FROM GENERAL RELATIVITY TO SCALE RELATIVITY THEORY THROUGH GROUP INVARIANCES OF SL(2,R) TYPE

Razvan-Vasile ABABEI[1], Maria-Alexandra PAUN[2, *], Mihail FRASILA[3], Cristina Marcela RUSU[4], Vladimir-Alexandru PAUN[5], Maricel AGOP [4,6], Viorel-Puiu PAUN[6,7]

Correlations between General Relativity and Scale Relativity theories in the description of the gravitational systems dynamics through groups invariances of SL(2,R) type are investigated. In this context, a new possible holographic principle (totally different in relation to holographic principle of string theories) becomes more suitable to operate in gravitational dynamics by means of Ernst's type potentials.

Keywords: Scale Relativity, gravitational systems dynamics, SL(2,R) group invariance, string theories, Ernst's type potentials

1. Introduction

The usual physical models (Newtonian and post-Newtonian models, General Relativity [1-3]) used in describing the dynamics of gravitational systems are based on the hypothesis of the differentiability of the physical quantities used to describe its evolution. As a consequence, the validity of these models must be understood gradually, in areas where differentiability and integrability are still functional [1-3]. However, when discussing nonlinearity and chaoticity in the dynamics of the gravitational system (e.g.; strange topologies such as black hole, worm holes etc. [4]) , many differentiable and integrable mathematical procedures are not valid. Therefore, in order to properly describe the dynamics of gravitational

1 . Research Center with Integrated Techniques for Atmospheric Aerosol Investigation in Romania, RECENT Air, Laboratory of astronomy and astrophysics, Astronomy Observatory, "Alexandru Ioan Cuza" University, 5-7 Mihail Sadoveanu, 700490 Iasi, Romania; razvan.ababei@uaic.ro
2. Division Radio Monitoring and Equipment, Section Market Access and Conformity, Federal Office of Communications OFCOM, Avenue de l'Avenir 44, CH-2501, Biel/Bienne, Switzerland
3. Faculty of Physics, Alexandru Ioan Cuza University of Iasi, Romania;frasilamihail@yahoo.com
4. Department of Physics, „Gheorghe Asachi" Technical University of Iasi, 700050 Iasi, Romania; cristina-marcela.rusu@academic.tuiasi.ro, magop@tuiasi.ro
5. Five Rescue Research Laboratory, 35 Quai d'Anjou, 75004, Paris, France; vladimir.alexandru.paun@ieee.org
6. Romanian Scientists Academy, 54 Splaiul Independentei, 050094 Bucharest, Romania
7. Physics Department, Faculty of Applied Sciences, University Politehnica of Bucharest, Romania; viorel.paun@physics.pub.ro
*Corresponding author, email: maria_paun2003@yahoo.com

systems, it is necessary to introduce the scale resolution both into the expressions of the physical variables as well as into the expressions of the fundamental equations governing these gravitational systems' dynamics [5-7].

Accepting the above affirmation, any physical variable (which might be used in the description of gravitational systems' dynamics), will depend on both the usual mathematical procedures on spatial and time coordinates as well as on a scale resolution. Specifically, instead of working with a single physical variable (a strictly non-differentiable mathematical function), it is possible to operate only with approximations of this mathematical function, resulting by averaging it at different scale resolutions. Thus, any physical variable used to describe the dynamics of gravitational systems dynamics will operate as the limit of a family of mathematical functions, the function being non-differentiable for zero scale resolution and differentiable for non-zero scale resolution [5-7].

This new method of describing the dynamics of complex systems, obviously implies the development of both new geometric structures and gravitational theories, consistent with these geometric structures, for which the laws of motion, invariant to time coordinate transformations are also invariant to transformations with respect to scale resolution. Such a geometric structure is the one based on the concept of the fractal/multifractal and the corresponding physical model described in the Scale Relativity Theory (SRT) [5]. From this perspective, holographic implementation in the description of gravitational system dynamics will be made explicitely based on continuous but non-differentiable curves (fractal/multifractal curves).

In the present paper, correlations between GR and SRT theories in the description of the gravitational systems dynamics through groups invariances of SL(2,R) type are highlighted. In this context, a new possible holographic principle (different than existing one [8]) becomes functional so that the above-mentioned correlations can be reductible to Ernst's type potentials.

2. Schrödinger type scenario and group invariances type of SL(2,R)

It is well known the fact that the dynamics gravitational systems dynamics in the SRT [7-9] can be described through the multifractal Schrödinger equation[6,7]:

$$\lambda^2(dt)^{\left[\frac{4}{f(\alpha)}\right]-2}\partial_l\partial^l\Psi + i\lambda(dt)^{\left[\frac{2}{f(\alpha)}\right]-1}\partial_t\Psi = 0 \tag{1}$$

where in Eq. (1) we used the operators:

$$\partial_t = \frac{\partial}{\partial_t}, \partial_l = \frac{\partial}{\partial x^l}, \partial_l\partial^l = \frac{\partial^2}{\partial x_l^2} \tag{2}$$

In the above relations, the meaning of the used quantities is the following:

- Ψ is the state function;
- dt is the scale resolution;
- x^l is the multifractal spatial coordinate;
- t is the non-multifractal temporal coordinate with the role of an affine parameter of the motion curves;
- λ is a parameter associated to the multifractal/non-multifractal scale transition;
- $f(\alpha)$ is the singularity spectrum with a singularity index of order $\alpha = \alpha(D_F)$[9,10];
- D_F is the fractal dimension of the motion curves [9,10];

The multifractal Schrödinger equation admits, besides the clasical Galilei group proper, an extra set of symmetries that, in general conditions, can be taken in a form involving just one space dimension and time, as a SL(2,R) type group in two variables with three parameters [11-13]. Limiting the general conditions, the space dimension can be chosen as the radial coordinate in a free fall, as in the case of Galilei kinematics, which can also be extended as such in general relativity [14,15], for instance in the case of free fall in a Schwarzschild field.

For our current necessities, it is necessary to start with the finite equations of the specific SL(2,R) group, and build gradually upon these [11,16], in order to discover the necessary equations. Working in the variables (t, r), the finite equations of this group are given by the transformations:

$$t \to \frac{\alpha t + \beta}{\gamma t + \delta}; \; r \to \frac{r}{\gamma t + \delta} \tag{3}$$

This transformation is a realization of the SL(2,R) structure in variables (t, r), with three essential parameters. Every vector in the tangent space SL(2,R) is a linear combination of three fundamental vectors, the infinitesimal generators:

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = t\frac{\partial}{\partial t} + \frac{r}{2}\frac{\partial}{\partial r}, \qquad X_3 = t^2\frac{\partial}{\partial t} + tr\frac{\partial}{\partial r} \tag{4}$$

satisfying the basic structure equations:

$$[X_1, X_2] = X_1, \qquad [X_2, X_3] = X_3, \qquad [X_3, X_3] = -2X_2 \tag{5}$$

which we take as standard commutation relations for this type of algebraic structure throughout the present work.

3. Gravity with an axial symmetry and group invariances of SL(2,R) type.

We are talking here about gravity as it is presented to us in the formalism of the theory of general relativity. The main reason is that we want to extract as much as possible on the path of a mathematical philosophy of physics. And in general relativity, for the first time in history, the field concept, considered on the side of its omnipresence and permanence, was subjected by Albert Einstein to a logical analysis based on the idea of the particle, considered in the extent to which the aspects of point in space and moment of time are involved in its concept. The result of this analysis is sublimated in Einstein's field equations. Rarely one can give a general solution, with positive profit for mathematical philosophy, of these equations. However, the vacuum and electromagnetic vacuum equations have such a solution, which can be brought to an elegant form in the case of stationary metrics. Frederic J. Ernst was the one who pointed out this form [17,18], for the case of the axially symmetric field. Later both he, but especially Israel and Wilson [24], whom we will follow here, have shown that it is possible to treat the general stationary case in a completely analogous way. We will follow this last work here, first because it seems a bit more explicit for what we want to bring out into evidence, then because it apparently has a fresh idea of bypassing the related indeterminacy of the metric tensor, leading to profitable results for mathematical philosophy. We still follow the general idea of Ernst's original works, namely that of posing the problem of the gravific field in connection with a variational principle, for reasons that will be immediately highlighted.

The main point of the cited work of Israel and Wilson is that, for a stationary space-time metric, conveniently written in the form

$$(ds)^2 = f(= f(dx^4 + \omega_m dx^m)^2 - f^{-1}(\gamma_{mn} dx^m dx^n) \tag{6}$$

where we use the convention of summation by repeated indices of different variance, Einstein's equations for the electromagnetic field in vacuum

$$G_{\alpha\beta} = -8T_{\alpha\beta} \tag{7}$$

take the form of the system of equations with nonlinear partial derivatives

$$\begin{aligned} \nabla^2\epsilon &= \nabla\epsilon \cdot (\nabla\epsilon + 2\Psi^*\nabla\Psi) \\ f\nabla^2\Psi &= \nabla\Psi \cdot (\nabla\epsilon + 2\Psi^*\nabla\Psi) \end{aligned} \tag{8}$$

Let us now explain the symbols: Greek indices go from 1 to 4, while Latin indices go from 1 to 3 and represent spatial indices. The spatio-temporal metric tensor is defined by

$$g_{44} = f,\ \gamma_{mn} = g_{4m}g_{4n} - g_{44}g_{mn},\ \omega_k = \frac{g_{4k}}{g_{44}} \tag{9}$$

and for raising and lowering the spatial indices, the metric (γmn) is used. All of these components do not depend on the time coordinate (stationarity property). A potential four-vector $(\boldsymbol{A}, A_4) \equiv (A_\gamma)$ describes the electromagnetic field whose intensities are given by its covariant curl:

$$F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha \tag{10}$$

This electromagnetic field satisfies the equation

$$-4\pi T_{\mu\nu} \equiv g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} - \frac{1}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \tag{11}$$

Further, $G_{\alpha\beta}$ is the Einstein tensor associated with the metric field, and defined by

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{12}$$

with $R_{\alpha\beta}$ the Ricci tensor of the metric, and R the scalar invariant of the curvature.

In relation to these symbols, then are defined the functions

$$\Psi \equiv A_4 + i\Phi;\ \epsilon \equiv f - \Psi^*\psi + i\phi;\ i = \sqrt{-1} \tag{13}$$

where Φ ia a magnetic potential and ϕ an arbitrary function. Once we know the functions ϵ, Φ and ϕ, we can construct the Ricci tensor corresponding to the metric (γ_{mn}) by

$$\begin{gathered} -f^2 R_{mn}(\gamma) = \frac{1}{2} \epsilon_{(m} \nabla_{n)} \epsilon^* + \Psi \nabla_{(m} \epsilon \cdot \nabla_{n)} \Psi^* \\ + \Psi^* \nabla_{(m} \epsilon^* \cdot \nabla_{n)} \Psi - (\epsilon + \epsilon^*) \nabla_{(m} \Psi \cdot \nabla_{n)} \Psi^* \end{gathered} \tag{14}$$

where the parentheses mean symmetrization in relation to the indices they contain.

As we said, F. J. Ernst [17] introduced the complex potential ϵ for the case especially of the gravitational field with axial symmetry. It was later proven that spatial symmetry is not a necessary condition for the existence of such a potential [18] but only the stationarity of the metric field (time independence). Despite this fact, it is not possible to solve the problem of gravity in the spirit in which Einstein first posed it [19], that is, given the energy tensor, solve the equations to find uniquely the metric tensor. In fact, this problem has never been solved as such. The deep reason is very simple in our opinion: the field is deprived here of what we would call a condition of universality, this being defined as the presence of the field in any interaction in space. More precisely, to find a solution for the gravitational potentials (metric tensor components), it is necessary to solve Einstein's equations (7). However, the right side of these equations contains the energy tensor, whose construction, although it explains the properties of interaction, requires a priori knowledge of the metric tensor.

This problem has been repeatedly iterated in theoretical physics, in one way or another, and among its settlement cases there are some remarkable for their contribution to knowledge of the nature of the gravitational field [19-22]. Special mention should be made of the conclusion that the theory of Einstein's non-symmetric field [19] is completely equivalent to nonlinear Born-Infeld electrodynamics [20] provided that the metric tensor be defined with an antisymmetric part representing the electromagnetic field [21]. Indeed, the metric tensor $(g_{\alpha\beta})$ might not be symmetric in general, because in the metric only its symmetrical part appears, by the very algebraic nature of the quadratic form which represents the metric. So, if we take a general metric tensor, non-symmetric, and accept that it is compatible with the space connection, then the antisymmetric part of the metric is identical by its definition with the Born-Infeld electromagnetic field. Next, the Born-Infeld equations actually represent the cancellation of the space-time torsion covector. There is obviously very much to say about this fascinating problem of modern theoretical physics, but we will limit now to the introduction of what Ernst's complex potential made possible. More precisely, we will show what this potential means for the measurement problem. Let's note that the problem of the gravitational field could be solved if there is a logic slightly deviated from the usual line, in the sense of allowing the metric γ to be arbitrary, so that it can be conveniently chosen. Indeed, Israel and Wilson [23] note that equations (14) should only be taken as compatibility conditions between a specially chosen spatial metric and the complex fields ϵ and Ψ. In the particular case of the ordinary Euclidean space, the conditions of compatibility are simply reduced to the linear equation

$$\Psi = a + b\epsilon,\ ab^* + a^*b = -\frac{1}{2} \tag{15}$$

and the whole construction comes back to solving the Laplace's equation

$$\nabla^2\xi = 0,\ \xi \equiv (1+\epsilon)^{-1} \tag{16}$$

Through equation (15), the gravitational field determines an electromagnetic field. This field is still not a transition field, as we usually know it, but it only reflects the property of omnipresence and permanence of the gravitational field. In one of the now classic works, Misner and Wheeler [24] admirably captured these attributes of the gravitational field (actually of space itself as an active physical entity) and studied their physical significance in detail. Here we are particularly interested in making it clear the fact that equation (14) is the mark of a measurement process, showing its relevance for the case of the harmonic oscillator. Ernst himself [17] noted the fact that a functional relationship between the complex gravitational and electromagnetic potentials solves the gravitational field problem.

4. Ernst's type potentials

In 1971, Ernst [18] proved that the theory providing equations (8) and (14) above, applied to the purely gravitational case, can be obtained from the variational principle

$$\delta \iiint \left\{ R(\gamma) + 2\frac{\gamma^{mn}(\nabla_m \epsilon)(\nabla_n \epsilon^*)}{(\epsilon + \epsilon^*)^2} \right\} \sqrt{\det(\gamma)}(d^3x) = 0 \qquad (17)$$

where $R(\gamma)$ is the scalar curvature of the metric γ. As such it can now be seen that in a Euclidean space this variational principle refers exclusively to Ernst's complex potential:

$$\delta \iiint \left\{ \frac{\gamma^{mn}(\nabla_m \epsilon)(\nabla_n \epsilon^*)}{(\epsilon + \epsilon^*)^2} \right\} (d^3x) = 0 \qquad (18)$$

In other words, only in cases where the gravitational field defines an electromagnetic field through a linear relationship of the type (15), that gravitational field can be described exclusively through a complex potential. Here we will limit ourselves to this last case of the gravity field in vacuum. The line of ideas that we have just presented opens an unexpected path for the solution of the problem of vacuum fields, because the variational principle (18) can be constructed in connection with the continuous group $\mathrm{SL}(2, \mathrm{R})$ that we have here in mind.

Richard Matzner and Charles Misner observed [25] that the principle variational (18) is actually a response to what, in somewhat more contemporary terms, constitutes the problem of harmonic applications, the modern version of Dirichlet's ancient problem, a fact explicitly recognized a little later by Misner himself [16]. From this point of view, equation (18) describes a harmonic application from Euclidean space to $\mathrm{SL}(2, \mathrm{R})$. This fact is much more palpable if, instead of the Ernst ϵ potential, we use as a field variable $h \equiv i\epsilon$, so that equation (18) becomes

$$\delta \iiint \left\{ \frac{\gamma^{mn}(\nabla_m h)(\nabla_n h^*)}{(h - h^*)^2} \right\} (d^3x) = 0 \qquad (19)$$

Obviously, this variational equation describes a harmonic application between the usual Euclidean space of metric (γ_{mn}) and the higher complex plane the Poincaré representation of the hyperbolic plane with the metric given by

$$(ds)^2 = -4\frac{(dh)(dh^*)}{(h-h^*)^2} \equiv \frac{(du)^2+(dv)^2}{v^2},\ h = u + iv \tag{20}$$

known as the invariant metric of the SL(2, R) group [26]. This is the main idea contained in Ernst's approach, as well as the reason why we follow it closely.

The complex potential h is somewhat closer to how this geometry is built, and this from several points of view. The most important of these is its possible physical meaning. Indeed, equation (13) gives us, for the case of the null electromagnetic field (pure gravitational field):

$$h = -\phi + if \tag{21}$$

so that the real part of the potential is arbitrary, while the imaginary part

$$v \equiv f = g_{44} \tag{22}$$

always has the attractive aspect of being positive with a fixed point of unity (the speed of light in a vacuum). These are essential qualities required by the geometry of the Poincaré half-plane. By himself this fact, the Poincaré metric is physically legitimized. Another attractive theoretical point of this potential is that the differential equations corresponding to the variational principle (19) known as the "Ernst equations" of the problem - take the shape

$$(h-h^*)(\nabla^2 h) = 2(\nabla h)(\nabla h) \tag{23}$$

and complex conjugate, obviously, and can be easily solved with the help of Laplace's equation. More precisely, the solution of equation (16) can be written in the form

$$h = -i\frac{\cosh\psi - e^{-i\alpha}\sinh\psi}{\cosh\psi + e^{-i\alpha}\sinh\psi},\ \nabla^2\psi = 0 \tag{24}$$

with α real and arbitrary. Therefore, here the solution to the problem of the stationary gravitational field is also reduced to solving Laplace's equation in the usual space of our experience, just like in classical Newtonian theory. In Figure 1 is presented the dependence of Re(h)=F(ω,t) as a function of dimensionless ω and t for coth(ψ)=1.3 [26-30]. Such behaviors can be found in gravitational systems dynamics such as irregular-shaped celestial bodies.

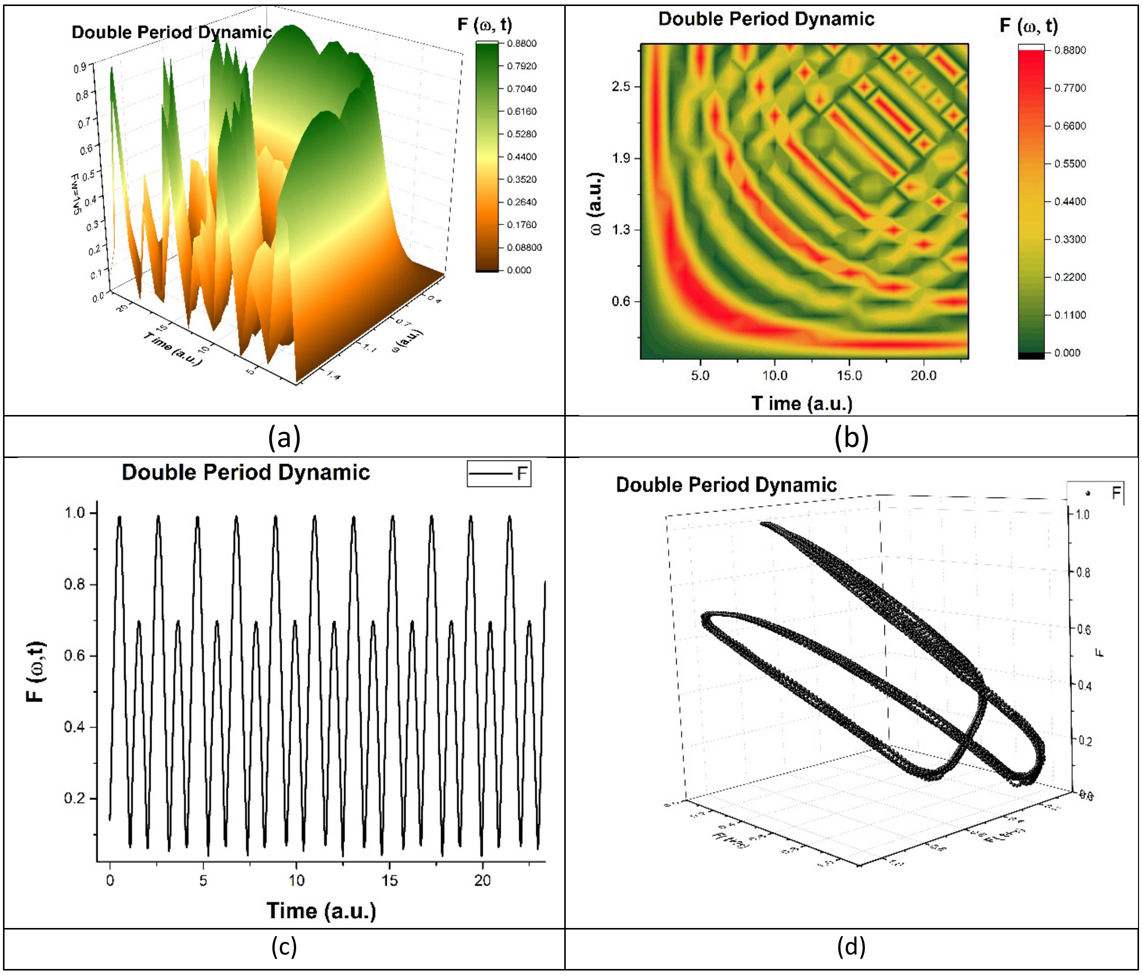

Fig. 1 a-d: The "double period dynamic modes" of gravitational system dynamics are presented: (a) - 3D diagram, (b) - contour plot, (c) - time series and (d) - reconstituted attractor for scale resolutions given by ω_{max}.

The software programs used to represent the four diagrams in Figure 1 (a-d) were developed and successfully used in previous works of the multifractal analysis field in various interest topics [31-35].

5. Conclusions

In this paper, correlations between the Space-Time Theory and the Scale Relativity Theory have been established. In such context, a new holographic principle based on depiction of gravitational system dynamics by means of fractal/multifractals curves becomes functional.

Then any description of gravitational systems dynamics though SRT and GR implies a special group invariances of SL(2,R) type. Moreover, the compatibility of descriptions in these two instances (GR and SRT) is dictated by Ernst's type potentials. In particular, for gravitation dynamics on Peano's type curves (i.e., non-differentiable curves in fractal dimensions D_F=2) at Compton scale resolution correlation between Quantum Mechanics and published GR works.

Finally, our theoretic model can be validated for gravitational dynamics of irregular-shaped celestial bodies (in our case from doubling period gravitational dynamics).

REFERENCES

[1]. *R. A. C. Dolcetta,* Classical Newtonian Gravity, Springer, Berlin, 2019.

[2]. *J. O'Leary*, General Relativistic and Post-Newtonian Dynamics for Near-Earth Objects and Solar Objects Bodies, Springer, Berlin, 2021.

[3]. *A. Das, A. DeBenedictis,* General Relativity Theory, Springer-Verlag, New York, 2012.

[4]. *C. W. Misner,* Gravitation, Princeton University Press, Princeton, 2017.

[5]. *L. Nottale,* "Scale Relativity and Fractal Space-Time: A New Approach to Unifying Relativity and Quantum Mechanics", Imperial College Press, London, UK, 2011.

[6]. *I. Merches, M. Agop*, Differentiability and Fractality in Dynamics of Physical Systems; World Scientific: New Jersey, NJ, USA, 2016.

[7]. *M. Agop, V. P. Paun*, On the new perspectives of fractal theory. In Fundaments and Applications; Romanian Academy Publishing House, Bucharest, Romania, 2017.

[8]. *M. Gasperini, J. Maharana,* String Theory and Fundamental Interactions, Springer-Verlag Berlin and Heidelberg, 2017.

[9]. *B. B. Mandelbrot*, Fractal and Chaos; Springer: Berlin/Heidelberg, Germany, 2004.

[10]. *M. A. Paun, and V. A. Paun*, "High-Frequency 3-D Model for the Study of Antennas in Cochlear Implants", 7th IEEE International Workshop on Advances in Sensors and Interfaces (IWASI), IEEE Transactions on Components Packaging and Manufacturing Technology, **8(7)**, 2018, pp.1135-1140.

[11]. *N. Mazilu, M. Agop*, Skyrmions. In A Great Finishing Touch to Classical Newtonian Philosophy, Ny Nova Science Publishers C, New York, NY, USA, 2012.

[12]. *D. Bigatti, L. Susskind*, The Holographic Principle. In: Thorlacius, L., Jonsson, T. (eds) M-Theory and Quantum Geometry. NATO Science Series, vol 556. Springer, Dordrecht, 2000.

[13]. Faulkner, T., Lewkowycz, A., & Maldacena, J. (2013). Quantum corrections to holographic entanglement entropy. Journal of High Energy Physics, 2013(11), 1-18.

[14]. *M. Talbot*, The Holographic Universe, HarperCollins Publishers, New York, 1991.

[15]. *S. H. Strogatz*, Nonlinear Dynamics and Chaos, 2nd ed.; CRC Press: Boca Raton, FL, USA, 2015.

[16]. *N. Mazilu, M. Agop, I. Merches,* Scale Transitions as Foundations of Physics; World Scientific: Singapore, Singapore, 2021.

[17]. *F.J. Ernst*, “New Formulation of the Axially Symmetric Gravitational Field Problem I”, Physical Review, **167**, 1968, p. 1175.
[18]. *F.J. Ernst*, “Exterior Algebraic Derivation of Einstein Field Equations Employing a Generalized Basis”, Journal of Mathematical Physics, **12**, 1971 p. 2395.
[19]. *A. Einstein*, Relativity: The Special and General Theory, Dover Publications, New York, 2010.
[20]. *M. Born, L. Infeld,* Foundations of the New Field Theory, Proceedings of the Royal Society of London, **A144**, 1934, pp. 425-451.
[21]. *C. W. Misner, J.A. Wheeler*, Classical Physics as Geometry: Gravitation, Electromagnetism, Unquantized Charge and Mass as properties of Curved Empty Space, Annals of Physics, **Vol. 2**, 1957, p. 525.
[22]. *V. A. Fock*, “Teoria Spatiului, Timpului si Gravitatiei”, Editura Academiei, Bucuresti, Romania, 1962.
[23]. *W. Israel, G. A. Wilson*, A Class of Stationary Electromagnetic Vacuum Fields, Journal of Mathematical Physics, Vol. 13, 1972, pp. 865-867.
[24]. *A. Matzner, C. W. Misner*, Gravitational Field Equations for Sources with Axial Symmetry and Angular Momentum, Physical Review, **Vol. 154**, 1967, pp.1229-1232.
[25]. *C. W. Misner*, Harmonic Maps as Models for Physical Theories, Physical Review D, **18**, 1978, pp. 45104524.
[26]. *P. Postolache, Z. Borsos, V.A. Paun, and V.P. Paun*, “New Way in Fractal Analysis of Pulmonary Medical Images”, in University Politehnica of Bucharest Scientific Bulletin-Series A-Applied Mathematics and Physics, **vol. 80,** no. 1, 2018, pp. 313-322.
[27]. *M.V. Nichita, M.A. Paun, V.A. Paun, and V.P. Paun,* “Fractal Analysis of Brain Glial Cells. Fractal Dimension and Lacunarity”, in University Politehnica of Bucharest Scientific Bulletin-Series A-Applied Mathematics and Physics, **vol. 81**, no. 1, 2019, pp. 273-284.
[28]. *V. P. Paun*, “Fractal surface analysis of Zircaloy-4 SEM micrographs using the time-series method”, 6th International Conference on Solid State Surfaces and Interfaces, CENTRAL EUROPEAN JOURNAL OF PHYSICS, **7(2)**, 2009, pp. 264-269.
[29]. *D. Bordescu, M.A. Paun, V.A. Paun, and V.P. Paun*, “Fractal analysis of Neuroimagistic. Lacunarity degree, a precious indicator in the detection of Alzheimer’s disease”, University POLITEHNICA of Bucharest Scientific Bulletin, Series A-Applied Mathematics and Physics, **vol. 80**, no. 4, 2018, pp. 309-320.
[30]. *M. A. Paun, and V. A. Paun*, “High-Frequency 3-D Model for the Study of Antennas in Cochlear Implants”, 7th IEEE International Workshop on Advances in Sensors and Interfaces (IWASI), IEEE Transactions on Components Packaging and Manufacturing Technology, **8(7)**, 2018, pp. 1135-1140.
[31]. *V. A. Paun, L. Ochiuz, M. Hortolomei, et al.*, “In Vitro Release Kinetics Evaluation of Erythromycin in Microemulsions for Dermal Applications”, Materiale Plastice, **53(4)**, 2016, pp. 699-702.
[32]. *M.V. Nichita, M.A. Paun, V.A. Paun, and V.P. Paun,* “Image Clustering Algorithms to Identify Complicated Cerebral Diseases”, Description and Comparison, IEEE ACCESS, **8**, 2020, pp. 88434-88442.
[33]. *N. Mazilu, M. Agop, I. Gatu, et al.*, “The classical theory of light colors: a paradigm for description of particle interactions”, International Journal of Theoretical Physics, **55(6)**, 2016, pp. 2773-2793.

[34]. *V.P. Paun, M.A. Paun, A. Toma, A, C. Ciucu, F. Popentiu,* "Transport phenomenon simulation for linear polymers through nanometer pores", Materiale Plastice, **45(1),** 2008, pp.57-60.
[35]. *M.A. Paun, J.M. Sallese, M. Kayal,* "A Circuit Model for CMOS Hall Cells Performance Evaluation including Temperature Effects", Advances in Condensed Matter Physics, **Vol. 2013**, 2013, Article Number 968647.