A NOTE ON SELF-SIMILAR VECTOR FIELDS IN STATIC SPHERICALLY SYMMETRIC SPACE-TIMES

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Algebraic and direct integration techniques are used in this paper to obtain self similar vector fields in static spherically symmetric space-times. It turns out that the above space-times admit tilted proper self similar vector field of first and second kinds. The above space-time also admit non tilted orthogonal proper self similar vector fields of zeroth kind and non tilted parallel proper self similar vector fields of infinite kind. The above space-time admits these self similar vector fields for a special choice of the metric functions.

Keywords: Algebraic and direct integration techniques; Proper self similar vector fields.

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1. Introduction

It is customary to study the physical and geometrical aspects of our universe through famous Einstein field equations. Over the past few years a considerable attention has been given to find some exact solutions of Einstein field equations. To obtain some physical solutions, it is common to impose some sort of symmetry conditions on metric tensor. A significant interest seems in literature to study different kind of symmetries like Killing, homothetic and conformal symmetries [1-9]. In general relativity a characteristic scale does not exist [10], therefore, a scale transformation does not affect the field equation when suitable matter field is considered. Thus scale invariant solutions of field equations exist and are known as self similar solutions. In general relativity self similar solutions are indispensable. Some inhomogeneous and dynamical solutions can be found by considering self similarity hypothesis [11]. Self similar solutions have some great applications in evolution of voids, universe expansion after big bang and collapsing stars [12].

A vector field X is said to be self similar if it satisfies the following two conditions [13]:

$$\underset{X}{L} u_a = \alpha u_a, \tag{1}$$

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$$\underset{X}{L}\, h_{ab} = \delta\, h_{ab}, \tag{2}$$

where u^a is the four velocity of the fluid satisfying $u^a u_a = \pm 1$ and $h_{ab} = g_{ab} \pm u_a u_b$ is the projection tensor and $\alpha, \delta \in \Re$. If $\delta \neq 0$, the ratio α / δ, which is scale independent, characterizes similarity transformation and known as similarity index. If the above ratio gives unity, X turns out to be a homothetic vector field, which is known as first kind self similarity. If $\alpha = 0$ and $\delta \neq 0$, similarity is called of the zeroth kind. If the ratio is neither zero nor one, it is referred to as self-similarity of the second kind. Self similarity is called of infinite kind when $\alpha \neq 0$ and $\delta = 0$. If both $\delta = \alpha = 0$, X turns out to be a Killing vector fields. A vector field X is said to be a proper self similar vector field if it is not Killing. Self similar vector field X can be tilted or non tilted to the four velocity vector u^a. When u^a is time like then $u^a u_a = -1$ and $h_{ab} = g_{ab} + u_a u_b$. In such case the self similar vector field X will be parallel, orthogonal or tilted (that is neither parallel nor orthogonal) to the time like vector field u^a when $X = F(u)\dfrac{\partial}{\partial u}$, $X = f(x)\dfrac{\partial}{\partial x}$ or $X = (\alpha u + \beta)\dfrac{\partial}{\partial u} + x\dfrac{\partial}{\partial x}$, respectively [10]. The above theory is also valid for space-like vector field u^a.

Much work has been done to find self-similar solutions of the space-times. H. Maeda et al [10] classify all spherically symmetric space-times admitting self-similar vector fields of the second, zeroth or infinite kind. They studied the cases in which the self-similar vector field is not only tilted but also parallel or orthogonal to the fluid flow. Coley [12] in his paper discussed various mathematical and physical properties of space-times admitting self-similarity. In [13] authors discussed some important symmetries of Bianchi type I space-times. They determined the self similarities of Bianchi I metrics without any constraint on the type of the fluid. They showed that Bianchi type I space-times admit self similarity of first and zeroth kind. In [14] the authors explored some properties of the self similar solutions of the first kind for spherically symmetric space-times and also checked the singularities of these solutions. The same authors [15] studied cylindrically symmetric solutions which admit self-similar vector fields of second, zeroth and infinite kinds, for the tilted fluid case, parallel and orthogonal cases. They showed that the parallel case gives contradiction both in perfect fluid and dust cases and the orthogonal perfect fluid case yields a vacuum solution while the orthogonal dust case gives contradiction. We developed a new approach to find self similar vector fields for static plane symmetric space-times [16]. We obtained self similar vector fields of first, second and infinite kind in tilted case and zeroth kind non tilted self similar vector field for static plane symmetric space-times. Now we wish to extend our findings for static spherically symmetric space-times. Using an algebraic and direct integration method we will investigate

proper self similar vector fields for static spherically symmetric space-times. Throughout in this paper M represents a four dimensional, connected, Hausdorff space-time manifold with Lorentz metric g of signature (-, +, +, +). The curvature tensor associated with g_{ab}, through the Levi-Civita connection, is denoted in component form by $R^a{}_{bcd}$, and the Ricci tensor components are $R_{ab} = R^c{}_{acb}$. The usual covariant, partial and Lie derivatives are denoted by a semicolon, a comma and the symbol L, respectively. Here, M is assumed non flat in the sense that the curvature tensor does not vanish over any non empty open subset of M.

2. Main results

Consider static spherically symmetric space-times in the usual coordinate system (t, r, θ, ϕ) (labeled by (x^0, x^1, x^2, x^3), respectively) with the line element [1]

$$ds^2 = -e^{A(r)}dt^2 + e^{B(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \qquad (3)$$

where A and B are functions of r only. The above space-times (3) admit at least four independent Killing vector fields [1] which are

$$\frac{\partial}{\partial t}, \ \frac{\partial}{\partial \phi}, \ \cos\phi\frac{\partial}{\partial \theta} - \cot\theta\sin\phi\frac{\partial}{\partial \phi}, \ \sin\phi\frac{\partial}{\partial \theta} + \cot\theta\cos\phi\frac{\partial}{\partial \phi}. \qquad (4)$$

We are taking the four-velocity vector as time like vector field u and define $u_a = e^{\frac{A(r)}{2}}\delta_a^0$, so that $u^a u_a = -1$. The line element (3) becomes

$$ds^2 = -du^2 + e^{B(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \qquad (5)$$

Solving equation (1) we get $X^0 = \alpha u + \beta$, where $\alpha, \beta \in \Re$. Writing (2) explicitly by using (5) we get

$$B'(r)X^1 + 2X^1{}_{,1} = \delta, \qquad (6)$$

$$r^2 X^2{}_{,1} + e^{B(r)}X^1{}_{,2} = 0, \qquad (7)$$

$$e^{B(r)}X^1{}_{,3} + r^2\sin^2\theta\, X^3{}_{,1} = 0, \qquad (8)$$

$$2X^1 + 2rX^2{}_{,2} = r\delta, \qquad (9)$$

$$\sin^2\theta\, X^3{}_{,2} + X^2{}_{,3} = 0, \qquad (10)$$

$$2X^1 + 2r\cot\theta\, X^2 + 2r\, X^3{}_{,3} = r\delta, \qquad (11)$$

where 'bar' denotes differentiation with respect to r. Solving equations (6), (7) and (8) we get

$$X^1 = \frac{\delta}{2} e^{\frac{-B}{2}} \int e^{\frac{B}{2}} dr + e^{\frac{-B}{2}} P^1(\theta,\phi), \qquad X^2 = -P_\theta^1(\theta,\phi)\int \frac{e^{\frac{B}{2}}}{r^2} dr + P^2(\theta,\phi),$$

$$X^3 = -\frac{P_\phi^1(\theta,\phi)}{\sin^2\theta}\int \frac{e^{\frac{B}{2}}}{r^2} dr + P^3(\theta,\phi),$$

(12)

where $P^1(\theta,\phi)$, $P^2(\theta,\phi)$ and $P^3(\theta,\phi)$ are functions of integration which are to be determined. If one proceeds further after some lengthy calculations one finds that there exists only one possibility when the above space-time admits proper self similar vector field. To avoid details here we shall write only the result as follows:

Case I:

In this case we have $B = \eta$, where $\eta \in \Re \setminus \{0,1\}$. The line element (5) becomes

$$ds^2 = -du^2 + e^\eta dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

(13)

The above space-time is $1+3$ decomposable but the rank of the 6×6 Riemann matrix is one. It is important to note that if we choose $\eta = 0$ or $\eta = 1$ the above space-times (13) become flat which gives contradict to our assumption. Here we already assumed that we are interested in non flat the space-time. Self similar vector fields in this case becomes

$$X^0 = \alpha u + \beta, \qquad\qquad X^1 = \frac{\delta r}{2},$$

$$X^2 = c_1 \cos\phi + c_2 \sin\phi, \quad X^3 = \cot\theta (c_2 \cos\phi - c_1 \sin\phi) + c_3,$$

(14)

where $c_1, c_2, c_3 \in \Re$. In view of equation (14) we need to consider two distinct cases which are:

Tilted Case:

If we choose $\alpha \neq 0$, $\delta \neq 0$ and $\alpha = \delta$ then the first kind proper self similar vector fields for static spherically symmetric space-times after subtracting Killing vector fields from (14) can be written as

$$X = (\alpha u, \frac{\alpha r}{2}, 0, 0).$$

(15)

If we choose $\alpha \neq 0$, $\delta \neq 0$ and $\alpha \neq \delta$ then the second kind proper self similar vector fields for static spherically symmetric space-times after subtracting Killing vector fields from (14) can be written as

$$X = (\alpha u, \frac{\delta r}{2}, 0, 0).$$

(16)

Non Tilted Case:

If we choose $\alpha = 0$ and $\delta \neq 0$, then the proper self similar vector fields for static spherically symmetric space-times after subtracting Killing vector fields from (14) can be written as

$$X = (0, \frac{\delta r}{2}, 0, 0). \tag{17}$$

In this case the proper self similar vector field is of zeroth kind orthogonal to the time like vector u^a. Now by choosing $\alpha \neq 0$ and $\delta = 0$ proper self similar vector fields for the above space-times after subtracting Killing vector fields from (14) can be written as

$$X = (\alpha u, 0, 0, 0). \tag{18}$$

In this case the proper self similar vector field is of infinite kind and parallel to the time like vector u^a.

3. Conclusion:

In this paper we used algebraic and direct integration techniques to study self similar vector fields in the static spherically symmetric space-times. From the above study we find there exists only one case when the above space-time admits proper self similar vector field in both the tilted and non-tilted cases. It is important to note that when the above space-times admit proper self similar vector field the rank of the 6×6 Riemann matrix is one.

R E F E R E N C E S

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