U.P.B. Sci. Bull., Series A, Vol. 76, Iss. 2, 2014 ISSN 1223-7027

ON BIPRODUCTS AND TERMINAL COALGEBRAS

Adriana Balan[1]

A category $\mathscr{C}$ with (countable) biproducts admits summation of countable families of arrows. If this summation is also idempotent, then a version of limit-colimit coincidence holds. In particular, for a $\mathscr{C}$-endofunctor H which is ω^{op}-continuous, the initial H-algebra and the terminal H-coalgebra coincide. This applies to the case of monadic category $\mathscr{C}$ with such biproducts as above, and the endofunctor H comes from a lifting.

Keywords: biproducts, fixed point, initial algebra, terminal coalgebra
MSC2010: 18A30

1. Introduction

Categories with countable biproducts are models of the partially additive categories introduced by Manes and Arbib ([3]) as an algebraic semantics for programming languages.

They have been also shown to closely relate to iteration theories and categories with fixed-point operations ([8]), as they provide a categorical method for the construction of a trace operator. More recently, it was noticed that strong unique decomposition categories can be characterized in terms of faithful embeddings into categories with countable biproducts ([10]).

Infinite data structures, like streams or infinite trees, are often used to model dynamical processes and to compute solutions for them, by means of coalgebraic techniques. They usually arise as solutions of recursive definitions, in the form $X \cong HX$, where H is a functor (a type constructor), that is, are fixed points of H (terminal coalgebras). Thanks to the properties of terminal coalgebras, proofs within infinite data structures can be conveniently managed by coinduction. But usually infinite data structures carry additional algebraic structure, coming from the existence of a lift of H to some category of algebras. If the category of algebras is conveniently chosen (has countable biproducts), proving things about them can be more easily managed by induction instead of coinduction, as soon as the lifted functor has the unique fixed point property.

The paper is organized as follows: Section 2 contains an introduction to categories with countable biproducts. One main feature of such categories is the existence of a summation operator, defined on countable families of arrows, with nice behaviour: associativity, commutativity, additive identity. Examples of such categories are Rel, the category of sets

[1]Lecturer, Dept. Mathematical Methods and Models, Faculty of Applied Sciences, University Politehnica of Bucharest, Romania; Email: `adriana.balan@mathem.pub.ro`.

and relations, and ωSLat, the category of countably sup-semilattices and continuous morphisms. For both examples, the summation on arrows is idempotent. However, it turns out that it does not hold in arbitrary categories with countable biproducts. If assumed separately, it is equivalent with enrichment over countably sup-semilattices. In particular, there is a natural partial order on arrows completely characterized by summation.

Section 3 contains the technical result of the paper. Under some mild assumptions of splitting idempotents, a version of the limit-colimit coincidence holds for ω-chains of embedding-projection pairs. In Section 4 we apply the above result to the initial-terminal chain of an endofunctor H on $\mathscr{C}$. Under the assumption of ω^{op}-continuity and preservation of the natural order on arrows, such a functor has the unique fixed point property, that is, the initial algebra and the final coalgebra coincide. In the sequel, we give some examples of such functors which arise as liftings when the category $\mathscr{C}$ is taken to be the category of algebras for a monad. The last Section contains some concluding remarks and thoughts on future research directions.

2. **Preliminaries: biproducts and enrichment**

Let $\mathscr{C}$ be a category. Objects of $\mathscr{C}$ will be denoted by $X, Y, \ldots$. We shall write id_X for the identity morphism of an object X of $\mathscr{C}$Assume $\mathscr{C}$ has a zero object denoted 0.[2] Then $\mathscr{C}$ has also zero morphisms defined as $0_{X,Y} : X \to 0 \to Y$, for each pair of objects X, Y.[3]

We consider that the category $\mathscr{C}$ has countable [4] coproducts and products. For a countable set I[5] and a family of objects $(X_i)_{i\in I}$, denote by $\mathsf{in}_i : X_i \to \sqcup_{i\in I} X_i$ the canonical morphisms into the coproduct, respectively by $\mathsf{pr}_i : \sqcap_{i\in I} X_i \to X_i$ the projections from the product to the components. Then for any indexing sets I, J, a morphism $f : \sqcup_{i\in I} X_i \to \sqcap_{j\in J} Y_j$ is uniquely determined by its components $f_{ij} : X_i \to Y_j$, where $f_{ij} = \mathsf{pr}_j f \mathsf{in}_i$. In particular, using the zero morphisms, one can construct a canonical map from the I-th coproduct into the product indexed by same set I, $\theta : \sqcup_{i\in I} X_i \to \sqcap_{i\in I} X_i$ given by

$$\mathsf{pr}_j \theta \mathsf{in}_i = \begin{cases} \mathsf{id}_{X_i} \text{ , if } i = j \\ 0 \quad \text{ , otherwise} \end{cases}$$

Definition 2.1. *$\mathscr{C}$ is said to have* countable biproducts *if θ is an isomorphism for all countable indexing sets I.*[6]

We shall identify in the sequel the coproduct and the product via this isomorphism and denote them by $\oplus_{i\in I} X_i$ (the biproduct of the family $(X_i)_{i\in I}$).

[2]A zero object is an object simultaneously initial and terminal ([17])

[3]That is, $\mathscr{C}$ is enriched over pointed sets and the enrichment is unique ([17]).

[4]By which we mean finite or denumbrable.

[5]For now on, all index sets will be assumed countable.

[6]In fact, any natural isomorphism will suffice – see [16] and [11].

From now on we shall assume $\mathscr{C}$ has countable biproducts. For a (countable) indexing set I and a family of objects $(X_i)_{i\in I}$, it is easy to see that the projections $\mathsf{pr}_i : \oplus_{i\in I}X_i \to X_i$ and the injections $\mathsf{in}_i : X_i \to \oplus_{i\in I}X_i$ satisfy $\mathsf{pr}_j\mathsf{in}_i = \begin{cases} \mathsf{id}_{X_i} & \text{, if } i = j \\ 0 & \text{, otherwise} \end{cases}$.

Let $\oplus_{i\in I}f_i : \oplus_{i\in I}X_i \to \oplus_{i\in I}X_i'$ be the arrow induced by a family of morphisms $(X_i \xrightarrow{f_i} X_i')_{i\in I}$. That is, $\mathsf{pr}_j(\oplus_{i\in I}f_i)\mathsf{in}_k = \begin{cases} f_i & \text{, if } i = j = k \\ 0 & \text{, otherwise} \end{cases}$.

We shall denote the biproduct of I copies of an object X as $X^{\oplus I}$. Let $\Delta : X \to X^{\oplus I}$ and by $\nabla : X^{\oplus I} \to X$ be the diagonal and the codiagonal morphisms determined by $\mathsf{pr}_i\Delta = \mathsf{id}_X$, respectively $\nabla\mathsf{in}_i = \mathsf{id}_X$ for all $i \in I$. Then for any family of maps $(f_i : X \to Y)_{i\in I}$, define $\sum_{i\in I} f_i$ to be the composite $X \xrightarrow{\Delta} X^{\oplus I} \xrightarrow{\oplus_{i\in I}f_i} Y^{\oplus I} \xrightarrow{\nabla} Y$. In particular, for I the empty set, the resulting arrow is the zero morphism. For any pair of arrows $f, g : X \to Y$, we shall write $f + g$ for the composite $X \xrightarrow{\Delta} X \oplus X \xrightarrow{f\oplus g} Y \oplus Y \xrightarrow{\nabla} Y$.

Then $\mathscr{C}$ becomes a partially additive category with all (countable) families of morphisms summable. We do not intend to recall here all the theory of partially additive category (for which we refer to [2], [8] for a complete and detailed exposition); we just emphasize that the summation is commutative and associative (more generally, invariant under permutation of terms), has the zero morphism as countably additive identity, and there are no additive inverses. Composition of arrows distributes over summation from left and right. The canonical injections and projections satisfy $\sum_{i\in I} \mathsf{in}_i\mathsf{pr}_i = \mathsf{id}$, for each indexing set I.

Example 2.1. *(i) In* Rel*, the category of sets and relations, arbitrary coproducts (disjoint unions) are also products, thus biproducts. In particular,* Rel *has countable biproducts and the induced summation on relations is their union.*

(ii) The category ωSLat *of countably sup-semilattices has countable biproducts. Recall that a countably sup-semilattice is a poset such that any countable subset has a supremum (in particular, the empty set has a supremum, thus the poset has a least element). A morphism of countably sup-semilattices is a function preserving all countable suprema; in particular, it preserves the least element and the order. The category of countably sup-semilattices is an (infinitary) variety, as it is the category of algebras for the countably power-set monad. Consequently, (categorical) products of countably-sup-semilattices exist and are formed as in* Set*. These are also coproducts, in case the indexing set is countable. Thus* ωSlat *has countable biproducts (this can be easily shown as in* [13]*, where the case of all (small) biproducts is considered).*

The above examples have one common feature, namely the summation on arrows is countably idempotent. It is worth noticing that not all categories with countable biproducts have this property: take S to be a countably complete semiring [7] which is not idempotent (for example, take $S = (\mathbb{N} \cup \{\infty\}, +, \cdot, 0, 1)$ with $+$ and $\cdot$ defined in an obvious

[7] That is, countable sums exist in S, are an extension of finite sums, are associative and commutative and satisfy the distributivity laws with respect to multiplication (see [7], [15]).

fashion - notice that $\infty \cdot 0 = 0 \cdot \infty = 0$) and $\mathscr{C}$ be the category of (left) countably complete S-semimodules and S-linear maps ([7]). $\mathscr{C}$ is monadic over Set, for the monad $MX = \{f : X \to S \mid \mathsf{supp}(f) \text{ countable}\}$,[8] thus products in $\mathscr{C}$ are formed as in Set, and it can be easily seen that countable products are also coproducts. Thus there is a well-defined countable summation on arrows. The homset of algebra maps $M1 \to M1$ can be identified with $M1 = S$, and the corresponding summation on arrows coincides with the original addition on S. But this was assumed non-idempotent.

In view of the above observations, it seems that idempotency of summation does not come for free. We shall assume it holds in $\mathscr{C}$ in the sequel: for each countable non-empty set I and for each morphism $f : X \to Y$, $\sum_{i\in I} f = f$.

The above assumption allows us to define a partial order on arrows: $f \sqsubseteq g$ for $f, g : X \to Y$ if there is $h : X \to Y$ such that $f + h = g$ (equivalently, if $f + g = g$). The supremum of two arrows two arrows f, g is their sum $f + g$. This also extends to countable family of arrows, and left or right composition preserves such supremum. The least element is the zero map, also preserved by composition. In other words, $\mathscr{C}$ is enriched over $\omega\mathsf{SLat}$.[9]

Remark 2.1. *If $\mathscr{C}$ is any $\omega\mathsf{SLat}$-enriched category with countable products (or countable coproducts), then one can reason as in the finite case (enrichment over commutative monoids, and existence of finite (co)products) to conclude that these are actually biproducts. The induced summation on arrows coincides with the supremum in each sup-semilattice homset, and the idempotency of summation of arrows is a consequence of enrichment. We preferred starting with an arbitrary $\mathscr{C}$ with countable biproducts specially to emphasize that idempotency (which will turn to be crucial in the sequel) does not come for free for arbitrary categories with countable biproducts.*

3. On (co)limits of ω-chains

Let $\mathscr{C}$ be an $\omega\mathsf{SLat}$-enriched category with countable biproducts. Consider the following diagram in $\mathscr{C}$

$$X_0 \underset{\varphi_0}{\overset{\psi_0}{\leftrightarrows}} X_1 \leftrightarrows \ldots \leftrightarrows X_n \underset{\varphi_n}{\overset{\psi_n}{\leftrightarrows}} \ldots \tag{1}$$

such that $\psi_n \varphi_n = \mathsf{id}_{X_n}$ for each $n \in \mathbb{N}$. For each $n, k \in \mathbb{N}$, denote by $\xi_{n,k} : X_n \to X_k$ the morphism given by

$$\xi_{n,k} = \begin{cases} id \text{ , if } n = k \\ \varphi_n \varphi_{n+1} \ldots \varphi_{k-1} \text{ , if } n < k \\ \psi_k \psi_{k+1} \ldots \psi_{n-1} \text{ , if } n > k \end{cases}$$

The family of arrows $(\xi_{n,k})_{n,k\in\mathbb{N}}$ verifies $\psi_k \xi_{n,k+1} = \xi_{n,k} = \xi_{n+1,k}\varphi_n$ for all n, k. Let $\xi : \oplus_{n\in\mathbb{N}} X_n \to \oplus_{n\in\mathbb{N}} X_n$ the arrow induced by this family. Thus $\mathsf{pr}_n \xi \mathsf{in}_k = \xi_{n,k}$ holds for all $n, k \in \mathbb{N}$.

[8] Here $\mathsf{supp}(f) = \{x \in X \mid f(x) \neq 0\}$.

[9] Note also that each homset of endomorphisms $\mathscr{C}(X, X)$ becomes a closed semiring ([14]) with the above summation as addition and composition as multiplication.

Lemma 3.1. *ξ is idempotent if and only if $\varphi_n\psi_n \sqsubseteq \mathsf{id}_{X_{n+1}}$ for all $n \in \mathbb{N}$.*

Proof. The proof is left to the reader, taking into account that the sum of morphisms is also their supremum. □

Proposition 3.1. *If ξ is idempotent and splits as $\oplus_{n\in\mathbb{N}}X_n \xrightarrow{r} L \xrightarrow{i} \oplus_{n\in\mathbb{N}}X_n$, then $(L,(X_n \xrightarrow{\mathsf{in}_n} \oplus_{n\in\mathbb{N}}X_n \xrightarrow{r} L)_{n\in\mathbb{N}})$ is the colimit of the chain $(X_n,\varphi_n)_{n\in\mathbb{N}}$, and the limit of the cochain $(X_n,\psi_n)_{n\in\mathbb{N}}$ is $(L,(L \xrightarrow{i} \oplus_{n\in\mathbb{N}}X_n \xrightarrow{\mathsf{pr}_n} X_n)_{n\in\mathbb{N}})$.*

Proof. We show only the first statement, the other one following by duality. Denote by j_n the morphism $X_n \xrightarrow{\mathsf{in}_n} \oplus_{n\in\mathbb{N}}X_n \xrightarrow{r} L$. Then

$$
\begin{aligned}
j_{n+1}\varphi_n &= r\,\mathsf{in}_{n+1}\varphi_n \\
&= r\xi\,\mathsf{in}_{n+1}\varphi_n \\
&= \sum_{n\in\mathbb{N}} r\,\mathsf{in}_k\mathsf{pr}_k\xi\,\mathsf{in}_{n+1}\varphi_n \\
&= \sum_{n\in\mathbb{N}} r\mathsf{in}_k\xi_{n+1,k}\varphi_n \\
&= \sum_{n\in\mathbb{N}} r\mathsf{in}_k\xi_{n,k} \\
&= \sum_{n\in\mathbb{N}} r\mathsf{in}_k\mathsf{pr}_k\xi\,\mathsf{in}_n \\
&= r\xi\,\mathsf{in}_n \\
&= j_n
\end{aligned}
$$

hence $(L,(j_n)_{n\in\mathbb{N}})$ is a compatible cone. We show now it is the colimiting cone: take $(A,(X_n \xrightarrow{a_n} A)_{n\in\mathbb{N}})$ another cone over the diagram $(X_n,\varphi_n)_{n\in\mathbb{N}}$ and denote by $a : \oplus_{n\in\mathbb{N}}X_n \to A$ the corresponding arrow given by the universality of coproducts. Then for the morphism $\bar{a} : L \xrightarrow{i} \oplus_{n\in\mathbb{N}}X_n \xrightarrow{a} A$ we have

$$
\begin{aligned}
\bar{a}j_n &= \bar{a}r\,\mathsf{in}_n \\
&= air\,\mathsf{in}_n \\
&= a\xi\,\mathsf{in}_n \\
&= \sum_{n\in\mathbb{N}} a\,\mathsf{in}_k\mathsf{pr}_k\xi\,\mathsf{in}_n \\
&= \sum_{n\in\mathbb{N}} a_k\xi_{n,k}
\end{aligned}
$$

Notice now that for $k \geq n$, we have $a_k\xi_{n,k} = a_k\varphi_{k-1}\ldots\varphi_n = a_n$, while for $k < n$, we obtain $a_k\xi_{n,k} = a_{k+1}\varphi_k\psi_k\ldots\psi_{n-1} \sqsubseteq a_{k+1}\psi_{k+1}\ldots\psi_{n-1} \sqsubseteq \ldots \sqsubseteq a_n$. Therefore $\bar{a}j_n = a_n$ for all n. Now, the unicity of the map $L \to A$ follows from construction: if $L \xrightarrow{b} A$ would be any morphism verifying $bj_n = a_n$, then

$$\begin{aligned} br &= br \sum_{n\in\mathbb{N}} \mathsf{in}_n \mathsf{pr}_n \\ &= \sum_{n\in\mathbb{N}} a_n \mathsf{pr}_n \\ &= a \sum_{n\in\mathbb{N}} \mathsf{in}_n \mathsf{pr}_n \\ &= a \end{aligned}$$

hence $b = bri = ai = \bar{a}$. □

Remark 3.1. *(i) The splitting condition of the idempotent ξ is a mild condition, automatically fulfilled in case $\mathscr{C}$ is Cauchy complete. This happens for example, when $\mathscr{C}$ has all (co)equalizers.*

(ii) As any countably sup-semilattice is in particular an ω-complete partial order, and any morphism of sup-semilattices preserves (countable) joins, in particular joins of ω-chains and bottom element (the zero morphism), it follows that $\mathscr{C}$ is $\omega\mathsf{cpo}$-enriched. Therefore, one can apply the limit-colimit coincidence from [18] *in order to obtain the above result, as for each n, (φ_n, ψ_n) is an embedding-projection pair. However, we preferred the above approach to make more visible the impact of biproducts.*

4. Biproducts and terminal coalgebras

Let $\mathscr{C}$ be a $\omega\mathsf{SLat}$-enriched category with countable biproducts, such that idempotent splits in $\mathscr{C}$. Consider an ω^{op}-continuous endofunctor functor $H : \mathscr{C} \to \mathscr{C}$.[10] Then the terminal H-coalgebra exists and can be obtained ([**?**]) as the limit of the sequence

$$0 \xleftarrow{!} H0 \leftarrow \ldots \leftarrow H^n \xleftarrow{H^n!} \ldots$$

where $! : 0 \to H0$ is the unique morphism to the zero object. Completing the above diagram with the arrows $H^n 0 \xrightarrow{H^n ¡} H^{n+1}0$, where $¡ : 0 \to H0$ is the (only) morphism from the zero object, produces the double chain

$$0 \underset{¡}{\overset{!}{\leftrightarrows}} H0 \leftrightarrows \ldots \leftrightarrows H^n \underset{H^n ¡}{\overset{H^n !}{\leftrightarrows}} \ldots \tag{2}$$

If for each n, $H^n ! H^n ¡ \sqsubseteq \mathsf{id}_{H^{n+1}0}$ holds, from the previous section we may conclude that the terminal H-coalgebra carries also the structure of initial H-algebra.[11]

In order to come up with some relevant examples, our first attempt was to look at functors H which are $\omega\mathsf{SLat}$-enriched (such H would preserve zero morphisms and (countable) summation of arrows, i.e. suprema; in particular it would preserve the order). However, this turned out to be rather deceiving, because preservation of zero morphisms immediately implies preservation of the zero object (as in any category with zero morphisms, the zero object is the only object for which the identity is also the zero morphism), therefore the

[10] That is, it preserves limits of ω^{op}-chains.

[11] Not only being realized on same object, but also the structural morphism of H-algebra is the inverse of the H-coalgebra structure morphism.

terminal H-coalgebra (thus also the initial H-algebra), obtained as explained before, will be trivial. So in order to insure the non-triviality of the sequence (2), it seems more-likely to require that H preserves only the order on arrows, that is, to be locally monotone. This happens in particular if H preserves binary sums of morphisms.

Countable biproducts in categories of algebras. Let $\mathscr{A}$ be a category with countable products and coproducts and M a monad on $\mathscr{A}$ for which $M0 = 1$. For each (non-empty) indexing set I and each $i \in I$, there is a map $\sqcup_{j\in I} X_j \to MX_i$, whose j-th component is the unit of the monad $X_i \to MX_i$, for $j = i$, and $X_j \to 1 = M0 \to MX_i$, for $j \neq i$. By the universal property of products, we obtain a map $\sqcup_{i\in I} X_i \to \sqcap_{i\in I} MX_i$. It extends uniquely over the multiplication of the monad to a natural transformation $\mathsf{bc} : M(\sqcup_{i\in I} X_i) \to \sqcap_{i\in I} MX_i$. As in [6], one can show that the natural transformation bc is an isomorphism for all (countable) indexing sets I precisely when the Kleisli category of the monad $\mathsf{Kl}(M)$ has (countable) biproducts, or equivalently, when the category of M-algebras $\mathsf{Alg}(M)$ has (countable) biproducts. Call such a monad ω-additive. Moreover, if $\mathscr{C}$ has equalizers and coequalizers, then the Eilenberg-Moore category is Cauchy complete (because it has equalizers), and the Kleisli category is also because it inherits coequalizers from base category. Thus idempotents split in both categories associated to the monad M. However, as explained in the first section and also mentioned in [12], this is not enough to ensure $\omega\mathsf{Slat}$-enrichment. For an example of ω-additive monad M such that both $\mathsf{Kl}(M)$ and $\mathsf{Alg}(M)$ are $\omega\mathsf{SLat}$-enriched, take any quantale Q and the monad $MX = \{f : X \to Q \mid \mathsf{supp}(f) \text{ countable}\}$ introduced in Section 2.

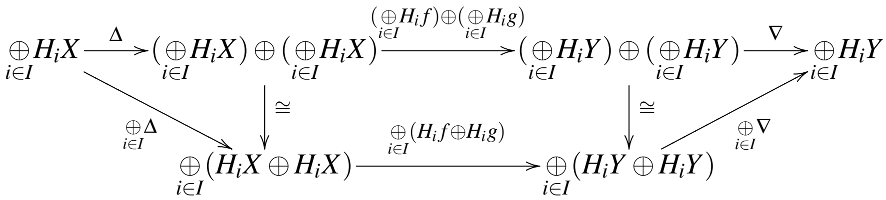
Locally monotone functors on categories of algebras. In the sequel, the category $\mathscr{C}$ is the category of algebras $\mathsf{Alg}(M)$ for a monad M as above. For a discussion concerning the Kleisli category, see [9]. First, a general remark: given any monad M on a category $\mathscr{A}$ (with products), the family of $\mathscr{A}$-endofunctors inductively defined by the grammar below always admit liftings to $\mathsf{Alg}(M)$:

$$H ::= \mathsf{Id} \mid K_A \mid \sqcap_{i\in I} H_i \mid MG \tag{3}$$

where K_A stands for the constant functor to an object $A \in \mathscr{A}$ carrying an M-algebra structure, and G is any $\mathscr{A}$-functor ([4], [5]). In particular, it includes also power functors like H^I, for (countable) index set I, and binary products of functors $H_1 \times H_2$. For each such H, the terminal H-coalgebra, if exists, naturally inherits an M-algebra structure making it the terminal coalgebra for the lifted endofunctor $\bar{H}$. Now, if the monad is ω-additive such that $\mathsf{Alg}(M)$ becomes $\omega\mathsf{SLat}$-enriched, and the lifted endofunctor $\bar{H}$ is ω^{op}-continuous and locally monotone, then $\bar{H}$ has the unique fixed point property. See [4] for a discussion on the ω^{op}-continuity of the lifted functor. Here, we would rather focus on the second aspect, the one of local monotonicity.

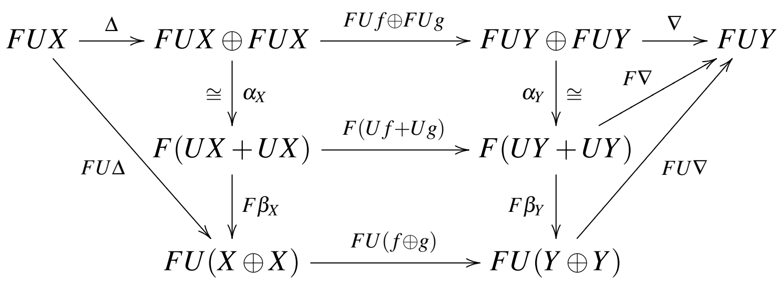
We show by induction of the above grammar that all functors of (3) lift to locally monotone functors on $\mathsf{Alg}(M)$. First, the $\mathscr{A}$-identity functor lifts to the identity on $\mathsf{Alg}(M)$ which is obviously locally monotone. The constant functor to the object $A \in \mathscr{A}$ lifts to the constant functor to the M-algebra A sending any arrow to the identity of A. Again, this is locally monotone.

Consider now a countable set I and a family $(H_i)_{i\in I}$ of $\mathscr{A}$-endofunctors having locally monotone liftings $(\bar{H}_i)i \in I$ on $\mathsf{Alg}(M)$. Then the lifting of $\sqcap_{i\in I} H_i$ is $\oplus_{i\in I}\bar{H}_i$. Consider $f,g: X \to Y$ with $f \sqsubseteq g$, equivalently $f+g=g$. Then $\bar{H}_i f \sqsubseteq \bar{H}_i g$ for all i by local monotonicity. The top line in the diagram below is $\oplus_{i\in I} H_i f + \oplus_{i\in I} H_i g$; the square in the middle commutes by commutativity and associativity of the biproduct, while the two outer triangles commute by naturality.



Thus the top and the bottom lines are equal, so $\oplus_{i\in I} H_i f + \oplus_{i\in I} H_i g = \oplus_{i\in I}(H_i f \oplus H_i g) = \oplus_{i\in I} H_i g$. Note that the countability of the indexing set I was essential in establishing the local monotonicity of the product functor.

Now let's look at M itself; if we denote by $F \dashv U : \mathsf{Alg}(M) \to \mathscr{A}$ the adjunction between the corresponding free algebra functor and the forgetful one, then M lifts on $\mathsf{Alg}(M)$ to FU, the associated comonad. Take again a pair of algebra morphisms $f,g: X \to Y$ with $f \sqsubseteq g$ (hence $f+g=g$). In the diagram below, the natural isomorphisms α_x, α_Y express that F preserves coproducts (being a left adjoint), while $\beta_X : UX+UX \to U(X\oplus X)$ and $\beta_Y : UY+UY \to U(Y\oplus Y)$ are given by the universality of coproducts; do not confuse $Uf+Ug$, which is the $\mathscr{A}$-morphism $UX+UX \to UY+UY$ universally induced by $UX \underset{Ug}{\overset{Uf}{\rightrightarrows}} UY$, with the summation of morphisms which holds in $\mathsf{Alg}(M)$.



All squares and triangles commute by naturality;[12] thus we obtain

$$\begin{aligned}
FUf + FUg =& \nabla(FUf \oplus FUg)\Delta \\
=& F\nabla\alpha_Y(FUf \oplus FUg)\Delta \\
=& F\nabla F(Uf + Ug)\alpha_X\Delta \\
=& FU\nabla F\beta_Y F(Uf + Ug)\alpha_X\Delta \\
=& FU\nabla FU(f \oplus g)F\beta_X\alpha_X\Delta \\
=& FU\nabla FU(f \oplus g)FU\Delta \\
=& FU(f + g) \\
=& FUg
\end{aligned}$$

In particular, $FUf \sqsubseteq FUg$, thus FU is locally monotone. By a similar argument one can show that the lifting of MG (where G is any $\mathscr{A}$-endofunctor), is locally monotone. We just point out that the lifted functor is FGU. Thus all functors as above allow for locally monotone liftings to $\mathsf{Alg}(M)$, therefore their associated final coalgebras also carry the structure on an initial algebra for the lifted endofunctor.

5. **Conclusions**

It remains an open question on how to handle the behavior of diagrams like (1) in absence of $\omega\mathsf{SLat}$-enrichment, that is, in absence of idempotency of summation. It is worth mentioning that there are such categories with (countable) biproducts and endofunctors on them which do not preserve the summation on arrows, but still have the unique fixed point property: for instance take $\mathscr{C} = \mathsf{SProc}$, the category of synchronous processes ([1]) and $H = \bigcirc$, the unit delay functor (monad).

Acknowledgement

This work has been supported by the CNCSIS-UEFISCSU project PD-56 no. 19/03.08.10.

REFERENCES

[1] *S. Abramsky*, Interaction Categories (Extended Abstract). In: G. Burn, S. Gay, M. Ryan (eds.), Theory and Formal Methods, Springer-Verlag, 1993, 57-69.

[2] *M. Arbib, E. Manes*, Partially Additive Categories and Flow-Diagram Semantics, J. Algebra. **2** (1980), 203-227.

[3] *M. Arbib, E. Manes*, Algebraic Approaches to Program Semantics. Springer-Verlag, New York, 1986.

[4] *A. Balan, A. Kurz*, On coalgebras over algebras, Theor. Comput. Sci. **412**(2011), No. 38, 4989-5005.

[5] *F. Bonchi, M. Bonsangue, J. J. M. M. Rutten, A. Silva*, Generalizing the powerset construction, coalgebraically. FSTTCS 2010, 272-283.

[6] *D. Coumans, B. Jacobs*. Scalars, monads, and categories, arXiv:1003.0585 (2010).

[12] Notice the lack of symmetry in the left and right triangles, as there is no natural arrow in $\mathscr{A}$ into the coproduct.

[7] *J. S. Golan*, Semirings and their Applications, Springer, 1999.

[8] *E. Haghverdi*, A Categorical Approach to Linear Logic, Geometry of Proofs and Full Completeness, PhD Thesis, University of Ottawa, 2000.

[9] *I. Hasuo, B. Jacobs, A. Sokolova* Generic Trace Semantics via Coinduction, Logical Methods in Computer Science **3**(2007), 4-11.

[10] *N. Hoshino* A Representation Theorem for Unique Decomposition Categories, ENTCS **286**(2012), 213-227.

[11] *M. C. Iovanov*, When Π is isomorphic to $\oplus$, Comm. Algebra **34**(2006), No. 12, 4551-4562.

[12] *B. Jacobs*, From Coalgebraic to Monoidal Traces. Electr. Notes Theor. Comput. Sci. **264**(2010), No. 2, 125-140.

[13] *A. Joyal, M. Tierney*, An extension of the Galois theory of Grothendieck, Mem. Am. Math. Soc. **309**(1984).

[14] *D. Kozen*, On Kleene algebras and closed semirings. In: B. Rovan (ed.), Mathematical Foundations of Computer Science 1990, LNCS **452**(1990) Springer Berlin/Heidelberg, 26-47.

[15] *W. Kuich*, Semirings and Formal Power Series: Their Relevance to Formal Languages and Automata. In: G. Rozenberg, A. Salomaa (eds.), Handbook of Formal Language Theory, Vol. 1, Springer, 1997, 609-677.

[16] *S. Lack*, Non-canonical isomorphisms, J. Pure Appl. Algebra, **216**(2012), No. 3, 593-597.

[17] *B. Pareigis*, Categories and Functors, Pure and Applied Mathematics **39**, Academic Press, New York, 1970.

[18] *M.B. Smyth, G.D. Plotkin*, The category theoretic solution of recursive domain equations, SIAM Journ. Comput. **11**(1982), 761-783.