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$\varphi - n -$APPROXIMATE WEAK AMENABILITY OF ABSTRACT SEGAL ALGEBRAS

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In this paper, we investigate $\varphi - n -$approximately weakly amenability and character inner amenability of abstract Segal algebra. Let B be an abstract Segal algebra in a Banach algebra A with a central approximate identity which is bounded in $\|.\|_A$. Suppose that $\varphi \in Hom(A)$ is such that φ_B $(\varphi|_B)$ is in $Hom(B)$. We prove that for each $n \in \mathbb{N}$, if A is $\varphi - n -$weakly amenable, then B is $\varphi|_B -$approximately weakly amenable.

Keywords: abstract Segal algebra, $\varphi -$ amenability, Banach algebra.

1. Introduction

A Banach algebra A is called *amenable* if for each Banach A-module X, every bounded derivation from A into a dual A-module X^* is an inner derivation. Recently, some authors have added a kind of twist to the amenability definition. Given a continuous homomorphism φ from A into A, they defined and studied φ*-derivations* and φ*-amenability* (see [3], [7], [18]and [21]).

Suppose that A is a Banach algebra and $\varphi \in Hom(A)$, consisting of all continuous homomorphisms from A into A.

Let X be a Banach A-bimodule, a linear operator $D: A \to X$ is a φ-*derivation* if it satisfies $D(ab) = D(a)\varphi(b) + \varphi(a)D(b)$ for all $a, b \in A$. A φ-derivation D is called a φ*-inner derivation* if there is $x \in X$ such that $D(a) = \varphi(a)x - x\varphi(a)$ for all $a \in A$. Let $\mathcal{Z}_\varphi^1(A, X)$ denote the set of all continuous φ-derivations and $\mathcal{N}_\varphi^1(A, X)$ be the set of all φ-inner derivations from A into X. The first cohomology group $\mathcal{H}_\varphi^1(A, X)$ is defined to be the quotient space $\mathcal{Z}_\varphi^1(A, X)/\mathcal{N}_\varphi^1(A, X)$.

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A Banach algebra A is called φ-*amenable* if $\mathcal{H}^1_\varphi(A, X^*) = \{0\}$ for all A-bimodules X and A is called φ-*weakly amenable* if $\mathcal{H}^1_\varphi(A, A^*) = \{0\}$. Note that every derivation of a Banach algebra A into an A-bimodule X is an id_A-derivation, where id_A is the identity operator on A. A φ-derivation D is called *approximate* φ-*inner derivation* if there is a net (x_α) in X such that $D(a) = \lim_\alpha (\varphi(a).x_\alpha - x_\alpha.\varphi(a))$. A Banach algebra A is called φ-*approximate amenable* if every φ-derivation is an approximate φ-inner derivation.

The aim of the present paper is to introduce and investigate $\varphi - n -$approximately weakly amenability of abstract Segal algebra.

2. The results

We start this section by introducing the following:

Let A be a Banach algebra and X, Y be Banach A-bimodules, then A-bimodule homomorphism from X to Y is a homomorphism $\varphi : X \to Y$ with

$$\varphi(a \cdot x) = a \cdot \varphi(x), \quad \varphi(x \cdot a) = \varphi(x) \cdot a \ \ (a \in A, x \in X).$$

Definition 2.1 *Let A be a Banach algebra with the norm $\|.\|_A$. Then a Banach algebra B with the norm $\|.\|_B$ is an abstract Segal algebra with respect to A if:*

(i) B is a dense left ideal in A;
(ii) there exists $M > 0$ such that $\|b\|_A \leq M\|b\|_B$ for all $b \in B$;
(iii) there exists $C > 0$ such that $\|ab\|_B \leq C\|a\|_A\|b\|_B$ for all $a, b \in B$.

Recall that a net $(e_\alpha)_{\alpha\in I}$ in A is central bounded approximate identity if $ae_\alpha = e_\alpha a \ (a \in A, \alpha \in I)$ and $(e_\alpha)_{\alpha\in I}$ is a bounded approximate identity for A.

Let A be a Banach algebra, the duals $A^{(n)}$ are Banach A-bimodule for each $n \in N$. We take $A^{(0)} = A$ and we denotes the restriction of φ to B by φ_B.

Theorem 2.2 *Let B be an abstract Segal algebra in a Banach algebra A with a central approximate identity which is bounded in $\|.\|_A$.Then for each $n \in \mathbb{N}$, and $\varphi \in Hom(A)$, if A is $\varphi - n -$weakly amenable, then B is $\varphi_B - n -$approximately weakly amenable*

Proof. Let $n \in \mathbb{N}$ and $D: B \to B^{(n)}$ be a continuous $\varphi-$derivation. Let X_n be the closed linear span of the set $\{\varphi_B(a) \cdot B^{(n)} \cdot \varphi_B(b) : a, b \in B\}$. Suppose that (e_α) is a central approximate identity for B. Since (e_α) is central, it follows that

$$D(B) \subseteq X_n.$$

Consequently, (e_α) is a multiplier-bounded, central approximate identity for X_n. In particular

$$\lim_\alpha \varphi_B(e_\alpha^2) \cdot D(b) = D(b) \quad (b \in B)$$

For each α define the map $\tau_\alpha : A \to B$ by

$$\tau_\alpha(a) = \varphi(a)e_\alpha \quad (a \in A).$$

Let $\theta : B \to A$ denote the inclusion map. Trivially, both τ_α and θ are linear and continuous left A-module morphisms and also continuous B-bimodule morphisms. It is clear that $\tau_\alpha\theta(b) = \varphi_B(b)e_\alpha = e_\alpha\varphi_B(b)$ for all $b \in B$, so by induction, for each $n \in \mathbb{N}$ we have

$$(\tau_\alpha\theta)^{(n)}(F) = \varphi_B^{(n)}(F) \cdot e_\alpha = e_\alpha \cdot \varphi_B^{(n)}(F) \quad (F \in B^{(n)}).$$

Define the continuous linear map $D_\alpha : A \to A^{(n)}$ by

$$D_\alpha(a) = \begin{cases} \theta^{(n)}[D(ae_\alpha) - \varphi(a) \cdot D(e_\alpha)] & if \quad n \quad is \quad even \\ \tau_\alpha^{(n)}[D(e_\alpha\, a) - D(e_\alpha) \cdot \varphi(a)] & if \quad n \quad is \quad odd \end{cases} \qquad (*)$$

for all $a \in A$. Thus for each $b \in B$ we have

$$D_\alpha(b) = \begin{cases} \theta^{(n)}(D(b) \cdot \varphi_B(e_\alpha)) & if \quad n \quad is \quad even \\ \tau_\alpha^{(n)}(\varphi_B(e_\alpha).D(b)) & if \quad n \quad is \quad odd \end{cases} \qquad (**)$$

Since (e_α) is central, it follows that $D_\alpha(bc) = \varphi_B(b) \cdot D_\alpha(c) + D_\alpha(b) \cdot \varphi_B(c)$ for all $b, c \in B$.

From the density of B in A, it follows that D_α is a $\varphi-$derivation from A into $A^{(n)}$. By assumption there exists $F_\alpha \in A^{(n)}$ such that

$$D_\alpha(a) = \varphi(a) \cdot F_\alpha - F_\alpha \cdot \varphi(a) \quad (a \in A).$$

In particular, by equation $(**)$ for each even $n \in \mathbb{N}$ we have

$$\begin{aligned} D(b) \cdot \varphi_B(e_\alpha^2) &= (\tau_\alpha\theta)^{(n)}(D(b) \cdot \varphi_B(e_\alpha)) \\ &= (\tau_\alpha)^{(n)} D_\alpha(b) \\ &= \varphi_B(b) \cdot (\tau_\alpha)^{(n)}(F_\alpha) - (\tau_\alpha)^{(n)}(F_\alpha) \cdot \varphi_B(b), \end{aligned}$$

and for every odd $n \in \mathbb{N}$ we have

$$\begin{aligned}\varphi_B(e_\alpha^2)\cdot D(b) &= (\tau_\alpha\theta)^{(n)}(\varphi_B(e_\alpha)\cdot D(b))\\ &= (\tau_\alpha)^{(n)} D_\alpha(b)\\ &= (\theta)^{(n)} D_\alpha(b)\\ &= \varphi_B(b)\cdot(\theta)^{(n)}(F_\alpha)-(\theta)^{(n)}(F_\alpha)\cdot\varphi_B(b),\end{aligned}$$

for all $b\in B$. For each even n put $G_\alpha=\tau_\alpha^{(n)}(F_\alpha)$ and for every odd n put $G_\alpha=\theta^{(n)}(F_\alpha)$ if n is odd. Trivially, $G_\alpha\in A^{(n)}$ for all α. By (*) we have

$$D(b)=\lim_\alpha \varphi_B(e_\alpha^2)\cdot D(b)=\lim_\alpha \varphi_B(b)\cdot G_\alpha - G_\alpha\cdot\varphi_B(b)\ \ (b\in B).$$

That is, D is φ_B −approximately inner. □

Before we present our next result we recall from [14] that a linear subspace $S^1(G)$ of the convolution group algebra $L^1(G)$ is said to be a Segal algebra on G if it satisfies the following conditions.

(i) $S^1(G)$ is dense in $L^1(G)$.

(ii) $S^1(G)$ is a Banach space under some norm $\|.\|_S$ and for each $f\in S^1(G)$

$$\|f\|_1\le\|s\|_S$$

(iii) $S^1(G)$ is left translation invariant and the map $x\mapsto\delta_x * f$ from G into $S^1(G)$ is continuous.

(iv) $\|\delta_x * f\|_S=\|f\|_S$ for all $f\in S^1(G)$ and $x\in G$.

That every Segal algebra is an abstract Segal algebra with respect to $L^1(G)$ but not conversely; see [22].

Corollary 2.3 *Let $S(G)$ be a Segal algebra on a locally compact SIN group G .Then for each $n\in\mathbb{N}$ and $\varphi\in Hom(L^1(G))$ with $\varphi(S(G))\subseteq S(G)$. If $L^1(G)$ is $\varphi-n-$weakly amenable, then $S(G)$ is approximately $\varphi|_{S(G)}-n-$weakl-y amenable*

Proof. Since G is a SIN group, then by a result of [17], $S(G)$ has a central approximate identity which is bounded in $\|.\|_1$. The result is now obvious from Theorem 2.2. □

Proposition 2.4 *Suppose* A *is a commutative Banach algebra which is generated by the idempotent elements in* A *and* φ is in $Hom(A)$; *then* A *is* $\varphi - n -$*weak amenable.*

Proof. Let P denote the set of all idempotent elements in A. Assume that $D: A \to A^{(n)}$ is a $\varphi-$derivation. For any $p \in P, D(p) = D(p^2) = D(p^3)$ and $D(p^2) = 2\varphi(p)D(p), D(p^3) = 3\varphi(p^2)D(p)$, so $D(p) = 0$. Since $p \in P$ is arbitrary and A is generated by P, it follows that $D(a) = 0$ for all $a \in A$. So, A is $\varphi - n -$weak amenable. □

We recall is called a *character* on A is a non-zero homomorphism from A into the scalar field. The set of all characters on A the character space of A, is denoted by Φ_A. The kernel of $\varphi \in \Phi_A \cup \{0\}$ is denoted by M_φ. Let $\varphi \in \Phi_A \cup \{0\}$. A linear functional d on A is called a point derivation at φ if

$$d(ab) = d(a)\varphi(b) + \varphi(a)d(b), (a, b \in A).$$

Proposition 2.5 *Let* A *be a Banach algebra ,* $\varphi \in \Phi_A$ *and* θ *be an idempotent homomorphism on* A. *Let* A *be* $\theta-$*approximately weak amenable and* $Im\theta \bigcap \ker\varphi = \varnothing$. *Then there are no non-zero, continuous point derivations at* φ.

Proof. Let d be a continuous point derivation of A at $\varphi \in \Phi_A$. Then the mapping $D: A \to A^*$ given by $D(a) = d(a)(\varphi \circ \theta), (a \in A)$ is a $\theta-$ derivation, since

$$\begin{aligned} D(ab) &= d(a)\varphi(b)(\varphi \circ \theta) + \varphi(a)d(b)(\varphi \circ \theta) \\ &= d(a)(\varphi \circ \theta) \cdot \theta(b) + \theta(a) \cdot d(b)(\varphi \circ \theta), (a, b \in A). \end{aligned}$$

Since A is $\theta-$approximately weak amenable, so D is $\theta-$approximately inner. Thus, there exists a net $(\theta_\alpha) \subset A^*$ such that for every $a \in A$,

$$D(a) = \lim_\alpha (\theta(a) \cdot \theta_\alpha - \theta_\alpha \cdot \theta(a)).$$

For every $a \in A$ we have

$$D(a)(\theta(a)) = \lim_\alpha (\theta(a) \cdot \theta_\alpha - \theta_\alpha \cdot \theta(a))(\theta(a)) = \lim_\alpha \theta_\alpha(\theta(a^2) - \theta(a^2)) = 0,$$

since $D(a)(a)=D(a)(\theta(a))\ (a\in A)$, $D(a)(a)=0$ and so $d\,|_{A/M_\varphi}=0$. Thus $d=0$.
□

Recently Jabbari et al. [11] have introduced the notation of φ-inner amenability $(\varphi\in\Phi_A)$. A Banach algebra A is said to be φ-inner amenable if there exists $m\in A^{**}$ satisfying $m(\varphi)=1$ and $m(f\cdot a)=m(a\cdot f)(f\in A^*, a\in A)$. Such m will sometimes be referred as a φ-inner mean, and A is said to be character inner amenable if A is φ-inner amenable for every $\varphi\in\Phi_A$. They also gave several characterizations of φ-inner amenability. For instance, as in the case of φ-amenability in [14, Theorem 1.4], they proved that a φ-inner mean is in fact some w^*-cluster point of a bounded net $(a_\alpha)\in A$ satisfying $\|a_\alpha a-aa_\alpha\|\to 0$, for all $a\in A$ and $\varphi(a_\alpha)=1$ for all α; [15, Theorem 2.1]. In this section we investigate character inner amenability of abstract Segal algebras.

Theorem 2.6 *Let A be a Banach algebra and let B be an abstract Segal algebra with respect to A. Suppose that there exists $b_0\in B$ such that $bb_0=b_0b$ for every $b\in B$. Then A is φ-inner amenable $(\varphi\in\Phi_A)$ if and only if B is $\varphi|_B$-inner amenable*

Proof. Suppose that A is φ-inner amenable. Then there is a bounded net $(a_\alpha)\in A$ such that $\|a_\alpha a-aa_\alpha\|_A\to 0$, for all $a\in A$ and $\varphi(a_\alpha)=1$ for all α. We may assume that $\varphi(b_0)=1$. For every α we put

$$b_\alpha := a_\alpha b_0\in B.$$

Since B is an abstract Segal algebra with respect to A, there exists $C>0$ such that for each $b\in B$,

$$\begin{aligned}\|b_\alpha b-bb_\alpha\|_B &= \|(a_\alpha b-ba_\alpha)b_0\|_B\\ &\le C\|a_\alpha b-ba_\alpha\|_A\|b_0\|_B\to 0,\end{aligned}$$

and

$$\varphi(b_\alpha)=\varphi(a_\alpha)=1.$$

Since (a_α) is $\|.\|_A$ −bounded, it follows that (b_α) is $\|.\|_B$ −bounded. Thus B is $\varphi|_B$-inner amenable.

Conversely Suppose that B is $\varphi|_B$-inner amenable. Then there is a bounded net $(b_\alpha)\in B$ such that $\|b_\alpha b-bb_\alpha\|_B\to 0$, for all $b\in B$ and $\varphi(b_\alpha)=1$ for all α. We can assume that $\varphi(b_0)=1$. Define

$$a_\alpha := b_0 b_\alpha$$

for all α. Since B is a dense left ideal in A, $ab_0 = b_0 a$ for every $a \in A$ and so B is an abstract Segal algebra with respect to A. Therefore there there exists $M > 0$ such that for each $b \in B$,

$$\|a_\alpha a - aa_\alpha\|_A = \|b_\alpha(ab_0) - (ab_0)b_\alpha)\|_A$$
$$\leq M\|b_\alpha(ab_0) - (ab_0)b_\alpha)\|_B \to 0,$$

and for every α

$$\varphi(a_\alpha) = \varphi|_B (b_\alpha) = 1.$$

Since $\|.\|_A \leq M\|.\|_B$, it follows that (a_α) is $\|.\|_A -$bounded. Therefor A is φ-inner amenable. □

As a consequence of Theorem 2.6 we have the following result.

Corollary 2.7 *Let B be an abstract Segal algebra with respect to a character inner amenable Banach algebra A. If there exists $b_0 \in B$ such that $bb_0 = b_0 b$ for every $b \in B$ then for all $\varphi \in \Phi_B$, B is φ-inner amenable*

Corollary 2.8 *Let B be an abstract Segal algebra with respect to a Banach algebra A with a central approximate identity. Then A is φ-inner amenable $(\varphi \in \Phi_A)$ if and only if B is $\varphi|_B$-inner amenable.*

Proof. Suppose that $(e_\alpha)_{\alpha \in I}$ is a central approximate identity for B. Fix $\alpha_0 \in I$. Since (e_α) is central, it follows that $e_{\alpha_0} b = be_{\alpha_0}$ $(b \in B)$. Now an application of Theorem 2.6 completes the proof. □

Proposition 2.9 *Let A be a Banach algebra and $\varphi \in \Phi_A$. If* $\ker\varphi$ *has a central approximate identity, then A is φ-inner amenable.*

Proof. Choose $a_0 \in A$ such that $\varphi(a_0) = 1$. Let $(e_\alpha)_{\alpha \in I}$ be a central approximate identity for $\ker\varphi$. Set $a_\alpha = a_0 - a_0 e_\alpha$. Then, for every $b \in \ker\varphi$

$$\|a_\alpha b - ba_\alpha\| = \|a_0 b - a_0 e_\alpha b - ba_0 + ba_0 e_\alpha\|$$
$$\leq \|ba_0 - ba_0 e_\alpha\| + \|a_0\|\|b - be_\alpha\| \to 0,$$

so for every $a \in A \setminus \ker\varphi$ we have $a_0 a - aa_0 \in \ker\varphi$ and

$$\|a_\alpha a - aa_\alpha\| = \|a_0 a - a_0 e_\alpha a - aa_0 + aa_0 e_\alpha\|$$

$$= \|(a_0 a - a a_0) - (a_0 a - a a_0) e_\alpha\| \to 0.$$

It follows that, $\varphi(a_\alpha) = 1$ and $\|a_\alpha a - a a_\alpha\| \to 0$, for all $a \in A$. Thus A is φ-inner amenable. □

Let A be a Banach algebra. We recall from [6] that a left multiplier on A is an element L in $L(A)$ (linear maps on A) such that $L(ab) = L(a)b, (a, b \in A)$ and a right multiplier on A is an element R in $L(A)$ such that $R(ab) = aR(b)$. A multiplier is a pair (L, R) where L and R are left and right multipliers on A, respectively, and $aL(b) = R(a)b \ (a, b \in A)$.

Proposition 2.10 *Let A be a character inner amenable Banach algebra. If there exists $b_0 \in A$ such that $R(b_0^2)a = aR(b_0^2)$ for every $a \in A$ and R is of closed range, then for each $\varphi \in \Phi_{R(A)}$ the Banach algebra $R(A)$ is φ-inner amenable.*

Proof. For arbitrary $\varphi \in \Phi_{R(A)}$ we can choose $\varphi(R(b_0^2)) = 1$. Define the linear functional $\tilde{\varphi}$ on A by $\tilde{\varphi}(a) := \varphi(aR(b_0^2)) \ (a \in A$. It is clear that $\tilde{\varphi}$ defines a non-zero multiplicative linear functional on A whose definition is independent of b_0. That is $\tilde{\varphi} \in \Phi_A$. As we mentioned in preliminaries, the $\tilde{\varphi}-$ amenability of A implies that there exist a net $(u_\alpha)_{\alpha \in I}$ in A such that $\tilde{\varphi}(u_\alpha) = 1$ for all $\alpha \in I$, and $\|u_\alpha a - a u_\alpha\| \to 0$ for each $a \in A$. Now for each $\alpha \in I$, set $v_\alpha := u_\alpha R(b_0^2)$. So we have $\varphi(v_\alpha) = 1$ and for each $a \in A$

$$\|v_\alpha R(a) - R(a) v_\alpha\| \le \|R(b_0^2)\| \|u_\alpha R(a) - R(a) u_\alpha\| \to 0.$$

This complete the proof. □

A similar argument is also valid for a left multiplier L on A.

Corollary 2.11 *Let A be a character amenable Banach algebra. If there exists $b_0 \in A$ such that $R(b_0^2)a = aR(b_0^2)$ for every $a \in A$ and R is of closed range, then for each $\varphi \in \Phi_{R(A)}$ the Banach algebra $R(A)$ is φ-inner amenable.*

Proof. Suppose that A is character amenable. Then A has a bounded approximate identity and so by [7, Corollary 2.2] A is a character inner amenable. By Proposition 2.10, $R(A)$ is φ-inner amenable for each $\varphi \in \Phi_{R(A)}$. □

In the case where A is commutative, Proposition 2.10 given the following result.

Corollary 2.12 *Let A be a commutative character inner amenable Banach algebra and suppose that* $R: A \to A$ *is of closed range. Then for each* $\varphi \in \Phi_{R(A)}$ *the Banach algebra* $R(A)$ *is* φ*-inner amenable.*

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