A CHEBYSHEVE COLLOCATION METHOD FOR THE SOLUTION OF HIGHER-ORDER FREDHOLM-VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS SYSTEM

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A numerical approximation method for the solution of Fredholm-Volterra Integro-Differential Equations (FVIDEs) system is presented. A Chebysheve collocation points together the Shannon approximation is proposed to transform FVIDEs system to an algebraic system. The convergence analysis of the proposed scheme is derived. Finally, the reliability and efficiency of the method are demonstrated by some numerical experiments.

Keywords: Chebysheve collocation points, Fredholm-Volterra Integro-Differential Equations, Shannon wavelets, Convergence analysis, Numerical treatment.

1. Introduction

Integral Equations (*IEs*) and Integro-Differential Equations (*IDEs*) arise in many mathematical modeling processes, e.g. control theory, financial mathematics, dynamic processes in chemical reactors, population dynamics, electromagnetics, heat conduction, viscoelasicity and many other practical applications [1, 2, 3].

Recently, significant progress has been made in numerical methods based on the Shannon wavelets. These methods are some of those very much successful methods for numerical solution of ODEs, PDEs and IEs.

Several authors have investigated the numerical solvability of IDEs system and other related equations. Ebadi et al. [4] in 2009, solved system of nonlinear Volterra Integro-Differential Equations by using the Tau method. Cerdik-Yaslan and Akyuz-Dascioglu in [5] used Chebysheve polynomial to solve nonlinear form of FVIDEs system. In [6] Maleknejad et al. presented a method for solving FVIDEs based on the Bernstein operational matrix. Numerical solution of Fredholm Integro-Differential Equations by the Tau method has been proposed by Pour-Mahmoud et al. in [7]. In [2] Yusufoglu presented a new quadrature method for solving FVIDEs system.

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In this paper we consider Fredholm-Volterra Integro-Differential Equations system of the form[8]:

$$\sum_{n=0}^{m}\sum_{j=1}^{l} F_{ij}^{n}(t) f_j^{(n)}(t) = g_i(t) + \int_{-1}^{1}\sum_{j=1}^{l} K_{ij}(t,s) f_j(s)ds + \int_{-1}^{t}\sum_{j=1}^{l} G_{ij}(t,s) f_j(s)ds, \qquad (1)$$

$$i = 1,\dots,l,$$

under the mixed conditions

$$\sum_{n=0}^{m-1}[\alpha_j^n f_j^{(n)}(-1) + \beta_j^n f_j^{(n)}(1) + \delta_j^n f_j^{(n)}(c)] = a_j,\ j = 1,\dots,\ l, -1 \le c \le 1. \qquad (2)$$

Where $f_j(t)$ is an unknown function, the functions $F_{ij}^n(t)$, $K_{ij}(t,s)$ and $G_{ij}(t,s)$ are defined on interval $-1 \le t, s \le 1$ and α_j^n, β_j^n, δ_j^n and a_j are constants.

In this research, we design a numerical algorithm based on the collocation method in Chebysheve points and the connection coefficients of the Shannon approximation. The outline of the paper is as follows. In Section 2, we are briefly introduced the Shannon wavelets family and their basic properties. These preliminaries allow us to design a numerical algorithm, which is implemented in Section 3. In Section 4, the convergence analysis of the proposed method is investigated. Finally, in Section 5 the method is applied to a few test examples to illustrate the accuracy and the implementation of the method.

1. Preliminaries

We begin by recalling the definitions of the Shannon scaling functions and mother wavelets from [9], as follows:

$$\varphi_{j,k}(t) = 2^{j/2} Sinc(2^j t - k) = 2^{j/2}\frac{\sin \pi(2^j t - k)}{\pi(2^j t - k)}, \qquad j,k \in Z,$$

$$\psi_{j,k}(t) = 2^{j/2}\frac{\sin \pi(2^j t - k - \frac{1}{2}) - \sin 2\pi(2^j t - k - \frac{1}{2})}{\pi(2^j t - k - \frac{1}{2})}, \qquad j,k \in Z. \qquad (3)$$

Where the *Sinc* function is defined on the whole real line by:

$$Sinc(t) = \begin{cases} \frac{\sin(\pi t)}{\pi t}, & t \ne 0, \\ 1, & t = 0. \end{cases}$$

Also, the following equations are the Fourier transforms of (3):

$$\begin{cases} \varphi_{j,k}(\omega) = \dfrac{2^{-j/2}}{2\pi} e^{-i\omega k/2^j} \chi(\dfrac{\omega}{2^j} + 3\pi), & j,k \in Z, \\ \psi_{j,k}(\omega) = -\dfrac{2^{-j/2}}{2\pi} e^{-i\omega(k+1/2)/2^j} [\chi(\dfrac{\omega}{2^{j-1}}) + \chi(-\dfrac{\omega}{2^{j-1}})], & j,k \in Z, \end{cases}$$

where the characteristic function $\chi(\omega)$ is defined as:

$$\chi(\omega) = \begin{cases} 1, & 2\pi \le \omega < 4\pi, \\ 0, & otherwise. \end{cases}$$

2. Outline of the method for FVIDEs system

Consider the Fredholm-Volterra Integro-Differential Equation system (1) where the following assumptions are satisfied [9, 10]:

I). Let $\varphi_{0,h}(t)$ and $\psi_{r,k}(t)$ be the Shannon scaling functions and mother wavelets, then:

$$\begin{aligned} &\bullet < \psi_{r,k}(t), \psi_{i,h}(t) >= \delta_{ri}\delta_{hk}, \\ &\bullet < \varphi_{0,k}(t), \varphi_{0,h}(t) >= \delta_{kh}, \\ &\bullet < \varphi_{0,k}(t), \psi_{r,h}(t) >= 0, \qquad r \ge 0. \end{aligned} \tag{4}$$

II). Let $f_j(t)$ be a class of functions such that the following integral exist and finite:

$$\begin{aligned} \eta_{j,k} &=< f_j(t), \varphi_{0,k}(t) >= \int_{-\infty}^{\infty} f_j(t)\varphi_{0,k}(t)dt, \\ \tau_{r,j,k} &=< f_j(t), \psi_{r,k}(t) >= \int_{-\infty}^{\infty} f_j(t)\psi_{r,k}(t)dt. \end{aligned} \tag{5}$$

Remark 1. δ_{ri} is the Kronecker delta.

2.1. Linear function approximation

In order to obtain the numerical solution $f_j(t)$ of (1), we recall the following information from [9,10], that will become instrumental in establishing our useful formulas.

Theorem 1 . *If* $f_j(t) \in L_2(\mathbb{R})$ *, then the series*

$$\sum_{k=-\infty}^{\infty} \eta_{j,k}\varphi_{0,k}(t) + \sum_{r=0}^{\infty}\sum_{k=-\infty}^{\infty} \tau_{r,j,k}\psi_{r,k}(t), \tag{6}$$

converges to $f_j(t)$, with $\eta_{j,k}$ and $\tau_{r,j,k}$ are define in (5).

Definition 1 . *The derivatives of* $f_j(t)$ *are computed in terms of the Shannon wavelets as*

$$f_j^{(n)}(t)=\sum_{k=-\infty}^{\infty}\eta_{j,k}\varphi_{0,k}^{(n)}(t)+\sum_{r=0}^{\infty}\sum_{k=-\infty}^{\infty}\tau_{r,j,k}\psi_{r,k}^{(n)}(t). \tag{7}$$

Definition 2 . $\varphi_{0,k}^{(n)}(t)$ *and* $\psi_{r,k}^{(n)}(t)$ *are defined by*

$$\begin{cases}\varphi_{0,k}^{(n)}(t)=\sum_{h=-\infty}^{\infty}\vartheta_{kh}^{(n)}\varphi_{0,h}(t)+\sum_{r=0}^{\infty}\sum_{h=-\infty}^{\infty}\alpha_{kh}^{(n)r}\psi_{r,h}(t),\\ \psi_{r,k}^{(n)}(t)=\sum_{h=-\infty}^{\infty}\beta_{kh}^{(n)r}\varphi_{0,h}(t)+\sum_{r=0}^{N}\sum_{h=-M}^{M}\kappa_{kh}^{(n)rr}\psi_{r,h}(t).\end{cases} \tag{8}$$

Remark 2. α, β, κ and ϑ are defined as the connection coefficients of the Shannon wavelets.

Theorem 2 . *Values of connection coefficients are:*

$$\alpha_{kh}^{(n)r}=\beta_{kh}^{(n)r}=0.$$

$$\vartheta_{kh}^{(n)}=\begin{cases}(-1)^{k-h}\dfrac{i^n}{2\pi}\sum_{s=1}^{n}\dfrac{n!\pi^s}{s![i(k-h)]^{n-s+1}}[(-1)^s-1], & k\neq h,\\ \dfrac{i^n\pi^{n+1}}{2\pi(n+1)}[1+(-1)^n], & k=h.\end{cases}$$

$$\kappa_{kh}^{(n)rr}=\begin{cases}\dfrac{i^n2^{rn}\pi^{n+1}}{2\pi(n+1)}[(1+(-1)^n)(2^{n+1}-1)], & k=h,\\ \dfrac{i^n2^{rn}}{2\pi}\sum_{s=1}^{n}(-1)^n\dfrac{n!\,\pi^s(2^s-1)}{s!\,[i(h-k)]^{n-s+1}}[(-1)^s-1], & k\neq h,\end{cases}$$

Using above statements, we have:

$$f_j(t)=\sum_{k=-\infty}^{\infty}\eta_{j,k}\varphi_{0,k}(t)+\sum_{r=0}^{\infty}\sum_{k=-\infty}^{\infty}\tau_{r,j,k}\psi_{r,k}(t), \tag{9}$$

$$f_j^{(n)}(t)=\sum_{k=-\infty}^{\infty}\eta_{j,k}\sum_{h=-\infty}^{\infty}\vartheta_{kh}^{(n)}\varphi_{0,h}(t)+\sum_{r=0}^{\infty}\sum_{k=-\infty}^{\infty}\tau_{r,j,k}\sum_{h=-\infty}^{\infty}\kappa_{kh}^{(n)rr}\psi_{r,h}(t). \tag{10}$$

2.2. The Chebysheve collocation scheme

This subsection is devoted to applying the Chebysheve collocation method to numerically solve the FVIDEs system. To do so, we consider a collocation method including the Shannon wavelets approximation representation of the equation.

We define a finite truncated series of $f_j(t)$ and $f_j^{(n)}(t)$ by

$$f_j(t) = \sum_{k=-M}^{M} \eta_{j,k}\varphi_{0,k}(t) + \sum_{r=0}^{N}\sum_{k=-M}^{M} \tau_{r,j,k}\psi_{r,k}(t), \tag{11}$$

$$f_j^{(n)}(t) = \sum_{k=-M}^{M} \eta_{j,k} \sum_{h=-M}^{M} \vartheta_{kh}^{(n)}\varphi_{0,h}(t) + \sum_{r=0}^{N}\sum_{k=-M}^{M} \tau_{r,j,k} \sum_{h=-M}^{M} \kappa_{kh}^{(n)rr}\psi_{r,h}(t). \tag{12}$$

Considering the system (1) with respect to (11) and (12), we obtain:

$$\begin{aligned}&\sum_{n=0}^{m}\sum_{j=1}^{l}F_{ij}^{n}(t)[\sum_{k=-M}^{M}\eta_{j,k}\sum_{h=-M}^{M}\vartheta_{kh}^{(n)}\varphi_{0,h}(t)+\sum_{r=0}^{N}\sum_{k=-M}^{M}\tau_{r,j,k}\\ &\sum_{h=-M}^{M}\kappa_{kh}^{(n)rr}\psi_{r,h}(t)] = g_i(t)+\int_{-1}^{1}\sum_{j=1}^{l}K_{ij}(t,s)[\sum_{k=-M}^{M}\eta_{j,k}\varphi_{0,k}(s)+\\ &\sum_{r=0}^{N}\sum_{k=-M}^{M}\tau_{r,j,k}\psi_{r,k}(s)]ds+\int_{-1}^{t}\sum_{j=1}^{l}G_{ij}(t,s)[\sum_{k=-M}^{M}\eta_{j,k}\varphi_{0,k}(s)+\\ &\sum_{r=0}^{N}\sum_{k=-M}^{M}\tau_{r,j,k}\psi_{r,k}(s)]ds,\end{aligned}$$

or equivalently

$$\begin{aligned}&\sum_{j=1}^{l}\sum_{k=-M}^{M}\eta_{j,k}[\sum_{n=0}^{m}F_{ij}^{n}(t)\sum_{h=-M}^{M}\vartheta_{kh}^{(n)}\varphi_{0,h}(t)-\int_{-1}^{1}K_{ij}(t,s)\varphi_{0,k}(s)ds-\\ &\int_{-1}^{t}G_{ij}(t,s)\varphi_{0,k}(s)ds]+\sum_{j=1}^{l}\sum_{r=0}^{N}\sum_{k=-M}^{M}\tau_{r,j,k}[\sum_{n=0}^{m}F_{ij}^{n}(t)\sum_{h=-M}^{M}\kappa_{kh}^{(n)rr}\\ &\psi_{r,k}(t)-\int_{-1}^{1}K_{ij}(t,s)\psi_{r,k}(s)ds-\int_{-1}^{t}G_{ij}(t,s)\psi_{r,k}(s)ds] = g_i(t).\end{aligned} \tag{13}$$

Defining:

$$\begin{aligned}\Upsilon_{j,k}^{i}(t) &= \sum_{n=0}^{m}F_{ij}^{n}(t)\sum_{h=-M}^{M}\vartheta_{kh}^{(n)}\varphi_{0,h}(t)-\int_{-1}^{1}K_{ij}(t,s)\varphi_{0,k}(s)ds\\ &-\int_{-1}^{t}G_{ij}(t,s)\varphi_{0,k}(s)ds,\\ \Omega_{r,j,k}^{i}(t) &= \sum_{n=0}^{m}F_{ij}^{n}(t)\sum_{h=-M}^{M}\kappa_{kh}^{(n)rr}\psi_{r,k}(t)-\int_{-1}^{1}K_{ij}(t,s)\psi_{r,k}(s)ds\\ &-\int_{-1}^{t}G_{ij}(t,s)\psi_{r,k}(s)ds.\end{aligned}$$

Note that, there are $l(2M+1)(N+2)$ unknowns components, so for solving (1) we need $l(2M+1)(N+2)$ equations. Using the given notation and substituting $t = t_x = \cos(\frac{x\pi}{p-1})$, $x = 0,\ldots, p-1$, $p = l(2M+1)(N+2) - ml$, in (13), this relation can be transformed to the
following matrix form:

$$\Gamma_i \mathbf{x} = \mathbf{g_i}, \qquad i = 1,\ldots,l, \tag{14}$$

where

$$\Gamma_i = \begin{bmatrix} \Upsilon^i_{1,-M}(t_0) & \ldots & \Upsilon^i_{l,M}(t_0) & \Omega^i_{0,1,-M}(t_0) & \ldots & \Omega^i_{N,l,M}(t_0) \\ \Upsilon^i_{1,-M}(t_1) & \ldots & \Upsilon^i_{l,M}(t_1) & \Omega^i_{0,1,-M}(t_1) & \ldots & \Omega^i_{N,l,M}(t_1) \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \Upsilon^i_{1,-M}(t_{p-2}) & \ldots & \Upsilon^i_{l,M}(t_{p-2}) & \Omega^i_{0,1,-M}(t_{p-2}) & \ldots & \Omega^i_{N,l,M}(t_{p-2}) \\ \Upsilon^i_{1,-M}(t_{p-1}) & \ldots & \Upsilon^i_{l,M}(t_{p-1}) & \Omega^i_{0,1,-M}(t_{p-1}) & \ldots & \Omega^i_{N,l,M}(t_{p-1}) \end{bmatrix},$$

$$\mathbf{x} = \begin{bmatrix} \eta_{1,-M} & \ldots & \eta_{l,M} & \tau_{0,1,-M} & \ldots & \tau_{N,l,M} \end{bmatrix}^T,$$

and

$$\mathbf{g_i} = \begin{bmatrix} g_i(t_0) & g_i(t_1) & \ldots & g_i(t_{p-2}) & g_i(t_{p-1}) \end{bmatrix}^T.$$

On the other hand, the mixed conditions can be written as:

$$\begin{aligned} &\sum_{n=0}^{m-1}[\alpha_j^n(\sum_{k=-M}^{M}\eta_{j,k}\sum_{h=-M}^{M}\vartheta_{kh}^{(n)}\varphi_{0,h}(-1)+\sum_{r=0}^{N}\sum_{k=-M}^{M}\tau_{r,j,k} \\ &\sum_{h=-M}^{M}\kappa_{kh}^{(n)rr}\psi_{r,h}(-1))+\beta_j^n(\sum_{k=-M}^{M}\eta_{j,k}\sum_{h=-M}^{M}\vartheta_{kh}^{(n)}\varphi_{0,h}(1)+ \\ &\sum_{r=0}^{N}\sum_{k=-M}^{M}\tau_{r,j,k}\sum_{h=-M}^{M}\kappa_{kh}^{(n)rr}\psi_{r,h}(1))+\delta_j^n(\sum_{k=-M}^{M}\eta_{j,k} \\ &\sum_{h=-M}^{M}\vartheta_{kh}^{(n)}\varphi_{0,h}(c)+\sum_{r=0}^{N}\sum_{k=-M}^{M}\tau_{r,j,k}\sum_{h=-M}^{M}\kappa_{kh}^{(n)rr}\psi_{r,h}(c))] = a_j, \end{aligned} \tag{15}$$

where $j = 1,\ldots,l$, $-1 \le c \le 1$.

Relation (13) in Chebysheve points and mixed conditions give $l(2M+1)(N+2) - ml$ and ml equations, respectively. Therefore, $l(2M+1)(N+2)$ equations are obtained whose solution give the unknown components of the $\eta_{j,k}$ and $\tau_{r,j,k}$.

Actually, the desired approximation to the solution $f_j(t)$ of (1) can be obtained from

$$f_j(t) = \sum_{k=-M}^{M} \eta_{j,k}\varphi_{0,k}(t) + \sum_{r=0}^{N}\sum_{k=-M}^{M} \tau_{r,j,k}\psi_{r,k}(t).$$

2.3. The Algorithm

Here we summarize the previous results for constructing the Shannon approximate solution of (1) as the following:

I. Choose $N, M, l, m,$ then input constants and functions: $\varphi_{0,k}(t)$, $\psi_{r,k}(t)$, $F_{ij}^{n}(t)$, $K_{ij}(t,s)$, $G_{ij}(t,s)$, $g_i(t)$, α_j^n, β_j^n, δ_j^n, a_j, $k=-M,...,M$, $r=0,...,N$, $i, j=1,...,l$, $n=0,...,m$.

II. Compute $\vartheta_{kh}^{(n)}$ and $\kappa_{kh}^{(n)rr}$ by using Theorem 2, and obtain $\Upsilon^{i}_{j,k}(t)$ and $\Omega^{i}_{r,j,k}(t)$.

III. Compute Γ_i and $\mathbf{g_i}$ in Chebysheve points, then obtain $\eta_{j,k}$ and $\tau_{r,j,k}$ from (15) and $\Gamma_i\mathbf{x} = \mathbf{g_i}$, $i=1,...,l,$.

IV. Substituting $\eta_{j,k}$ and $\tau_{r,j,k}$ in $\sum_{k=-M}^{M}\eta_{j,k}\varphi_{0,k}(t) + \sum_{r=0}^{N}\sum_{k=-M}^{M}\tau_{r,j,k}\psi_{r,k}(t)$, $j=1,...,l$ and an approximate solution of (1) will be obtained.

3. Convergence analysis

The aim of this section is to provide a convergence analysis of the presented method for the FVIDEs system (1). Our strategy is mainly based on Theorem 4.2 from our recent paper [9].

Theorem 3. *Consider the FVIDEs system (1). Assume that* $f_j(t)$, $i=1,...,l$ *be the approximate solutions of the system (1) and* $\eta_{j,k}$ *and* $\tau_{r,j,k}$ *are given by (5). If* $f_j^{(n)}(t) \in L_2(\mathbb{R})$, *then the obtained approximation solutions of the proposed method converges to the exact solutions.*

Proof. Note that

$$\begin{aligned} f_j(t) &= \sum_{k=-\infty}^{\infty} \langle f_j(t), \varphi_{0,k}(t)\rangle \varphi_{0,k}(t) + \sum_{r=0}^{N-1}\sum_{k=-\infty}^{\infty} \langle f_j(t), \psi_{r,k}(t)\rangle \psi_{r,k}(t) \\ &= \sum_{r=-\infty}^{N-1}\sum_{k=-\infty}^{\infty} \langle f_j(t), \psi_{r,k}(t)\rangle \psi_{r,k}(t). \end{aligned}$$

As stated in [9, pp. 2675], we have

$$\left\| D^{(n)}\Big[\sum_{r=-\infty}^{N-1}\sum_{k=-\infty}^{\infty}\langle f_j(t),\psi_{r,k}(t)\rangle\psi_{r,k}(t)-f_j(t)\Big]\right\|_2 \to 0,\quad as N\to\infty,$$

or

$$\lim_{N\to\infty}\left\|\sum_{k=-\infty}^{\infty}\underbrace{\langle f_j(t),\varphi_{0,k}(t)\rangle}_{\eta_{j,k}}\varphi_{0,k}^{(n)}(t)+\sum_{r=-\infty}^{N-1}\sum_{k=-\infty}^{\infty}\underbrace{\langle f_j(t),\psi_{r,k}(t)\rangle}_{\tau_{r,j,k}}\psi_{r,k}^{(n)}(t)-f_j^{(n)}(t)\right\|_2\to 0.$$

Due to definitions of $\eta_{j,k}$, $\tau_{r,j,k}$ and relation (8), we can write:

$$\lim_{N\to\infty}\left[\sum_{k=-\infty}^{\infty}\eta_{j,k}\sum_{h=-\infty}^{\infty}\vartheta_{kh}^{(n)}\varphi_{0,h}(t)+\sum_{r=0}^{N-1}\sum_{k=-\infty}^{\infty}\tau_{r,j,k}\sum_{h=-\infty}^{\infty}\kappa_{kh}^{(n)rr}\psi_{j,h}(t)\right]=f_j^{(n)}(t).$$

Finally, the proof follows immediately from above relation and Theorem 1.

5. Numerical results

The numerical algorithm based on the Shannon approximation together with a Chebysheve collocation points described in Section 3 has been implemented to the following problems, taken from [8]:

5.1. Example 1:

$$\begin{cases} f_1^{(3)}(t)+f_2^{(2)}(t)+f_1^{(1)}(t)+e^{-t}f_2(t) & =\int_{-1}^{1}(\sinh(s)f_1(s)+\cosh(s)f_2(s))ds \\ & +\int_{-1}^{t}e^{t-s}f_2(s)ds-1-te^{t}-2e^{-t}, \\ f_2^{(3)}(t)-e^{t}f_1^{(3)}(t)-f_2^{(1)}(t)+tf_2(t)+f_1(t) & =\int_{-1}^{1}(3e^{s}f_1(s)-ts^{t+1}f_2(s))ds \\ & +\int_{-1}^{t}(-sf_1(s)+e^{-s}f_2(s))ds-6-t-te^{-t}+te^{t+2}, \end{cases} \quad (16)$$

with the mixed conditions

$f_1(1)=e^{-1}$, $f_1(0)=1$, $f_2(0)=1$, $f_1(-1)=e, f_2(-1)=e$, $f_2(1)=e$,

and the exact solution $f_1(t)=e^{-t}$, $f_2(t)=e^{t}$.

5.2. Example 2:

$$\begin{cases} f_1^{(1)}(t) + tf_2^{(1)}(t) + 3f_2(t) & = \int_{-1}^{1}(t-s)f_1(s) + t^2 f_2(s))ds \\ & + \int_{-1}^{t}(4sf_1(s) - f_2(s))ds + 20t^3 + 2t^2 + \frac{t}{3} - 3, \\ t^2 f_1^{(1)}(t) - f_2^{(1)}(t) + tf_1(t) + f_2(t) & = \int_{-1}^{1}(3sf_1(s) + (s^2 - 4t)f_2(s))ds \\ & + \int_{-1}^{t} 6f_1(s)ds + 5t^3 - 15t^2 - 8t + \frac{2}{3}, \end{cases} \tag{17}$$

with the mixed conditions

$f_1^{(1)}(0) - f_1(1) = -1,\ 2f_2(1) + f_2(-1) = 1,$

and the exact solution $f_1(t) = t^2 + 3t$, $f_2(t) = 4t^3 - 1$.

We consider FVIDE system (16) for numerical implementation of proposed method. In this equation, we have $l = 2$. Thus, the numbers of unknowns $\eta_{j,k}$ and $\tau_{r,j,k}$ are $l(2M+1)(N+2) = 2(2M+1)(N+2)$. For ease of exposition, we take $M = N = 1$. So, we have to evaluate the following unknowns, for $j = 1,2$, $k = -1,0,1$ and $r = 0,1$ as:

$$\begin{array}{llllll} \eta_{0,-1} & ,\eta_{0,0} & ,\eta_{0,1} & ,\eta_{1,-1} & ,\eta_{1,0} & ,\eta_{1,1}, \\ \tau_{0,0,-1} & ,\tau_{0,0,0} & ,\tau_{0,0,1} & ,\tau_{0,1,-1} & ,\tau_{0,1,0} & ,\tau_{0,1,1}, \\ \tau_{1,0,-1} & ,\tau_{1,0,0} & ,\tau_{1,0,1} & ,\tau_{1,1,-1} & ,\tau_{1,1,0} & ,\tau_{1,1,1}. \end{array}$$

Actually, we need 18 equations for solving this test problem. As we expected,
there are 6 relations with respect to the mixed conditions. Now, we compute Chebysheve
points as follows:

$$\begin{cases} p = l(2M+1)(N+2) - ml, \\ x = 0,\dots,p-1, \\ t_x = \cos(\frac{x\pi}{p-1}). \end{cases} \tag{19}$$

Table 1.

Approximate and exact solutions for (16).

x	t_x	Approximate solution $f_1(t_x)$	Exact solution $f_1(t_x)$
0	$\cos(0)$	0.367870781	0.367879441
2	$\cos(\frac{2\pi}{11})$	0.431169512	0.431169699
4	$\cos(\frac{4\pi}{11})$	0.660066148	0.660066287
6	$\cos(\frac{6\pi}{11})$	1.152937034	1.152939581
8	$\cos(\frac{8\pi}{11})$	1.924874416	1.924874429
10	$\cos(\frac{10\pi}{11})$	2.61037499	2.61037261

Table 2. Approximate and exact solutions for (16).

x	t_x	Approximate solution $f_2(t_x)$	Exact solution $f_2(t_x)$
1	$\cos(\frac{1\pi}{11})$	2.61037291	2.61037261
3	$\cos(\frac{3\pi}{11})$	1.924873081	1.924874429
5	$\cos(\frac{5\pi}{11})$	1.152939580	1.152939581
7	$\cos(\frac{7\pi}{11})$	0.660066102	0.660066287
9	$\cos(\frac{9\pi}{11})$	0.431162039	0.431169699
11	$\cos(\frac{11\pi}{11})$	0.367879159	0.367879441

Therefore, we can write

$$t_0 = \cos(\frac{0}{11}) \quad ,t_1 = \cos(\frac{\pi}{11}) \quad ,t_2 = \cos(\frac{2\pi}{11}) \quad ,t_3 = \cos(\frac{3\pi}{11}),$$
$$t_4 = \cos(\frac{4\pi}{11}) \quad ,t_5 = \cos(\frac{5\pi}{11}) \quad ,t_6 = \cos(\frac{6\pi}{11}) \quad ,t_7 = \cos(\frac{7\pi}{11}),$$
$$t_8 = \cos(\frac{8\pi}{11}) \quad ,t_9 = \cos(\frac{9\pi}{11}) \quad ,t_{10} = \cos(\frac{10\pi}{11}) \quad ,t_{11} = \cos(\frac{11\pi}{11}).$$

Substituting these points in (13) and according to the mixed conditions, 18 equations are obtained. By solving these relations, we can computed unknown component of $\eta_{j,k}$ and $\tau_{r,j,k}$. Finally, the approximate solutions of (16) will be obtained by the following equations:

$$f_1(t) = \sum_{k=-1}^{1}\eta_{1,k}\varphi_{0,k}(t) + \sum_{r=0}^{1}\sum_{k=-1}^{1}\tau_{r,1,k}\psi_{r,k}(t),$$
$$f_2(t) = \sum_{k=-1}^{1}\eta_{2,k}\varphi_{0,k}(t) + \sum_{r=0}^{1}\sum_{k=-1}^{1}\tau_{r,2,k}\psi_{r,k}(t).$$

the computational results of Example 1, for M=N=1 in Chebycheve points, have been reported in Table 1 and 2. Table 3 and 4, represented the error estimates of the proposed method for different values of M and N. The maximum errors listed, show that we can achieve to good numerical results with small N and M.

Table 3.

Numerical results for(16)

M	N	Maximal error $f_1(t)$	Maximal error $f_2(t)$
2	3	1.02E-8	1.78E-8
3	4	1.87E-9	2.17E-10
4	6	2.59E-12	1.06E-12
5	7	1.14E-14	1.67E-15
5	8	3.34E-16	2.93E-17

Table 4.

Numerical results for(17)

M	N	Maximal error $f_1(t)$	Maximal error $f_2(t)$
2	4	3.45E-10	2.88E-9
4	5	2.90E-13	3.56E-13
5	7	0.98E-14	1.23E-15
6	5	1.49E-16	1.09E-16
8	6	1.94E-18	0.48E-17

4. Conclusions

In this research, we have applied the Shannon approximation for solving FVIDEs system. Shannon approximation method has become very popular in solving initial and boundary value problems of ordinary or partial differential equations as well as the approximate solution of integral equations. Also, the method employed here can be probably extended to investigate the approximate solution of other classes of IDEs. We have shown that the proposed method has produced highly numerical results.

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