APPROXIMATING COMMON FIXED POINTS OF NONEXPANSIVE MAPPINGS IN CAT(0) SPACES

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In this paper, we construct a new iteration process in the setting of CAT(0) spaces involving two nonexpanisve mappings. We prove strong and delta convergence results for approximating common fixed points via newly defined iteration process. Further, we reconfirm our results by examples and tables.

Keywords: CAT(0) space; Common fixed point; Δ-convergence; Nonexpansive mapping.

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1. **Introduction**

Fixed point theory is an exciting branch of mathematics. It is a mixture of analysis, topology and geometry. Over the last 50 years or so the theory of fixed points has been revealed as a very powerful and important tool in the study of nonlinear phenomena. It has numerous applications in almost all areas of mathematical sciences. For example, proving the existence of solutions of ordinary and partial differential equations, variational inequalities, integral equations, system of linear equations, closed orbit of dynamical systems and zero of monotone operators. Owing to it's importance fixed point theory is attracting young researchers across the world. Metric fixed point theory is one of the active branch of fixed point theory in which geometric properties of underlying space play a significant role. Approximation of fixed points in uniformly convex Banach space for different classes of nonlinear mappings using the various iterative processes is the thrust and active research field so that many iterative algorithms have been presented to solve nonlinear problems, see for example [5, 11, 12,18, 20, 25, 26, 27, 28, 29]. By now, there exists an extensive literature on the iterative fixed points for various classes of mappings.

Recently, many authors have introduced three step iteration process for approximation of fixed points of nonexpansive mapping in various spaces like [1, 19, 22, 24].

In 2016, Thakur et al. [23] obtained the following new iteration scheme for approximation of fixed points of nonexpansive mappings in the framework of Banach space.

Let K be a nonempty closed convex subset of a uniformly convex Banach space E and $S : K \to K$ be a nonexpansive map. For any arbitrary $u_1 \in K$ construct a sequence $\{u_n\}$ by:

$$\begin{aligned} v_n &= (1-\eta_n)u_n + \eta_n S u_n \\ w_n &= (1-\lambda_n)v_n + \lambda_n S v_n \\ u_{n+1} &= (1-\delta_n)S v_n + \delta_n S w_n \end{aligned} \quad (1)$$

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for $n \in \mathbb{N}$, where $\{\eta_n\}$, $\{\lambda_n\}$ and $\{\delta_n\}$ are real sequences in $(0,1)$. Further, they showed that the new iteration process is faster than a number of existing iteration processes.
It is always matter of attraction to extend the result of linear space to the nonlinear space. Due to absence of natural linear and convex structure, many problems cannot be studied in metric space. Therefor we are restricting our study to a spacial class CAT(0) space of a metric spaces which properly includes classes of Hilbert spaces and some of Banach spaces. The term CAT(0) space was first coined by M. Gromov. A metric space E is said to be a CAT(0) space if it is geodesically connected and if every geodesic triangle in E is at least as thin as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature and the complex Hilbert ball with a hyperbolic metric [9] is a CAT(0) space. Other examples include pre-Hilbert spaces, $\mathbb{R}$- trees [2] and Euclidean buildings [3]. For a thorough discussion of these spaces and of the fundamental role they play in geometry, see Bridson and Haefliger [2]. Also, one can refer Burago et al. [4] for more elementary information and Gromov [10] for comparatively deeper study about these spaces.
It was Kirk who initiated the study of fixed point theory in CAT(0) spaces [13, 14]. He obtained that one can always find a fixed point for every nonexpansive (single valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space. Since, then several authors studied extensively this class of spaces and numerous fixed point results involving various mappings have been obtained.
We now present (1) in a CAT(0) space for two nonexpansive mappings as follows:
Let K be a nonempty closed convex subset of a complete CAT(0) space E and $S_1, S_2 : K \to K$ be two nonexpansive mappings. Let $u_1 \in K$ be any arbitrary, then the sequence $\{u_n\}$ is generated iteratively by:

$$\begin{aligned} v_n &= (1-\eta_n)u_n \oplus \eta_n S_1 u_n \\ w_n &= (1-\lambda_n)v_n \oplus \lambda_n S_2 v_n \\ u_{n+1} &= (1-\delta_n)S_2 v_n \oplus \delta_n S_2 w_n \end{aligned} \tag{2}$$

for $n \in \mathbb{N}$, where $\{\eta_n\}$, $\{\lambda_n\}$ and $\{\delta_n\}$ are real sequences in $(0,1)$.
The purpose of this paper is to study newly defined iteration process for two nonexpansive mappings in the setting of CAT(0) spaces and obtain strong and Δ-convergence theorems for the above mentioned iteration scheme.

2. **Preliminaries**

We begin by recalling some known facts in the existing literature of CAT(0) space.

Definition 2.1. *Let K be a non empty subset of a CAT(0) space E. Then, a mapping $S : K \to K$ is said to be nonexpansive if*

$$d(Su, Sv) \le d(u, v) \ \ for \ \ all \ \ u, v \in K.$$

A point $u \in K$ is said to be a fixed point of S if $Su = u$. We will denote the set of fixed points of S by $F(S)$.
First we state the following lemmas to be used later on.

Lemma 2.1. [8] *Let (E, d) be a CAT(0) space. For $u, v \in E$ and $t \in [0, 1]$, there exists a unique $w \in [u, v]$ such that*

$$d(u, w) = td(u, v) \ \ and \ \ d(v, w) = (1-t)d(u, v).$$

We use the notation $(1-t)u \oplus tv$ for the unique point w of the above lemma.

Lemma 2.2. [8] *Let (E, d) be a CAT(0) space. For $u, v, w \in E$ and $t \in [0, 1]$ we have*

$$d((1-t)u \oplus tv, w) \leq (1-t)d(u, w) + td(v, w).$$

Lemma 2.3. [8] *Let E be a CAT (0) space. Then*

$$d^2((1-t)u \oplus tv, w) \leq (1-t)d^2(u, w) + td^2(v, w) - t(1-t)d^2(u, v)$$

for all $u, v, w \in E$ and $t \in [0, 1]$.

Now, we collect some basic geometric properties, which are instrumental throughout the discussions.
Let $\{u_n\}$ be a bounded sequence in a complete CAT(0) space E. For $u \in E$ write:

$$r(u, \{u_n\}) = \limsup_{n\to\infty} d(u, u_n).$$

The asymptotic radius $r(\{u_n\})$ is given by

$$r(\{u_n\}) = \inf\{r(u, u_n) : u \in E\}$$

and the asymptotic center $A(\{u_n\})$ of $\{u_n\}$ is defined as:

$$A(\{u_n\}) = \{u \in E : r(u, u_n) = r(\{u_n\})\}.$$

In 2006, Dhompongsa, Kirk and Sims proved that $A(\{u_n\})$ consists of exactly one point if E is a CAT(0) space (Proposition 5 of [6]).
In 2008, Kirk and Panyanak [15] obtained an analogue result of weak convergence in Banach space and restriction of Lim's [17] concept of convergence to CAT(0) spaces which is known as Δ-convergence.

Definition 2.2. *A sequence $\{u_n\}$ in E is said to be Δ-convergent to $u \in E$ if u is the unique asymptotic center of $\{v_n\}$ for every subsequence $\{v_n\}$ of $\{u_n\}$. In this case, we write $\Delta - \lim_n u_n = u$ and read as u is the Δ-limit of $\{u_n\}$.*

From the definition of Δ-convergence it can be easily seen that every CAT(0) space satisfies Opial's property.
Now, we list few results which will be frequently used throughout the text.

Lemma 2.4. *The following assertions hold in a CAT(0) space:*

(i) *([15]) Every bounded sequence in a complete CAT(0) space admits a Δ-convergent subsequence.*
(ii) *([8]) If K is a closed convex subset of a complete CAT(0) space E and if $\{u_n\}$ is a bounded sequence in K, then the asymptotic center of $\{u_n\}$ is in K.*
(iii) *([7]) Let K be a nonempty bounded closed convex subset of a complete CAT(0) space (E, d) and $S : K \to K$ be a nonexpansive mapping. If $\{u_n\}$ is a bounded sequence in K such that $\Delta - \lim_n u_n = u$ and $\lim_{n\to\infty} d(Su_n, u_n) = 0$ then u is a fixed point of S.*

Lemma 2.5. [8] *If $\{u_n\}$ is a bounded sequence in a complete CAT(0) space with $A(\{u_n\}) = \{u\}$, $\{v_n\}$ is a subsequence of $\{u_n\}$ with $A(\{v_n\}) = \{v\}$ and the sequence $\{d(u_n, v)\}$ converges, then $v = u$.*

The following lemma is a consequence of Lemma 2.9 of [?] which will be used to prove our main result.

Lemma 2.6. [16] *Let (E, d) be a complete CAT(0) space and $u \in E$. Suppose $\{t_n\}$ is a sequence in $[b, c]$ for some $b, c \in (0, 1)$ and $\{u_n\}$, $\{v_n\}$ are sequences in E such that $\limsup_{n\to\infty} d(u_n, u) \leq r$, $\limsup_{n\to\infty} d(v_n, u) \leq r$ and $\lim_{n\to\infty} d(t_n v_n \oplus (1-t_n)u_n, u) = r$ hold for some $r \geq 0$, then $\lim_{n\to\infty} d(u_n, v_n) = 0$.*

3. Some Δ-convergence and strong convergence theorems

Let us begin with the following important lemma.

Lemma 3.1. *Let $S_1, S_2 : K \to K$ be two nonexpansive mappings defined on a closed convex subset K of a complete CAT(0) space E with $F(S_1) \cap F(S_2) \neq \emptyset$. If $\{u_n\}$ is a sequence defined by (2), then $\lim_{n\to\infty} d(u_n, q)$ exists for all $q \in F(S_1) \cap F(S_2)$.*

Proof. For any $q \in F(S_1) \cap F(S_2)$, we have

$$\begin{aligned} d(v_n, q) &= d((1-\eta_n)u_n \oplus \eta_n S_1 u_n, q) \\ &\leq (1-\eta_n)d(u_n, q) + \eta_n d(S_1 u_n, q) \\ &\leq (1-\eta_n)d(u_n, q) + \eta_n d(u_n, q) \\ &= d(u_n, q) \end{aligned} \tag{3}$$

and

$$\begin{aligned} d(w_n, q) &= d((1-\lambda_n)v_n \oplus \lambda_n S_2 v_n, q) \\ &\leq (1-\lambda_n)d(v_n, q) + \lambda_n d(S_2 v_n, q) \\ &\leq (1-\lambda_n)d(v_n, q) + \lambda_n d(v_n, q) \\ &\leq (1-\lambda_n)d(u_n, q) + \lambda_n d(u_n, q) \\ &= d(u_n, q). \end{aligned} \tag{4}$$

Using (3) and (4), we get

$$\begin{aligned} d(u_{n+1}, q) &= d((1-\delta_n)S_2 v_n \oplus \delta_n S_2 w_n, q) \\ &\leq (1-\delta_n)d(S_2 v_n, q) + \delta_n d(S_2 w_n, q) \\ &\leq (1-\delta_n)d(v_n, q) + \delta_n d(w_n, q) \\ &\leq (1-\delta_n)d(u_n, q) + \delta_n d(u_n, q) \\ &= d(u_n, q). \end{aligned}$$

Thus, $\{d(u_n, q)\}$ is a decreasing sequence of reals which is bounded below by zero and hence convergent. Therefore, $\lim_{n\to\infty} d(u_n, q)$ exists for all $q \in F(S_1) \cap F(S_2)$. □

Lemma 3.2. *Let $S_1, S_2 : K \to K$ be two nonexpansive mappings defined on a closed convex subset K of a complete CAT(0) space E with $F(S_1) \cap F(S_2) \neq \emptyset$. If $\{u_n\}$ is a sequence defined by (2), then $\lim_{n\to\infty} d(S_1 u_n, u_n) = 0$ and $\lim_{n\to\infty} d(S_2 u_n, u_n) = 0$.*

Proof. By Lemma 3.1, it follows that $\lim_{n\to\infty} d(u_n, q)$ exists for all $q \in F(S_1) \cap F(S_2)$, say $\lim_{n\to\infty} d(u_n, q) = c$.
From (3) and (4) we have

$$\limsup_{n\to\infty} d(w_n, q) \leq c \tag{5}$$

$$\limsup_{n\to\infty} d(v_n, q) \leq c. \tag{6}$$

Since S_1 and S_2 are nonexpansive mappings, we have
$d(S_1 u_n, q) \leq d(u_n, q)$, $d(S_1 v_n, q) \leq d(v_n, q)$, $d(S_1 w_n, q) \leq d(w_n, q)$,
$d(S_2 u_n, q) \leq d(u_n, q)$, $d(S_2 v_n, q) \leq d(v_n, q)$ and $d(S_2 w_n, q) \leq d(w_n, q)$
which implies that

$$\limsup_{n\to\infty} d(S_1 u_n, q) \leq c, \tag{7}$$

$$\limsup_{n\to\infty} d(S_1 v_n, q) \leq c, \tag{8}$$

$$\limsup_{n\to\infty} d(S_1 w_n, q) \le c, \tag{9}$$

$$\limsup_{n\to\infty} d(S_2 u_n, q) \le c, \tag{10}$$

$$\limsup_{n\to\infty} d(S_2 v_n, q) \le c \tag{11}$$

and

$$\limsup_{n\to\infty} d(S_2 w_n, q) \le c \tag{12}$$

Now, Using (3) and (4), we have $d(u_{n+1}, q) \le d(v_n, q) \le d(u_n, q)$ which gives

$$\lim_{n\to\infty} d(v_n, q) = c. \tag{13}$$

Owing to Lemma 2.6, (7) and (13), we get

$$\lim_{n\to\infty} d(S_1 u_n, u_n) = 0. \tag{14}$$

Since, $c = \lim_{n\to\infty} d(u_{n+1}, q) = d((1-\delta_n)S_2 v_n \oplus \delta_n S_2 w_n, q)$.
By using Lemma 2.6, (11) and (12) we get

$$\lim_{n\to\infty} d(S_2 v_n, S_2 w_n) = 0. \tag{15}$$

Now,

$$\begin{aligned} d(u_{n+1}, q) &= d((1-\delta_n)S_2 v_n \oplus \delta_n S_2 w_n, q) \\ &\le (1-\delta_n) d(S_2 v_n, q) + \delta_n d(S_2 w_n, q) \\ &\le (1-\delta_n) d(S_2 v_n, q) + \delta_n d(S_2 v_n, S_2 w_n) + \delta_n d(S_2 v_n, q) \\ &= d(S_2 v_n, q) + \delta_n d(S_2 v_n, S_2 w_n) \end{aligned}$$

which yields that

$$c \le \liminf_{n\to\infty} d(S_2 v_n, q). \tag{16}$$

Owing to Equations (11) and (16), we get

$$\lim_{n\to\infty} d(S_2 v_n, q) = c. \tag{17}$$

Also, we have

$$\begin{aligned} d(S_2 v_n, q) &\le d(S_2 v_n, S_2 w_n) + d(S_2 w_n, q) \\ &\le d(S_2 v_n, S_2 w_n) + d(w_n, q) \end{aligned}$$

which on using (15) and (17)gives

$$c \le \liminf_{n\to\infty} d(w_n, q). \tag{18}$$

Now, by using (5) and (18), we get

$$\lim_{n\to\infty} d(w_n, q) = c. \tag{19}$$

In view of Lemma 2.6, (6), (11) and (19), we obtain

$$\lim_{n\to\infty} d(S_2 v_n, v_n) = 0. \tag{20}$$

Now,

$$\begin{aligned} d(v_n, u_n) &= d((1-\eta_n)u_n \oplus \eta_n S_1 u_n, u_n) \\ &\le (1-\eta_n) d(u_n, u_n) \oplus \eta_n d(S_1 u_n, u_n) \end{aligned}$$

which on using (14) gives

$$\lim_{n\to\infty} d(v_n, u_n) = 0. \tag{21}$$

Consider,

$$\begin{aligned} d(u_n, S_2u_n) &\leq d(u_n, v_n) + d(v_n, S_2v_n) + d(S_2v_n, S_2u_n) \\ &\leq d(u_n, v_n) + d(v_n, S_2v_n) + d(v_n, u_n). \end{aligned}$$

Owing to (20) and (21), we get

$$\lim_{n\to\infty} d(u_n, S_2u_n) = 0.$$

□

Now, we prove the Δ convergence of iteration process (2).

Theorem 3.1. *Let $S_1, S_2 : K \to K$ be two nonexpansive mappings defined on a closed convex subset K of a complete CAT(0) space E with $F(S_1) \cap F(S_2) \neq \emptyset$. If $\{u_n\}$ is a sequence defined by (2), then $\{u_n\}$ Δ-converges to a common fixed point of S_1 and S_2.*

Proof. From Lemma 3.1 and 3.2, we have $\lim_{n\to\infty} d(u_n, q)$ exists for each $q \in F(S_1) \cap F(S_2)$ so that the sequence $\{u_n\}$ is bounded, $\lim_{n\to\infty} d(u_n, S_1u_n) = 0$ and $\lim_{n\to\infty} d(u_n, S_2u_n) = 0$.
Let $W_\omega(\{u_n\}) =: \cup A(\{b_n\})$, where union is taken over all subsequences $\{b_n\}$ over $\{u_n\}$. In order to show the Δ-convergence of $\{u_n\}$ to a common fixed point of S_1 and S_2, firstly we will prove $W_\omega(\{u_n\}) \subset F(S_1) \cap F(S_2)$ and thereafter argue that $W_\omega(\{u_n\})$ is a singleton set. To show $W_\omega(\{u_n\}) \subset F(S_1) \cap F(S_2)$, let $y \in W_\omega(\{u_n\})$. Then, there exists a subsequence $\{y_n\}$ of $\{u_n\}$ such that $A(\{y_n\}) = y$. By (i) and (ii) of Lemma 2.4, there exists a subsequence $\{z_n\}$ of $\{y_n\}$ such that $\Delta - \lim_n z_n = z$ and $z \in K$. Since $\lim_{n\to\infty} d(S_1u_n, u_n) = 0$ and $\{z_n\}$ is a subsequence of $\{u_n\}$, so $\lim_{n\to\infty} d(z_n, S_1z_n) = 0$. In view of Lemma 2.4(iii), we have $z = S_1z$ and hence $z \in F(S_1)$.
Similarly, $z \in F(S_2)$ so $z \in F(S_1) \cap F(S_2)$. By Lemma 2.5, we obtain $y = z$ which shows that $W_\omega(\{u_n\}) \subset F(S_1) \cap F(S_2)$. Now it is left to show that $W_\omega(\{u_n\})$ consists of single element only. For this, let $\{y_n\}$ be a subsequence of $\{u_n\}$. Again, by using Lemma 2.4, we can find a subsequence $\{z_n\}$ of $\{y_n\}$ such that $\Delta - \lim_n z_n = z$. Let $A(\{y_n\}) = y$ and $A(\{u_n\}) = u$. It is enough to show that $z = u$. If $z \neq u$ then, since $z \in F(S_1) \cap F(S_2)$, by Lemma 3.2, $\{d(u_n, z)\}$ is convergent. Again, by Lemma 2.5, we have $z = u$ which proves that $W_\omega(\{u_n\})$ is a singleton set. Hence the conclusion follows. □

Next, we establish some strong convergence results for iteration process (2).

Theorem 3.2. *Let $S_1, S_2 : K \to K$ be two nonexpansive mappings defined on a closed convex subset K of a complete CAT(0) space E with $F(S_1) \cap F(S_2) \neq \emptyset$. If $\{u_n\}$ is a sequence defined by (2), then $\{u_n\}$ converges to a common fixed point of S_1 and S_2 if and only if* $\liminf_{n\to\infty} d(u_n, F(S_1) \cap F(S_2)) = 0$.

Proof. If the sequence $\{u_n\}$ converges to a point $u \in F(S_1) \cap F(S_2)$, then it is obvious that $\liminf_{n\to\infty} d(u_n, F(S_1) \cap F(S_2)) = 0$.
For converse part, assume that $\liminf_{n\to\infty} d(u_n, F(S_1) \cap F(S_2)) = 0$. From Lemma 3.1, we have

$$d(u_{n+1}, q) \leq d(u_n, q) \quad for\ \ any\ \ q \in F(S_1) \cap F(S_2)$$

which yields

$$d(u_{n+1}, F(S_1) \cap F(S_2)) \leq d(u_n, F(S_1) \cap F(S_2)). \tag{22}$$

Thus, $\{d(u_n, F(S_1) \cap F(S_2))\}$ forms a decreasing sequence which is bounded below by zero as well, so we get that $\lim_{n\to\infty} d(u_n, F(S_1) \cap F(S_2))$ exists. As, $\liminf_{n\to\infty} d(u_n, F(S_1) \cap F(S_2)) = 0$ so that $\lim_{n\to\infty} d(u_n, F(S_1) \cap F(S_2)) = 0$.

Now, we prove that $\{u_n\}$ is a Cauchy sequence in K. Let ϵ >0 be arbitrarily chosen. Since $\liminf_{n\to\infty} d(u_n, F(S_1) \cap F(S_2)) = 0$, there exists n_0 such that for all $n \geq n_0$, we have

$$d(u_n, F(S_1) \cap F(S_2)) < \frac{\epsilon}{4}.$$

In particular,

$$\inf\{d(u_{n_0}, q) : q \in F(S_1) \cap F(S_2)\} < \frac{\epsilon}{4},$$

so there must exist a $r \in F(S_1) \cap F(S_2)$ such that

$$d(u_{n_0}, r) < \frac{\epsilon}{2}.$$

Thus, for $m, n \geq n_0$, we have

$$d(u_{n+m}, u_n) \leq d(u_{n+m}, r) + d(u_n, r) < 2d(u_{n_0}, r) < 2\frac{\epsilon}{2} = \epsilon$$

which shows that $\{u_n\}$ is a cauchy sequence. Since K is a closed subset of a complete metric space E, so K itself is a complete metric space and therefore $\{u_n\}$ must converge in K. Let $\liminf_{n\to\infty} u_n = g$.

Now, using $\lim_{n\to\infty} d(S_1 u_n, u_n) = 0$, we get

$$\begin{aligned} d(g, S_1 g) &\leq d(g, u_n) + d(u_n, S_1 u_n) + d(S_1 u_n, S_1 g) \\ &\leq d(g, u_n) + d(u_n, S_1 u_n) + d(u_n, g) \end{aligned}$$

$$\to 0 \ \ as \ \ n \to \infty$$

and hence $g = S_1 g$. Similarly, we can show that $g = S_2 g$. Thus, $g \in F(S_1) \cap F(S_2)$. □

Two mappings $S_1, S_2 : K \to K$ are said to satisfy the Condition (A)[**?**] if there exists a nondecreasing function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ and $g(r) > 0$ for all $r \in (0, \infty)$ such that

$$d(u, S_1 u) \geq g(d(u, F(S_1) \cap F(S_2)))$$

or

$$d(u, S_2 u) \geq g(d(u, F(S_1) \cap F(S_2)))$$

for all $u \in K$.

Theorem 3.3. *Let E be a complete CAT(0) space and K be a nonempty closed convex subset of E. Let $S_1, S_2 : K \to K$ be two nonexpansive mappings such that $F(S_1) \cap F(S_2) \neq \emptyset$ and $\{u_n\}$ be the sequence defined by (2). If S_1 and S_2 satisfies Condition (A), then $\{u_n\}$ converges strongly to a common fixed point of S_1 and S_2.*

Proof. From (22) we get $\lim_{n\to\infty} d(u_n, F(S_1) \cap F(S_2))$ exists.

Also, by Lemma 3.2, we have $\lim_{n\to\infty} d(u_n, S_1 u_n) = \lim_{n\to\infty} d(u_n, S_2 u_n) = 0$.

It follows from Condition (A) that

$$\lim_{n\to\infty} g(d(u_n, F(S_1) \cap F(S_2))) \leq \lim_{n\to\infty} d(u_n, S_1 u_n) = 0$$

or

$$\lim_{n\to\infty} g(d(u_n, F(S_1) \cap F(S_2))) \leq \lim_{n\to\infty} d(u_n, S_2 u_n) = 0$$

Iteration Number	*When* $u_1 = 2$	*When* $u_1 = 15$	*When* $u_1 = 23$
1	*2*	*15*	*23*
2	*5.58393776403457*	*10.7055368022956*	*17.5604324181304*
3	*5.93152540890778*	*7.68023641121788*	*12.5536925818119*
4	*5.98903831742232*	*6.37498607890297*	*8.7007403912898*
5	*5.99833695210071*	*6.06186110821241*	*6.68600635298456*
6	*5.99975919026045*	*6.00909152785473*	*6.11595232054481*
7	*5.9999664068792*	*6.00127118425478*	*6.01664010954467*
8	*5.99999545165397*	*6.00017216946208*	*6.00226257696089*
9	*5.99999939899435*	*6.0000227510451*	*6.00029914870638*
10	*5.99999992217448*	*6.00000294610066*	*6.00003874054349*
11	*5.99999999009259*	*6.00000037504734*	*6.00000493183451*
12	*5.99999999875701*	*6.00000004705388*	*6.00000061875451*
13	*5.99999999984601*	*6.00000000582953*	*6.00000007665784*
14	*5.99999999998113*	*6.00000000071432*	*6.00000000939329*
15	*5.99999999999771*	*6.00000000008669*	*6.00000000113992*
16	*5.99999999999973*	*6.00000000001043*	*6.00000000013715*
17	*5.99999999999997*	*6.00000000000125*	*6.00000000001638*
18	*6.00000000000000*	*6.00000000000015*	*6.00000000000194*
19	*6.00000000000000*	*6.00000000000002*	*6.00000000000023*
20	*6.00000000000000*	*6.00000000000000*	*6.00000000000003*
21	*6.00000000000000*	*6.00000000000000*	*6.00000000000000*

Table 1.

so that $\lim_{n\to\infty} g(d(u_n, F(S_1) \cap F(S_2))) = 0$.
Since g is a non-decreasing function satisfying $g(0) = 0$ and $g(r) > 0$ for all $r \in (0, \infty)$, therefore $\lim_{n\to\infty} d(u_n, F(S_1) \cap F(S_2)) = 0$.
By Theorem 3.2., the sequence $\{u_n\}$ converges strongly to a point of $F(S_1) \cap F(S_2)$. □

4. **Numerical example**

To illustrate our results and the convergence behavior of our iteration process (2), we furnish following two examples. First example is in the setting of one dimensional Euclidean space while second example is in the setting of two dimensional Euclidean space.

Example 4.1. *Let* $E = \mathbb{R}$ *and* $K = [1, 50]$. *Let* $S_1, S_2 : K \to K$ *be mappings defined as* $S_1(u) = \sqrt{u^2 - 9u + 54}$ *and* $S_2(u) = \sqrt{u^2 - 7u + 42}$ *for all* $u \in K$. *Clearly,* $u = 6$ *is the common fixed point of* S_1 *and* S_2. *Set* $\eta_n = \frac{n}{n+1}$, $\lambda_n = \frac{1}{n+1}$ *and* $\delta_n = 0.75$ *for all* $n \in \mathbb{N}$. *Then, we get the following table of iteration values and graph for three different initial points.*

It is evident from the Table 1. and Figure 1. that our iteration process (2) converges to the common fixed point of S_1 *and* S_2.

Now, we present another example to illustrate the utility of our newly proved result:

Example 4.2. *Let* $E = \mathbb{R}^2$ *equipped with the Euclidean norm. Let* $x = (x_1, x_2) \in \mathbb{R}^2$, *then the squared distance of* x *from the origin is:*

$$\|x\|^2 = {x_1}^2 + {x_2}^2$$

Consider K *as the closed unit disk:*

$$K = \{(x_1, x_2) \in \mathbb{R}^2 : {x_1}^2 + {x_2}^2 \le 1\}$$

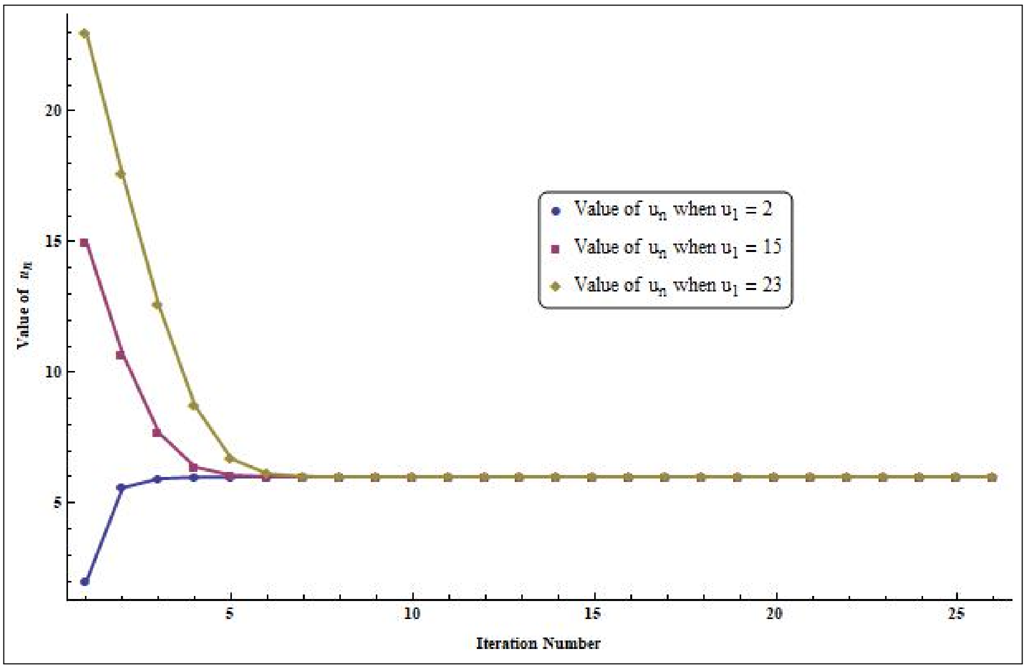

Fig. 1. Graph corresponding to Table 1.

which is bounded, closed and convex in E. We define the mapping $Rot_\theta : K \to K$ by:

$$Rot_\theta(x_1, x_2) = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Let $\theta = \frac{\pi}{4}$. Then,

$$Rot_{\frac{\pi}{4}}(x_1, x_2) = \frac{1}{\sqrt{2}} \begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \end{bmatrix}.$$

Also, for $(x_1, x_2), (y_1, y_2) \in K$, we have

$$\begin{aligned} \|Rot_{\frac{\pi}{4}}(x_1, x_2) - Rot_{\frac{\pi}{4}}(y_1, y_2)\| &= \tfrac{1}{\sqrt{2}} \left\| \begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \end{bmatrix} - \begin{bmatrix} y_1 - y_2 \\ y_1 + y_2 \end{bmatrix} \right\| \\ &= \tfrac{1}{\sqrt{2}} \left\| \begin{bmatrix} (x_1 - y_1) - (x_2 - y_2) \\ (x_1 - y_1) + (x_2 - y_2) \end{bmatrix} \right\| \\ &= \tfrac{1}{\sqrt{2}} \sqrt{[(x_1 - y_1) - (x_2 - y_2)]^2 + [(x_1 - y_1) + (x_2 - y_2)]^2} \\ &= \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \\ &= \|x - y\|. \end{aligned}$$

So, $Rot_{\frac{\pi}{4}}$ is nonexpansive. Similarly, we can show that Rot_θ is nonexpansive for $\theta = \frac{\pi}{2}$. Let $S_1 = Rot_{\frac{\pi}{4}}$ and $S_2 = Rot_{\frac{\pi}{2}}$. Then, S_1 and S_2 are nonexpansive mappings and zero is

Iteration Number	*Values of* $(u_{(n)_1}, u_{(n)_2})$
0	$(0.02, 0.02)$
1	$(-0.0142807765030734, -0.012557426932523)$
5	$(-0.00347359764140002, -0.00174169924319616)$
10	$(0.000528307227174463, 0.0000766890111449752)$
15	$(-0.0000722024152888611, 0.0000128754619898957)$
20	$(8.85645098799705 \times 10^{-6}, -4.80491510472172 \times 10^{6})$
25	$(-9.46684911822667 \times 10^{-7}, 1.00993830140719 \times 10^{-6})$
30	$(7.97033086207781 \times 10^{-8}, -1.72667253983818 \times 10^{-7})$
35	$(-2.91585691362794 \times 10^{-9}, 2.59637256076364 \times 10^{-8})$
40	$(-7.43986882842456 \times 10^{-10}, -3.511462629803 \times 10^{-9})$
45	$(2.49040224404431 \times 10^{-10}, 4.256202605727 \times 10^{-10})$
50	$(-5.08987438501134 \times 10^{-11}, -4.47105913312863 \times 10^{-11})$
55	$(8.57215654351895 \times 10^{-12}, 3.62568886834775 \times 10^{-12})$
60	$(-1.27464165147906 \times 10^{-12}, -1.01573491818395 \times 10^{-13})$
65	$(1.70588740042888 \times 10^{-13}, -4.19453479760621 \times 10^{-14})$
70	$(-2.04256460880254 \times 10^{-14}, 1.28549952063688 \times 10^{-14})$
75	$(2.10655657004786 \times 10^{-15}, -2.56044400483511 \times 10^{-15})$
80	$(-1.6379722615829 \times 10^{-16}, 4.25045439774781 \times 10^{-16})$
85	$(2.95289117447557 \times 10^{-18}, -6.25104577870752 \times 10^{-17})$

TABLE 2.

the common fixed point. In this case, our algorithm is the following:

$$\begin{array}{lll} u_{(1)} & = & (u_{(1)_1}, u_{(1)_2}) \in K \\ (v_{(n)_1}, v_{(n)_2}) & = & (1-\eta_n)(u_{(n)_1}, u_{(n)_2}) + \eta_n Rot_{\frac{\pi}{4}}(u_{(n)_1}, u_{(n)_2}) \\ (w_{(n)_1}, w_{(n)_2}) & = & (1-\lambda_n)(v_{(n)_1}, v_{(n)_2}) + \lambda_n Rot_{\frac{\pi}{2}}(v_{(n)_1}, v_{(n)_2}) \\ (u_{(n+1)_1}, u_{(n+1)_2}) & = & (1-\delta_n) Rot_{\frac{\pi}{2}}(v_{(n)_1}, v_{(n)_2}) + \delta_n Rot_{\frac{\pi}{2}}(w_{(n)_1}, w_{(n)_2}) \end{array}$$

Now, by setting $\eta_n = \lambda_n = \delta_n = 0.75$ *for all* $n \in \mathbb{N}$ *and taking* $u_{(1)} = (0.02, 0.02)$*, we get the following iteration table and graph:*

Thus, Table 2. and Figure 2. shows that our iteration process (2) converges to the common fixed point of S_1 *and* S_2*.*

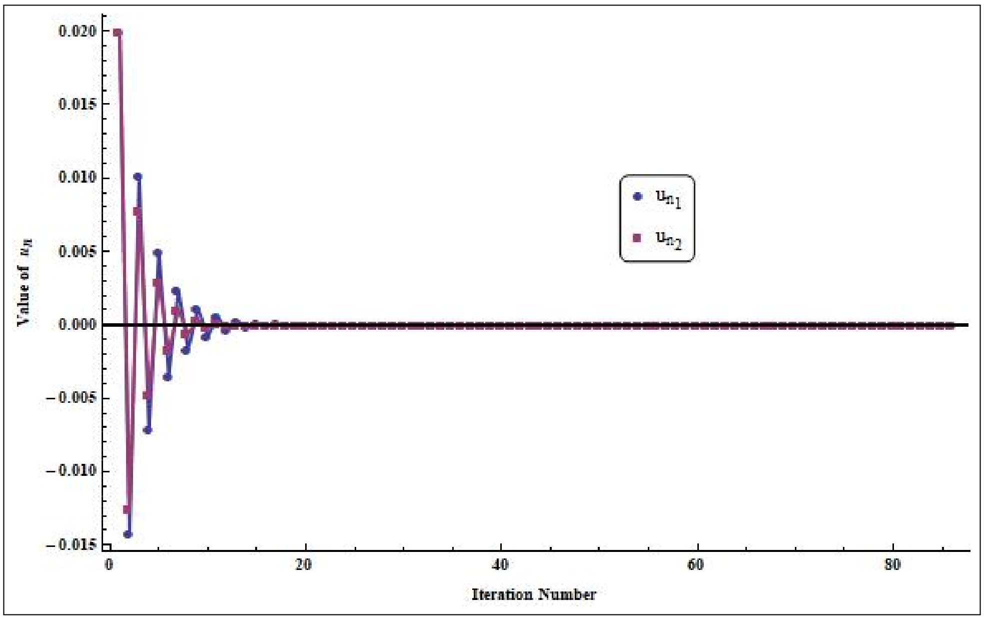

Fig. 2. Graph corresponding to Table 2.

5. Conclusion

We have proved the Δ and strong convergence of newly introduced iteration process (2) under suitable conditions. We have added two non trivial examples to support our claim. So, iteration process (2) can be used to approximate common fixed points of two nonexpansive mappings in CAT(0) space.

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