CONTRIBUTIONS TO THE STUDY OF LARGE STRAIN SOLID

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Acest articol se ocupă cu studiul unei probleme nelineare de mecanica corpurilor solide. Aceste probleme conduc, în general, la rezolvarea unor ecuaţii sau sisteme de ecuaţii diferenţiale destul de dificil de rezolvat in anumite condiţii la frontieră. De aceea, de cele mai multe ori acest lucru nici nu este posibil şi se impun rezolvări numerice sau aproximative, bazate pe neglijarea unor termeni din ecuaţii.

This paper deals with the study of a nonlinear problem in the mechanics of solids. Generally, these problems lead to differential equations or systems of equations which are quite difficult to solve in some boundary value conditions. It is not always possible to solve these equations and numerical or approximate approaches are required for solving, based on the neglecting of some equations' terms.

Keywords: symmetric deformation, constitutive equations, numerical solutions.

1. Introduction

Below we intend to illustrate the nature of solutions to elasticity problems with large shape changes. Generally the following pieces of information are known: the geometry of the solid; a constitutive law for the material (i.e. the hyperelastic strain energy potential); the body force density per unit mass (if any); prescribed boundary tractions and/or boundary displacements. With these assumptions, generally we wish to calculate the displacement field, the left Cauchy-Green deformation tensor and the stress field, satisfying the following equations: displacement strain relation; incompressibility condition; stress strain relation; equilibrium equation; boundary conditions.

We study the spherical symmetric deformation of a spherical shell, based on [1]. We assume the coating material to be elastic, homogenous, isotropic and the spherically symmetric deformation is specified using some functions which depend on the position vector. These functions will be determined later by specifying the equilibrium equations. For such a material, one can obtain nonlinear constitutive equations by the generalization of Hooke's linear equation, valid in the case of small deformations, adding additional terms reflecting the second order effects, third order effects, etc. By specifying the boundary

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conditions, we numerically solve the first boundary problem obtained from the equations of equilibrium. We will also study the existence and uniqueness of this problem's solutions.

In this area of study, there are some researches on similar themes with those presented in this paper. In [2] there are theoretically depicted some numerical methods applicable for the study of deformation of solids and also some cases in which one may determine analytical solutions for problems related to deformation of spherical shells, much easier than the cases presented in this paper. Some uniaxial deformations of spherical domains are considered in paper [3]. In [4] the problem is considered within the framework of the geometrical nonlinear theory of elasticity. Minimal restrictions are found under which the zero solution is unique for zero loads. Under these restrictions, the uniqueness of the solution for the tensile forces is proved, and the critical compressive force is found for which uniqueness is destroyed. In the current paper there are also specified and verified some conditions for existence and uniqueness of the solution for the studied cases and the constitutive equations chosen. In [5], Ting studies the remarkable nature of a sphere of nonlinear elastic material subjected to a uniform pressure at the surface of the sphere. Then he analyses a spherically uniform linear anisotropic elastic material. Conditions for the materials that are capable of a radially symmetric deformation to possess one or more symmetry planes there are also presented. Unlike these studies, this paper develops both theoretical aspects (related to the existence and uniqueness of solutions) and practical implementation (concluding with the obtaining of the numerical solutions of considered problems).

2. About the spherically symmetric deformation of a spherical shell

A representative spherically symmetric problem is illustrated in Fig. 1.

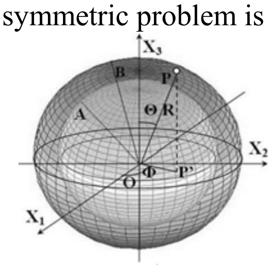


Fig. 1. The thick-walled spherical shell in the reference geometry.

For a finite deformation problem, we need a way to characterize the position of material particles in both the undeformed and deformed solid. In order to do this, we let (R,Θ,Φ) identify a material particle in the undeformed solid. The coordinates of the same point in the deformed solid is identified by a new set of spherical-polar coordinates (r,θ,φ). One way to describe the deformation would be to specify each of the deformed coordinates (r,θ,φ) in terms of the reference coordinates (R,Θ,Φ) .

We consider for the thick-walled spherical shell the reference geometry defined by:

$$0 \le A \le R \le B,\ 0 \le \Theta \le \pi,\ 0 \le \Phi \le 2\pi \tag{1}$$

in the terms of spherical polar coordinates (R,Θ,Φ), or

$$X_1 = R\sin\Theta\cos\Phi,\ \ X_2 = R\sin\Theta\sin\Phi,\ \ X_3 = R\cos\Theta \tag{2}$$

We describe the current geometry by:

$$0 \le a \le r \le b,\ 0 \le \theta \le \pi,\ 0 \le \varphi \le 2\pi \tag{3}$$

in the terms of spherical polar coordinates (r,θ,φ). We consider this hollow subjected to spherically symmetric loading (i.e. internal body forces, as well as tractions or displacements applied to the surface, independently of θ and ϕ, and act in radial direction only). Before deformation, the sphere has the inner radius *A* and the outer radius *B*. After deformation, the sphere has the inner radius *a* and the outer radius *b*.

For a spherically symmetric deformation, points only move radially, so the deformation is depicted by:

$$\vec{\mathbf{x}} = f(R)\vec{\mathbf{X}} \tag{4}$$

so that

$$r = f(R)R,\ \ \theta = \Theta,\ \ \varphi = \Phi \tag{5}$$

Since the material is unconstrained, the function *f* is unknown and, given appropriate boundary conditions, the essential problem is to find it. The deformation gradient **F** is given by:

$$\mathbf{F} = \nabla \otimes \vec{\mathbf{x}} = f(R)\mathbf{1} + \frac{1}{R} f'(R)\vec{\mathbf{X}} \otimes \vec{\mathbf{X}} \tag{6}$$

the symmetry of the geometry ensuring that **F=U,** the right stretch tensor **[**6]. In finite deformation problems vectors and tensors can be expressed as components in a basis $(\vec{\mathbf{e}}_R, \vec{\mathbf{e}}_\Theta, \vec{\mathbf{e}}_\Phi)$ associated with the position of material points in the undeformed solid, or, if more convenient, in a basis $(\vec{\mathbf{e}}_r, \vec{\mathbf{e}}_\theta, \vec{\mathbf{e}}_\varphi)$ associated with material points in the deformed solid. For spherically symmetric deformations the two bases are identical consequently, so we can write the position vector in the undeformed solid $\vec{\mathbf{X}} = R \cdot \vec{\mathbf{e}}_R$, the position vector in the deformed solid $\vec{\mathbf{x}} = r \cdot \vec{\mathbf{e}}_R$ and the displacement vector $\vec{\mathbf{u}} = \vec{\mathbf{x}} - \vec{\mathbf{X}} = (r - R) \cdot \vec{\mathbf{e}}_R$.

The principal stretches, corresponding to coordinate directions (r,θ,φ) respectively are:

$$\lambda_1 = Rf'(R) + f(R),\ \ \lambda_2 = \lambda_3 = f(R) \tag{7}$$

In order to simplify the problem, we will assume that:

- the solid is stress free in its undeformed configuration;
- temperature changes during deformation are neglected;
- the solid is incompressible;
- the material is isotropic relative to the reference configuration.

From the spherical symmetry it follows that the nominal stress $\mathbf{\Theta}$ is given by $\mathbf{\Sigma} = \mathbf{\Theta}$, where the Biot stress $\mathbf{\Sigma}$ may be written:

$$\mathbf{\Sigma} = \sigma_1 \frac{1}{R^2} \vec{\mathbf{X}} \otimes \vec{\mathbf{X}} + \sigma_2 \left(\mathbf{1} - \frac{1}{R^2} \vec{\mathbf{X}} \otimes \vec{\mathbf{X}} \right) \tag{8}$$

in terms of its principal components $\sigma_1, \sigma_2 = \sigma_3$. On use of the equilibrum equation:

$$Div\mathbf{\Theta}^T = 0 \tag{9}$$

in the terms of spherical polar coordinates, applied with (R, Θ, Φ) as independent variables, we obtain the equilibrum equation:

$$\frac{d\sigma_1}{dR} + \frac{2}{R}(\sigma_1 - \sigma_2) = 0 \tag{10}$$

We consider now a constitutive law for the material (i.e. the hyperelastic strain energy potential) and we introduce the strain energy function $w = W(\lambda_1, \lambda_2, \lambda_3)$, where w is a symmetric function. In this case, the principal components of the Biot stress $\mathbf{\Sigma}$ are:

$$\sigma_i = \frac{\partial W}{\partial \lambda_i}, \quad i = 1,2,3 \tag{11}$$

evaluated for $(\lambda_1, \lambda_2, \lambda_3)$ given by (7). Substitution of (11) and (7) into (10) yields a non-linear second order ordinary differential equation for $f(R)$,

$$\frac{\partial^2 W}{\partial \lambda_1^2} R f''(R) + 2\left(\frac{\partial^2 W}{\partial \lambda_1^2} + \frac{\partial^2 W}{\partial \lambda_1 \partial \lambda_2} \right) f'(R) + \frac{2}{R}\left(\frac{\partial W}{\partial \lambda_1} - \frac{\partial W}{\partial \lambda_2} \right) = 0 \tag{12}$$

where each of the derivatives of W depends non-linearly on $f(R)$ and $f'(R)$. In order to solve this equation, an explicit form of W needs to be chosen. In general it is not possible to obtain analytic solutions even for relatively simple forms of W. Below are presented some situations in wich a suitable choice of the strain energy function leads to a simplified form of (12) and in this case it is possible to obtain the analytic solutions of this equation.

3. Some examples of constitutive equations

In [1] it is proposed an example of a strain energy function,

$$W = G(I_1) - \nu I_2 + \mu I_3 \tag{13}$$

where $I_1 = \lambda_1 + \lambda_2 + \lambda_3, I_2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1, I_3 = \lambda_1\lambda_2\lambda_3$ are the principal invariants of **U;** ν and μ are constants, G is a function whose properties will be specified later. The form (13) chosen for W represents a generalization to three dimensions of the class of so-called “harmonic” materials used in plane-strain theory. The principal Biot stresses are calculated as:

$$\begin{aligned}\sigma_1 &= \frac{\partial W}{\partial \lambda_1} = G'(I_1) - \nu(\lambda_2 + \lambda_3) + \mu\lambda_2\lambda_3 \\ \sigma_2 = \sigma_3 &= \frac{\partial W}{\partial \lambda_2} = G'(I_1) - \nu(\lambda_2 + \lambda_3) + \mu\lambda_1\lambda_3\end{aligned} \tag{14}$$

Since the reference configuration is stress free, from (13) and (14) we obtain:

$$G(3) - 3\nu + \mu = 0, G'(3) - 2\nu + \mu = 0 \tag{15}$$

Substitution of (14) into (12) leads to a simplified form of (12),

$$G''(I_1)\frac{dI_1}{dR} = 0 \tag{16}$$

an equation that can be easy integrated. There are two posibilities:

a) $G''(I_1) = 0$, from wich we obtain $G(I_1) = (2\nu + \mu)I_1 - (3\nu + 4\mu)$. In this situation, no restriction is imposed on *f(R)* i.e. the equilibrum equations are satisfied for arbitry *f(R)* in respect of the strain-energy function (13) with the above mentioned G.

b) $G''(I_1) \neq 0$, from wich we obtain $dI_1/dR = 0$ or $\frac{d}{dR}\left[Rf'(R) + 3f(R)\right] = 0$.

The general solution of this equation is expressible as $f(R) = \alpha/3 + \beta/R^3$, where α and β are constants, with $I_1 = \alpha$. Since the principal stretches must be positive, it follows that $I_1 = \alpha > 0$ while $\lambda_1 = \frac{\alpha}{3} - \frac{2\beta}{R^3} > 0, \lambda_2 = \lambda_3 = \frac{\alpha}{3} + \frac{\beta}{R^3} > 0$. If λ_1 and λ_2 are to be positive for $0 \le A \le R \le B$, the constant β must be restricted according to $-\alpha A^3/3 < \beta < \alpha A^3/6$. Constants α and β may be calculated from the boundary conditions on $R = A$ and $R = B$ [1].

A second particular case of constitutive equations is the situation of a pressurized hollow rubber shell made from an incompressible Mooney-Rivlin solid [2], a hyperelastic material model where the strain energy density function W is a linear combination of two invariants of the left Cauchy-Green deformation tensor ***B***. Melvin Mooney and Ronald Rivlin proposed the model in two independent papers in 1952. The strain energy density function for an incompressible Mooney-Rivlin material is:

$$W = C_1(\bar{I}_1 - 3) + C_2(\bar{I}_2 - 3) \tag{17}$$

where C_1 and C_2 are empirically determined material constants, and $\bar{I}_1$ and $\bar{I}_2$ are the first and the second invariant of the deviatoric component of the left Cauchy-Green deformation tensor. This is a situation in which it is possible to obtain analytic solutions for equation (12), considering that no body forces act on the sphere and that the inner and outer surfaces of the sphere are subjected to given pressures [2].

Another example of constitutive equations proposed in the study of the spherically symmetric deformation of a spherical shell is presented in [7]. In this paper, it is chosen for the strain energy the function:

$$W(F) = \frac{a^2}{2\mu}\mathrm{tr(F)}^{\frac{\mu}{a}} + \frac{a^2}{\lambda}(\mathrm{detF})^{-\frac{\lambda}{2a}} \tag{18}$$

with $\mu > 0$ and $\lambda > 0$ the Lamé parameters and $a > 0$. This is a generalization of Hooke's law for isotropic materials. Considering a spherical shell made from such a material, equation (12) can be easy integrated.

4. Application for an elastic material with large deformations

In the following, we will consider the spherical shell constructed of an elastic material, isotropic, with large deformations. The constitutive equation will be chosen to generalize Hooke's law for isotropic materials [8]:

$$\Pi = a_1 E + (a_2 - a_1)\frac{trE}{3} I \tag{19}$$

where Π is the second Piola-Kirchhoff stress tensor,

$$a_1 = 2\mu, a_2 = 3\lambda + 2\mu \tag{20}$$

$\mu > 0$ and $\lambda > 0$ are the Lamé parameters, E is a tensor given by $E = (1/2)(FF^T - I)$ and F the deformation gradient defined by (6). In this case, the first Piola Kirchhoff stress tensor is :

$$S = F\Pi = F \cdot \left[a_1 E + (a_2 - a_1)\frac{trE}{3} I \right] \tag{21}$$

Introducing components of S in spherical coordinates S_{RR}, S_{TT} şi S_{FF}, the equlibrum equation $DivS = 0$, written in spherical coordinates [9] leads to the system:

$$\begin{cases} \frac{\partial S_{RR}}{\partial R} + \frac{2}{R} S_{RR} + \frac{1}{R}\frac{\partial S_{TR}}{\partial \theta} + \frac{1}{R\sin\theta}\frac{\partial S_{FR}}{\partial \phi} + \frac{ctg\theta}{R} S_{TR} - \frac{1}{R}(S_{TT} + S_{FF}) = 0 \\ \frac{\partial S_{RT}}{\partial R} + \frac{2}{R} S_{RT} + \frac{1}{R} S_{TR} + \frac{1}{R}\frac{\partial S_{TT}}{\partial \theta} + \frac{1}{R\sin\theta}\frac{\partial S_{FT}}{\partial \phi} + \frac{ctg\theta}{R}(S_{TT} - S_{FF}) = 0 \\ \frac{\partial S_{RF}}{\partial R} + \frac{2}{R} S_{RF} + \frac{1}{R} S_{FR} + \frac{1}{R}\frac{\partial S_{TF}}{\partial \theta} + \frac{1}{R\sin\theta}\frac{\partial S_{FF}}{\partial \phi} + \frac{ctg\theta}{R}(S_{TF} + S_{FT}) = 0 \end{cases} \quad (22)$$

Considering features of the problem, the system is reduced to a single relationship that is not identically satisfied,

$$\frac{\partial S_{RR}}{\partial R} + \frac{2}{R}(S_{RR} - S_{TT}) = 0 \quad (23)$$

Substituting the components of S in spherical coordinates S_{RR}, S_{TT} and S_{FF} one can obtain from (23) a second-order differential equation:

$$f''(R) = -\frac{f'}{R} \cdot \frac{8(2k+1)R^2 f'^2 + 2(17k+10)Rff' + 4(4k+5)f^2 - 12}{3(2k+1)R^2 f'^2 + 6(2k+1)Rff' + (4k+5)f^2 - 3} \quad (24)$$

where $k = a_1/a_2$ is a dimensionless constant depending on the choosing of the sphere's material. Elementary calculations show that the denominator of equation (24) is nonzero within the spherical shell defined by (1). Equation (24) can be solved taking into account boundary conditions for the spherical shell. These conditions could be traction boundary conditions on parts of the boundary where tractions are known and displacement boundary conditions on parts of the boundary where displacements are known. In this paper, it has been chosen the version:

$$f(A) = a/A, f(B) = b/B \quad (25)$$

The problem (24), (25) is a first boundary problem obtained from the equations of equilibrium. In the following we present some considerations on the the existence and uniqueness of problem's solution, problem wich has been obtained in the case of a spherically symmetric deformation of a spherical shell of elastic isotropic material, with large deformations. In [10] and [11] are presented the existence and the uniqueness theorems for the first boundary problems. Therefore, the problem (24), (25) has an unique solution [6].

5. Numerical solutions

Though the previous first boundary problem's solution exists and is unique, its analytical form is difficult to find. Therefore, a numerical approach was chosen here. The numerical solution is obtained using the ODE23 function from the MATLAB software package. This function uses the Runge-Kutta algorithm of orders 2 and 3 and is useful for solving Cauchy problems for differential equations (or systems of differential equations) of first order. We note that problem (24), (25) from the preceding paragraph is not of the type mentioned above because, on the one hand, the equation (24) is not one of the first order, and secondly, the conditions (25) are boundary conditions and not initial conditions. Both issues will be treated below and we will show how to eliminate these difficulties.

Obviously, the second order differential equation (24) may be canonical associated to a first order system of differential equations equivalent to this equation. Thus, noting $f_1 = f', f_2 = f$ we obtain the system:

$$f_1' = -\frac{f_1}{R}\cdot\frac{8(2k+1)R^2 f_1^2 + 2(17k+10)Rf_1 f_2 + 4(4k+5)f_2{}^2 - 12}{3(2k+1)R^2 f_1^2 + 6(2k+1)Rf_1 f_2 + (4k+5)f_2{}^2 - 3}, f_2' = f_1 \tag{26}$$

This system was used in the numerical approach of the problem mentioned below, instead of equation (24). As noted above, the studied problem is not a Cauchy problem but a first boundary problem. Therefore, we used the 'shooting method' [10], in order to associate to the problem mentioned below a series of Cauchy problems and thus for obtaining the solution. The 'shooting method' consists of sequentially solving Cauchy problems (associated with the first boundary problem), of the type :

$$f''(R) = -\frac{f'}{R}\cdot\frac{8(2k+1)R^2 f'^2 + 2(17k+10)Rff' + 4(4k+5)f^2 - 12}{3(2k+1)R^2 f'^2 + 6(2k+1)Rff' + (4k+5)f^2 - 3} \tag{27}$$

$$f(A) = a/A, \quad f'(A) = u$$

According to [10], the Cauchy problem (27) admits unique solution. After numerically solving this problem, the appropriate value of $f(B)$ is calculated for the solution found and the process is repeated until the value of $f(B)$ approaches the value required by (25) with some precision previously established. Based on the theorem of existence and uniqueness of the solution, this is just the problem's (24), (25) solution. In the small strain situation, when the constitutive law for the material reduces to the Hooke's law of linear elasticity, the considered problem becomes simpler because some terms in the equation (24) are negligible and then its solution can be also analytically determined. Thus, by neglecting second order terms, the problem (24), (25) leads to:

$$Rf''(R) + 4f'(R) = 0, f(A) = a/A, f(B) = b/B \tag{28}$$

and the solution of this problem is:

$$f(R) = A^2 B^2 \frac{bA - aB}{A^3 - B^3} R^{-3} + \frac{aA^2 - bB^2}{A^3 - B^3} \tag{29}$$

In this case we made a comparative study of the analytical solution (denoted by ε) given by (29) with the numerical one, obtained using the shooting method, (denoted by rn) and noticed a very good approximation, i.e. the small strain solution is accurate. On the other hand, the equation (28) can be also found by direct calculation. We denote by ε the Cauchy's strain tensor in linear elasticity, $\varepsilon = (1/2)(\nabla\vec{u} + \nabla^T\vec{u})$, $\vec{u} = \vec{x} - \vec{X}$ and by σ the Cauchy tensile stress. The constitutive law (in a natural reference configuration, in which $\varepsilon = 0 \Rightarrow \sigma = 0$) is $\sigma = \lambda tr\varepsilon I + 2\mu\varepsilon$. Introducing this relation in the equilibrium equation written in spherical coordinates, leads to equation (28) above. This is a verification of the correctness of calculations leading to equations (24) and (28).

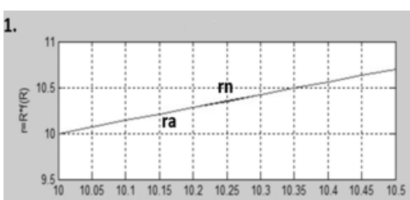
The numerical values chosen for A, B, a, b are those mentioned in Table 1, some cases corresponding to small deformations and some to the large ones:

Table 1

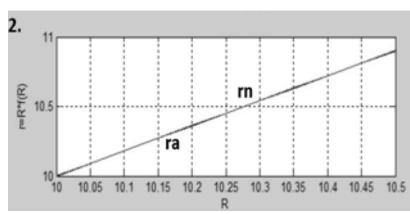
Numerical values for A, B, a, b

Nr.	A(m)	B(m)	a(m)	b(m)	Nr.	A(m)	B(m)	a(m)	b(m)
1.	0.10	0.105	0.10	0.107	3.	0.20	0.225	0.21	0.235
2.	0.10	0.105	0.10	0.109	4.	0.20	0.225	0.18	0.205

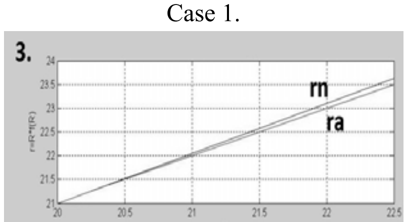
The sphere walls were subjected to stretch or compression. The hollow sphere was considered made from a cellular rubber [8] characterized by $\lambda = 0.12MPa$, $\mu = 0.14MPa$, $a_1 = 0.28MPa$, $a_2 = 0.64MPa$, $k = a_1/a_2 = 0.44$. It was highlighted the dependence of the final radius of a point belonging to the spherical shell depending on the initial one through the function *f(R)*. Some numerical results obtained are represented in Fig. 2:



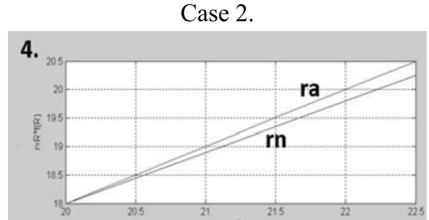
Case 1.



Case 2.



Case 3.



Case 4.

Fig. 2. Some numerical solutions.

6. Conclusions

The study of the spherically symmetric deformation of a spherical shell, a nonlinear problem in the mechanics of solids, leads to differential equations which are quite difficult in some boundary value conditions. Unlike other studies, this paper has developed both theoretical aspects related to the existence and uniqueness of solutions and practical implementation, ending with the obtaining of numerical solutions of the considered problems. In the small strain situation one can notice a very good approximation of the analytical solution given by (29) with the numerical one (obtained using the shooting method). The results obtained above bring original contributions in this field and on the other hand these studies remain open for further research.

REFERENCES

[1] *R. Ogden,* Non-Linear Elastic Deformations, J. Wiley and Sons, New York, 1984.

[2] *A.F.Bower,* Applied Mechanics of Solids, CRC Press, Oxford, 2009.

[3] *N.A. Bert, A. L. Kolesnikova, A. E. Romanov, V. V. Chaldyshev,* "Elastic behavior of a spherical inclusion with a given uniaxial dilatation", Journal Physics of the Solid State, **vol. 44**, no. 12, february 2006, pp. 2240-2250.

[3] *V. G. Osmolovskii* , "The nonlinear problem of the symmetric deformation of a hollow sphere", Journal of Mathematical Sciences, **vol. 10**, no. 1, march 2005, pp. 104-109.

[4] *C. T. Ting,* "The remarkable nature of radially symmetric deformation of spherically uniform linear anisotropic elastic solids", Journal of elasticity, **vol. 53**, no. 1, 1998-1999, pp. 47-64.

[5] *D. Petroşanu,* "Sur la Déformation d'un Réservoir Sphérique", Revue Roumaine des Science Techniques, Série de Mécanique Appliquée, **tome 45**, no. 3, 2000, pp. 309-320.

[6] *D. Petroşanu,* "Exemples de grandes déformations pour un reservoir sphérique et un tube cylindrique", University Politehnica of Bucharest, Scientific Bulletin , Series A : Applied Mathematics and Physics, **vol.66**, no. 1, 2004, pp. 37-46.

[7] *Eric Lainé,* Lois de comportement isotropes non-linéaires en grandes déformations. Application au caoutchouc, Ph. D. Thesis, scientific coordinators *C.Vallée, D. Fortuné*, University of Poitiers, 1998.

[8] *E. Soós, C. Teodosiu,* Calcul tensorial cu aplicații în mecanica solidelor (Tensor calculus with applications in solid mechanics), Editura Ştiințifică şi Enciclopedică, Bucureşti, 1983.

[9] *C. Udrişte, V. Iftode, M. Postolache,* Metode numerice de calcul (Numerical methods of calculating), Editura Tehnică, Bucureşti, 1996.

[10] *H. Keller,* Numerical Methods for Two-Point Boundary-Value Problems, Blaisdell Publishing Company, London, 1968.