-1}\varphi_{k+1,k},\ \delta_k = \alpha_{k-1} + \alpha_{k+1}$$

$$\theta^{\delta_k}_{y} = \begin{bmatrix} \cos\delta_k & 0 & -\sin\delta_k \\ 0 & 1 & 0 \\ \sin\delta_k & 0 & \cos\delta_k \end{bmatrix}, \qquad (1)$$

where $\varphi_{k,k-1}$ and $\varphi_{k+1,k}$ denote the relative angles of rotation of the carrier T_k and the planet gear T_{k+1}, respectively, while α_{k-1}, α_{k+1} are the angles that characterize the geometry of the connected gears T_{k-1} and T_{k+1}.

The ratio of a gear pair is defined as

$$n_{k+1,k-1} = r_{k+1} / r_{k-1} = z_{k+1} / z_{k-1}, \tag{2}$$

where r_{k-1}, r_{k+1} and z_{k-1}, z_{k+1} are the radius and the number of teeth of the two gears, respectively (Fig. 3).

The motions of the five parts of the gear train are concurrent rotations around the fixed point O. To simplify the graphical image of the kinematic scheme of the gear mechanism, in what follows we will represent the intermediate reference systems by only two axes, as it is used in most of the robotics papers. The z_k axis is represented, of course, for each component T_k. It is noted that the relative rotation with angle $\varphi_{k,k-1}$ of the body T_k must always be pointed about the direction of the z_k axis. Consequently, four appropriate frames for the first kinematical chain and five frames for the second circuit are fixed in a same origin O (Fig. 4). We consider the rotation angles φ_{10}^A, φ_{10}^C of the actuators as parameters giving the instantaneous position of the mechanism. Seven relative angles of rotation φ_{10}^B, φ_{21}^B, φ_{10}^D, φ_{21}^D, φ_{32}^D, φ_{10}^E, φ_{21}^E are the variables in the inverse kinematics problem.

Starting from the reference origin O and pursuing five independent serial circuits $0-1a$, $0-1b-2b$, $0-1c$, $0-1d-2d-3d$ and $0-1e-2e$, we obtain the following successive matrices of transformation ([9], [10]):

$$a_{10} = a_{10}^{\varphi}\theta_1,\ b_{10} = b_{10}^{\varphi}\theta_2,\ b_{21} = b_{21}^{\varphi}\theta_1,\ c_{10} = c_{10}^{\varphi}\theta_1$$

$$d_{10} = d_{10}^{\varphi},\ d_{21} = d_{21}^{\varphi}\theta_2,\ d_{32} = d_{32}^{\varphi}\theta_1,\ e_{10} = e_{10}^{\varphi}\theta_2,\ e_{21} = e_{21}^{\varphi}\theta_1, \tag{3}$$

where

$$\theta_1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},\ \theta_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$p_{k,k-1}^{\varphi} = \begin{bmatrix} \cos\varphi_{k,k-1}^i & \sin\varphi_{k,k-1}^i & 0 \\ -\sin\varphi_{k,k-1}^i & \cos\varphi_{k,k-1}^i & 0 \\ 0 & 0 & 1 \end{bmatrix},\ (p = a,b,c,d,e)\ \ (i = A,B,C,D,E) \tag{4}$$

$$p_{k0} = \prod_{s=1}^{k} p_{k-s+1,k-s},\ (k = 1, 2, 3).$$

Let us suppose that the absolute motion of the end-effector $2e$ attached at the planet gear $2b = 3d$ is a rotation around the center O. In the inverse geometric problem however, the orientation of the end-effector in the fixed frame is known through of the two Euler angles $\varphi_{10}^{E}=\phi_1$, $\varphi_{21}^{E}=\phi_2$, that are expressed by two analytical functions

$$\phi_l=\phi_l^{*}[1-\cos(\frac{\pi}{6}t)], \quad (l=1,2), \tag{5}$$

where $2\phi_l^{*}$ represents the maximum value of the angle of rotation ϕ_l.

Constraint geometric conditions for the rotation of the end-effector are given by the identities

$$b_{20}=d_{30}=e_{20}. \tag{6}$$

From these equations, we obtain the real-time evolution of all characteristic joint angles, as follows:

$$\varphi_{10}^{A}=\frac{\phi_1}{n_1},\ \varphi_{10}^{B}=\phi_1,\ \varphi_{21}^{B}=\phi_2,\ \varphi_{10}^{C}=\frac{1}{n_2}(-\phi_1+\frac{\phi_2}{n_3})$$

$$\varphi_{10}^{D}=-\phi_1+\frac{\phi_2}{n_3},\ \varphi_{21}^{D}=\frac{\phi_2}{n_3},\ \varphi_{32}^{D}=\phi_2,\ \varphi_{10}^{E}=\phi_1,\ \varphi_{21}^{E}=\phi_2. \tag{7}$$

$$n_1=\frac{r_1}{r_2},\ n_2=\frac{r_3}{r_4},\ n_3=\frac{r_4}{r_5}.$$

In the design of power transmission mechanisms, it is often necessary to analyze the velocity ratios between their input and output parts and angular velocities or angular accelerations of the intermediate parts.

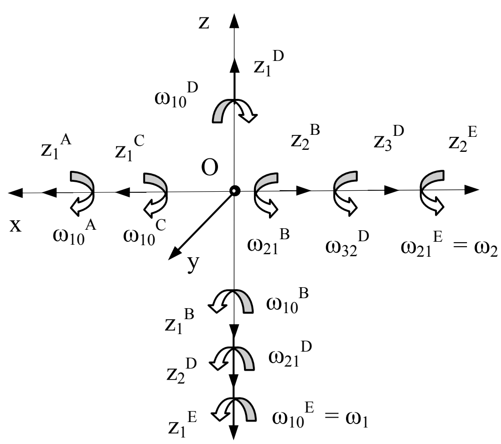

Fig. 4 Moving frames sequence

The analysis of the kinematics of bevel-gear wrist mechanisms of gyroscopic structure is very complex, due to the fact that the carriers and planet gears may possess simultaneous angular velocities about nonparallel axes. The conventional tabular or analytical method, which concentrates on planar epicyclical gear trains, is no longer applicable. To overcome this difficulty, Freudenstein, Longman and Chen [11] applied the dual relative velocity and dual matrix of transformation for the analysis of epicyclical bevel-gear trains. Tsai, Chen and Lin [12], Chang and Tsai [13] and Hedman [14] showed that the kinematic analysis of geared robotic mechanisms can be accomplished by applying the theory of fundamental circuits.

Since a kinematic chain is an assemblage of links and joints, these can be symbolized in a more abstract form known as an equivalent graph representation (Fig. 5). For the reason that will be clear later we use the *associated graph* to represent the topology of the mechanism. In the kinematic graph representation we denote the links by vertices and the joints by edges (Yan and Hsieh [15], [16]). Two small concentric circles label the vertex denoting the fixed forearm 0.

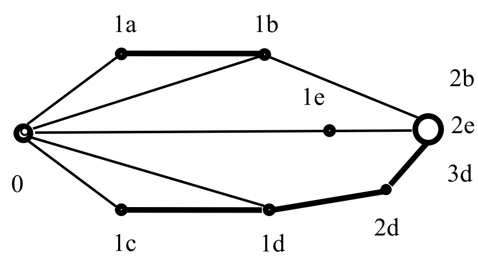

Fig. 5 Associated graph of the mechanism

To distinguish the difference between the pair types, the gear pairs $1a$ - $1b$, $1c$ - $1d$, $1d$ - $2d$ - $3d$ are drawn by thick edges and the revolute joints 0 - $1a$, 0 - $1b$, 0 - $1c$, 0 - $1d$, $1b$ - $2b$ by thin edges. There are two significant independent loops, five serial kinematic chains and we identify the end-effector $2b = 3d = 2e$ and one *fictitious* carrier $2d$.

The kinematics of an element for each circuit is characterized by skew-symmetric matrices given by the recursive relations [17], [18]:

$$\tilde{\omega}_{k0}^{i} = p_{k,k-1}\tilde{\omega}_{k-1,0}^{i}p_{k,k-1}^{T} + \tilde{\omega}_{k,k-1}^{i}$$
$$\tilde{\omega}_{k,k-1}^{i} = \dot{\varphi}_{k,k-1}^{i}\tilde{u}_{3}, (p = a,b,c,d,e), (i = A, B, C, D, E), \tag{8}$$

where $\tilde{u}_3$ is a skew-symmetric matrix associated with the unit vector $\vec{u}_3$. These matrices are *associated* to the angular velocities

$$\vec{\omega}_{k0}^{i} = p_{k,k-1}\vec{\omega}_{k-1,0}^{i} + \vec{\omega}_{k,k-1}^{i}, \vec{\omega}_{k,k-1}^{i} = \dot{\varphi}_{k,k-1}^{i}\vec{u}_{3}. \tag{9}$$

Knowing the rotate motion of the end-effector by the relations (5), one develops the inverse kinematic problem and determines the velocities $\vec{v}_{k0}^{i}, \vec{\omega}_{k0}^{i}$ and accelerations $\vec{\gamma}_{k0}^{i}, \vec{\varepsilon}_{k0}^{i}$ of each moving link.

Based on the important remark

$$\omega_{k,k-1} = n_{k+1,k-1}\omega_{k+1,k}\,, \tag{10}$$

the derivatives with respect to time of the relations (7) lead to the relative angular velocities of all links as function of the angular velocities $\omega_1 = \dot{\phi}_1,\ \omega_2 = \dot{\phi}_2$ of the end-effector:

$$\omega_{10}^{A} = \frac{\dot{\phi}_1}{n_1}\,, \omega_{10}^{B} = \dot{\phi}_1\,, \omega_{21}^{B} = \dot{\phi}_2\,, \omega_{10}^{C} = \frac{1}{n_2}(-\dot{\phi}_1 + \frac{\dot{\phi}_2}{n_3})$$

$$\omega_{10}^{D} = -\dot{\phi}_1 + \frac{\dot{\phi}_2}{n_3}\,, \omega_{21}^{D} = \frac{\dot{\phi}_2}{n_3}\,, \omega_{32}^{D} = \dot{\phi}_2\,, \omega_{10}^{E} = \dot{\phi}_1\,, \omega_{21}^{E} = \dot{\phi}_2\,. \tag{11}$$

Starting from these results, a complete expression of the Jacobian of the mechanism is easily written in an invariant form. This square invertible matrix is an essential element for the analysis of singularity loci into robot workspace.

Let us assume now that the mechanism has successively two independent virtual motions. Characteristic virtual velocities expressed as function of robot's position are given by the relations (11). First, we consider the following input angular velocities $\omega_{10a}^{Av} = 1\,, \omega_{10a}^{Cv} = 0$ and we obtain a set of virtual velocities:

$$\omega_{10a}^{Bv} = n_1\,,\ \omega_{21a}^{Bv} = n_1 n_3$$

$$\omega_{10a}^{Dv} = 0\,,\ \omega_{21a}^{Dv} = n_1\,,\ \omega_{32a}^{Dv} = n_1 n_3\,,\ \omega_{10a}^{Ev} = n_1\,,\ \omega_{21a}^{Ev} = n_1 n_3\,. \tag{12}$$

A second virtual motion is defined by the input velocities $\omega_{10c}^{Cv} = 1\,, \omega_{10c}^{Av} = 0$ and the following results:

$$\omega_{10c}^{Bv} = 0\,,\ \omega_{21c}^{Bv} = n_2 n_3$$

$$\omega_{10c}^{Dv} = n_2\,,\ \omega_{21c}^{Dv} = n_2\,,\ \omega_{32c}^{Dv} = n_2 n_3\,,\ \omega_{10c}^{Ev} = 0\,,\ \omega_{21c}^{Ev} = n_2 n_3\,. \tag{13}$$

Concerning the relative angular accelerations of the compounding elements of the mechanism, these are immediately given by deriving the relations of the velocities (9): $\varepsilon_{k,k-1}^{i} = \dot{\omega}_{k,k-1}^{i}$.

The angular accelerations $\vec{\varepsilon}_{k0}^{\,i}$ and the useful square matrices $\tilde{\omega}_{k0}^{i}\tilde{\omega}_{k0}^{i} + \tilde{\varepsilon}_{k0}^{i}$ are calculated with the following formulae [19], [20], [21]:

$$\vec{\varepsilon}_{k0}^{\,i} = p_{k,k-1}\vec{\varepsilon}_{k-1,0}^{\,i} + \varepsilon_{k,k-1}^{i}\vec{u}_3 + \omega_{k,k-1}^{i} p_{k,k-1}\tilde{\omega}_{k-1,0}^{i} p_{k,k-1}^{T}\vec{u}_3$$

$$\tilde{\omega}_{k0}^{i}\tilde{\omega}_{k0}^{i} + \tilde{\varepsilon}_{k0}^{i} = p_{k,k-1}\left(\tilde{\omega}_{k-1,0}^{i}\tilde{\omega}_{k-1,0}^{i} + \tilde{\varepsilon}_{k-1,0}^{i}\right)p_{k,k-1}^{T} +$$

$$+ \omega_{k,k-1}^{i}\omega_{k,k-1}^{i}\tilde{u}_3\tilde{u}_3 + \varepsilon_{k,k-1}^{i}\tilde{u}_3 + 2\omega_{k,k-1}^{i} p_{k,k-1}\tilde{\omega}_{k-1,0}^{i} p_{k,k-1}^{T}\tilde{u}_3. \tag{14}$$

The velocity $\vec{v}_k^{\,Ci}$ and the acceleration $\vec{\gamma}_k^{\,Ci}$ of mass centre of the T_k^i rigid body are calculated from two basic matrix relations

$$\vec{v}_k^{\,Ci} = \tilde{\omega}_{k0}^{i}\vec{r}_k^{\,Ci}\,,\ \vec{\gamma}_k^{\,Ci} = \{\tilde{\omega}_{k0}^{i}\tilde{\omega}_{k0}^{i} + \tilde{\varepsilon}_{k0}^{i}\}\vec{r}_k^{\,Ci}\,. \tag{15}$$

For simulation purposes let us consider a mechanism which has the following characteristics

$$
\begin{aligned}
&l_1 = 0.03\,\text{m}\,,\ l_2 = 0.055\,\text{m}\\
&r_1 = 0.025\,\text{m}\,,\ r_2 = 0.04\,\text{m}\\
&r_3 = 0.02\,\text{m}\,,\ r_4 = 0.035\,\text{m}\,,\ r_5 = 0.015\,\text{m}\\
&m_1 = 0.25\,\text{kg}\,,\ m_2 = 0.4\,\text{kg}\\
&m_3 = 0.2\,\text{kg}\,,\ m_4 = 0.35\,\text{kg}\,,\ m_5 = 1\,\text{kg} \qquad (16)\\
&\phi_1^* = \frac{\pi}{4}, \phi_2^* = \pi\,,\ M_r = 0.01\ \text{Nm}\,,\ \Delta t = 6\ \text{s}\,.
\end{aligned}
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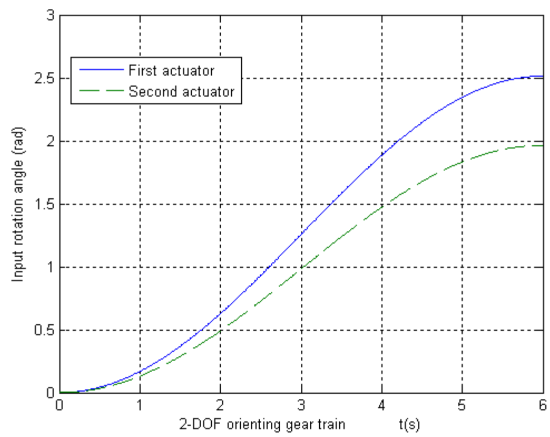


Fig. 6 Input rotation angles $\varphi_{10}^A, \varphi_{10}^C$ of two actuators

A software program which implements the suggested algorithm is developed in MATLAB to solve, first, the inverse kinematics of the orienting gear train. For illustration, it is assumed that for a period of six second the end-effector starts from rest, its initial position, and is moving on a known rotation motion. A numerical study of the robot kinematics is carried out by computation of the input angles of rotation φ_{10}^A, φ_{10}^C (Fig. 6) of the two revolute actuators.

3 Equations of motion

3.1. *Principle of virtual work*

Two torques of moment $\vec{m}_{10}^A = m_{10}^A \vec{u}_3$, $\vec{m}_{10}^C = m_{10}^C \vec{u}_3$ control by intermediate of electric motors the motion of the orienting gear train. The derivation of a dynamic model has a very important effect in the determination of the actuator torques.

In the inverse dynamic problem, in the present paper, one applies the principle of virtual work in order to establish some recursive matrix relations for the torques and the powers of the active systems.

The parallel mechanism can be artificially transformed in a set of five open serial chains C_i $(i = A, B, C, D, E)$ subjected to the constraints. This is possible by removing successively the joints for the end-effector and taking their effects into account by introducing the corresponding constraint conditions.

Considering that the end-effector motion is given, the position, angular velocity, angular acceleration as well as the velocity and acceleration of the centre of mass are known for each element. The force of inertia of an arbitrary rigid body T_k^A, for example

$$\vec{f}_{k0}^{\,inA} = -m_k^A \left[\vec{\gamma}_{k0}^A + \left(\tilde{\omega}_{k0}^A \tilde{\omega}_{k0}^A + \tilde{\varepsilon}_{k0}^A \right) \vec{r}_k^{\,CA} \right] \tag{17}$$

and the resulting moment of the forces of inertia

$$\vec{m}_{k0}^{\,inA} = -[m_k^A \tilde{r}_k^{CA} \vec{\gamma}_{k0}^A + \hat{J}_k^A \vec{\varepsilon}_{k0}^A + \tilde{\omega}_{k0}^A \hat{J}_k^A \vec{\omega}_{k0}^A] \tag{18}$$

are determined with respect to the common centre of rotation O. On the other hand, the wrench of two vectors $\vec{f}_k^{\,*A}$ and $\vec{m}_k^{\,*A}$ evaluates the influence of the action of the external and internal forces applied to the same element T_k^A or of its weight $m_k^A \vec{g}$, for example:

$$\vec{f}_k^{\,*A} = 9.81 m_k^A a_{k0} \vec{u}_3 \,,\; \vec{m}_k^{\,*A} = 9.81 m_k^A \tilde{r}_k^{CA} a_{k0} \vec{u}_3 \;\; (k = 1,2,...,5)\,. \tag{19}$$

Finally, two recursive relations generate the vectors [22], [23]

$$\begin{aligned} \vec{F}_k^A &= \vec{F}_{k0}^A + a_{k+1,k}^T \vec{F}_{k+1}^A \\ \vec{M}_k^A &= \vec{M}_{k0}^A + a_{k+1,k}^T \vec{M}_{k+1}^A + \tilde{r}_{k+1,k}^A a_{k+1,k}^T \vec{F}_{k+1}^A \,, \end{aligned} \tag{20}$$

where

$$\vec{F}_{k0}^A = -\vec{f}_{k0}^{\,inA} - \vec{f}_k^{\,*A} \,,\; \vec{M}_{k0}^A = -\vec{m}_{k0}^{\,inA} - \vec{m}_k^{\,*A} \,. \tag{21}$$

In the context of the real-time control, neglecting the frictional forces and considering the gravitational effect and the action of a resistant torque M_r through the vectors $\vec{f}_{2r}^{\,*E} = \vec{0}$, $\vec{m}_{2r}^{\,*E} = -M_r \vec{u}_3$, the relevant objective of a dynamic model is to determine the input torques, which must be exerted by the actuators in order to produce a given trajectory of the end-effector.

The fundamental principle of virtual work states that a mechanism is under dynamic equilibrium if and only if the virtual work developed by all external, internal and inertia forces vanish during any general virtual displacement, which is compatible with the constraints imposed on the mechanism. Applying *the fundamental equations of parallel robots dynamics* [24], [25], following compact matrix relations, it results

$$m_{10}^A = \vec{u}_3^T \left[\omega_{10a}^{Av} \vec{M}_1^A + \omega_{10a}^{Bv} \vec{M}_1^B + \omega_{10a}^{Cv} \vec{M}_1^C + \omega_{10a}^{Dv} \vec{M}_1^D + \omega_{10a}^{Ev} \vec{M}_1^E + \omega_{21a}^{Ev} \vec{M}_2^E \right] \tag{22}$$

for the torque of first actuator, and

$$m_{10}^{C}=\vec{u}_{3}^{T}\left[\omega_{10c}^{Av}\vec{M}_{1}^{A}+\omega_{10c}^{Bv}\vec{M}_{1}^{B}+\omega_{10c}^{Cv}\vec{M}_{1}^{C}+\omega_{10c}^{Dv}\vec{M}_{1}^{D}+\omega_{10c}^{Ev}\vec{M}_{1}^{E}+\omega_{21c}^{Ev}\vec{M}_{2}^{E}\right] \qquad (23)$$

for the torque of second actuator.

The relations (20), (22) and (23) represent the *inverse dynamic model* of the 2-DOF orienting gear train. The procedure leads to very good estimates of the actuators torques for given displacement of end-effector, provided that the inertial properties of the gears are known with sufficient accuracy and that friction is not significant. This new dynamic approach developed here can be extended to any gyroscopic bevel-gear train with revolute actuators.

3.2. *Equations of Lagrange*

A solution of the dynamics problem of the orienting gear train can be developed based on the Lagrange equations of second kind. The generalized coordinates of the robot are represented by the rotation angles of the two actuators: $q_1=\varphi_{10}^{A}$, $q_2=\varphi_{10}^{C}$.

The Lagrange equations are expressed by two differential relations

$$\frac{d}{dt}\{\frac{\partial E}{\partial \dot{q}_j}\}-\frac{\partial E}{\partial q_j}=Q_j \quad (j=1,2), \qquad (24)$$

that contain the following generalized forces

$$Q_1=m_{10}^{A}-M_r n_1 n_3\,,\; Q_2=m_{10}^{C}-M_r n_2 n_3\,. \qquad (25)$$

The components of the general expression of the total kinetic energy $E=E_A+E_B+E_C+E_D+E_E$ are expressed as analytical functions of first derivatives with respect to time of the generalized coordinates:

$$E_A=\frac{1}{2}\vec{\omega}_{10}^{AT}\hat{J}_1\vec{\omega}_{10}^{A}\,, E_B=\frac{1}{2}\vec{\omega}_{10}^{BT}\hat{J}_2\vec{\omega}_{10}^{B}$$
$$E_C=\frac{1}{2}\vec{\omega}_{10}^{CT}\hat{J}_3\vec{\omega}_{10}^{C}\,, E_D=\frac{1}{2}\vec{\omega}_{10}^{DT}\hat{J}_4\vec{\omega}_{20}^{D} \qquad (26)$$
$$E_E=\frac{1}{2}\vec{\omega}_{20}^{ET}\hat{J}_5\vec{\omega}_{20}^{E}\,,$$

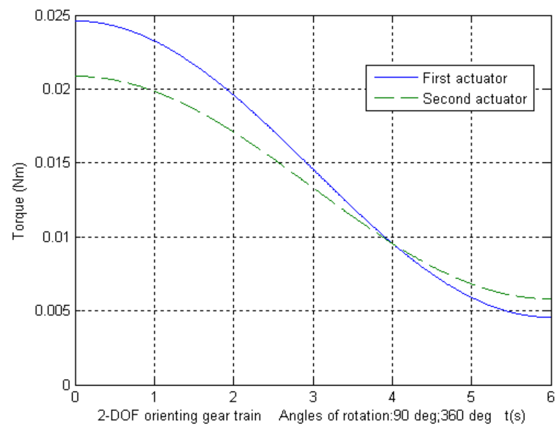
where the absolute angular velocities have the expressions:

$$\vec{\omega}_{10}^{A}=\dot{q}_1\vec{u}_3\,,\vec{\omega}_{10}^{B}=\dot{q}_1 n_1\vec{u}_3$$
$$\vec{\omega}_{10}^{C}=\dot{q}_2\vec{u}_3\,,\vec{\omega}_{10}^{D}=\dot{q}_2 n_2\vec{u}_3 \qquad (27)$$
$$\vec{\omega}_{20}^{E}=-\dot{q}_1 n_1\vec{u}_1+(n_1\dot{q}_1+n_2\dot{q}_2)n_3\vec{u}_3\,.$$

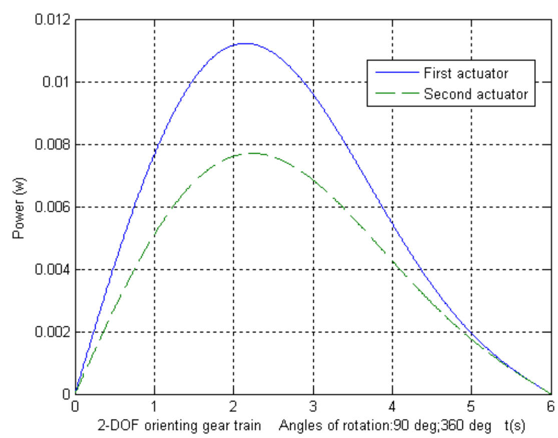
In the inverse dynamics problem, a long calculus of the derivatives with respect to time $\frac{d}{dt}\{\frac{\partial E}{\partial \dot{q}_j}\}$ $(j=1,2)$ of all above functions leads finally to the same expressions (22), (23) for the input torques m_{10}^{A}, m_{10}^{C} required by the actuators, now given as analytical solutions:

$$m_{10}^{A} = n_1 n_3 M_r + \{\frac{J_{1z}}{n_1} + J_{2z} n_1 + J_{5z} n_1\}\ddot{\phi}_1 + J_{5z} n_1 n_3 \ddot{\phi}_2$$

$$m_{10}^{C} = n_2 n_3 M_r - \{\frac{J_{3z}}{n_2} + J_{4z} n_2\}\ddot{\phi}_1 + \{\frac{J_{3z}}{n_2 n_3} + \frac{J_{4z} n_2}{n_3} + J_{5z} n_2 n_3\}\ddot{\phi}_2 . \quad (28)$$



Fig, 7 Input torques m_{10}^{A} , m_{10}^{C} of the two actuators



Fig, 8 Input powers P_{10}^{A} , P_{10}^{C} of the two actuators

Based on the algorithm derived from the above relations (22), (23) or (28), a computer program solves the inverse dynamics modeling of the robot, using the MATLAB software. Assuming that a resistant torque of constant moment $M_r = 0.01$ Nm applied at the end-effector and the weights $m_k^A \vec{g}$ of compounding rigid bodies constitute the external forces acting on the mechanism during its evolution, a numerical computation in the dynamics is developed, based on the determination of the two input torques m_{10}^{A}, m_{10}^{C} (Fig. 7) and their active powers $P_{10}^{A}=\omega_{10}^{A}m_{10}^{A}$ and $P_{10}^{C}=\omega_{10}^{C}m_{10}^{C}$ (Fig. 8). The time-history evolution of the torques and powers required by two active systems are shown for a period of six second of motion.

4 Conclusions

Within the inverse kinematics analysis, some exact matrix relations giving the position, velocity and acceleration of each link for a 2-DOF orienting gear train have been established.

Based on the principle of virtual work, the new approach described above is very efficient and establishes a direct recursive determination of the variation in real-time of torques and powers of the actuators. The matrix relations, given by this dynamic simulation, can be transformed in a model for automatic control of the gear mechanism.

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