U.P.B. Sci. Bull., Series A, Vol. 86, Iss. 4, 2024 ISSN 1223-7027

THE GEOMETRICAL-CONVEXITY OF A FUNCTION RELATED TO THE MINC-SATHRE RATIO

Cristinel Mortici[1], Mihai Postolache[2]

In this paper we give a new proof for the fact that the function $\Gamma(x)^{\frac{1}{x-1}}$ is geometrically convex. As an application, we give an improvement of Minc-Sathre inequality.

Keywords: Geometrically convex function, Minc-Sathre inequality.
MSC2020: 26D15; 41A60.

1. Introduction

The Gamma function defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad x > 0,$$

was introduced by the mathematician Leonhard Euler in the 18th century. It serves as a natural extension of the factorial function to complex and real numbers, playing a main role in various branches of mathematics, including calculus, complex analysis, and number theory.

From its elegant formulation to its surprising applications, the Gamma function continues to captivate mathematicians and scientists alike, bridging gaps between discrete and continuous structures in a profound way.

[1] Professor, Valahia University of Târgovişte, Târgovişte, Romania; National University of Science & Technology Politehnica Bucharest, Department of Mathematics and Informatics, Bucharest 060042, Romania; Academy of Romanian Scientists, 3 Ilfov, 050044, Bucharest, Romania. e-mail: cristinel.mortici@hotmail.com

[2] (Corresponding author). Professor, National University of Science & Technology Politehnica Bucharest, Department of Mathematics and Informatics, Bucharest 060042, Romania; Gheorghe Mihoc-Caius Iacob Institute of Mathematical Statistics and Applied Mathematics of the Romanian Academy, 050711 Bucharest, Romania; The Key Laboratory of Intelligent Information and Big Data Processing of NingXia Province, and Health Big Data Research Institute, North Minzu University, 750021 Yinchuan, China. e-mail: mihai.postolache@upb.ro

Chu et al. [12] proved that the function

$$f(x) = \Gamma(x)^{\frac{1}{x-1}}, \quad x > 1,$$

is geometrically convex. As an application, they have established the following inequality, for every $x \in \mathbb{R}$, $x \geq 1$:

$$\frac{\Gamma(x+2)^{\frac{1}{x+1}}}{\Gamma(x+1)^{\frac{1}{x}}} < \left(\frac{x+2}{x+1}\right)^{\frac{4x+3}{4x+4}}.$$

Its discrete form [12, rel. 1.8]:

$$\frac{((n+1)!)^{\frac{1}{n+1}}}{(n!)^{\frac{1}{n}}} < \left(\frac{n+2}{n+1}\right)^{\frac{4n+3}{4n+4}}, \quad n = 1, 2, 3, ...,$$

is an improvement of the Minc-Sathre inequality:

$$1 < \frac{((r+1)!)^{\frac{1}{r+1}}}{(r!)^{\frac{1}{r}}} < \frac{r+1}{r}, \quad r = 1, 2, 3... .$$

Minc-Sathre inequality was first introduced in 1964 (see [16]) and since then, many improvements and additional results were stated. It is of great use in the theory of permanents of a matrix, volume of the n-dimensional unit ball etc. See [2]-[8], [14]-[17].

2. The geometric convexity of $\Gamma(x)^{\frac{1}{x-1}}$

Chu et al. [12] proved that the function

$$f(x) := (\Gamma(x))^{\frac{1}{x-1}}, \quad x \in (1, \infty), \tag{2.1}$$

is geometrically convex. Recall that a function $f : I \subseteq \mathbb{R} \to (0, \infty)$ is geometrically convex if

$$f(\sqrt{x_1 x_2}) \leq \sqrt{f(x_1) f(x_2)}, \tag{2.2}$$

for every x_1, $x_2 \in I$. This class of functions was introduced by Montel [18], then several results were stated. See, e.g., [13], [15].

Geometrically convex functions are related to convex function in the following way. Let us consider the logarithm in (2.2),

$$\ln f(\sqrt{x_1 x_2}) \leq \frac{\ln f(x_1) + \ln f(x_2)}{2}.$$

Now, by replacing x_1 by e^{x_1} and x_2 by e^{x_2}, we get

$$\ln f\left(e^{\frac{x_1+x_2}{2}}\right) \leq \frac{\ln f(e^{x_1}) + \ln f(e^{x_2})}{2},$$

which means that the function $\ln f\left(e^{x}\right)$ is convex. If function f is differentiable, then this means that the derivative

$$\left(\ln f\left(e^{x}\right)\right)^{\prime}=\frac{e^{x} f^{\prime}\left(e^{x}\right)}{f\left(e^{x}\right)}$$

is increasing. In consequence, a differentiable function f is geometrically convex if and only if the function

$$y \mapsto \frac{y f^{\prime}(y)}{f(y)}, \quad\left(y=e^{x}\right)$$

is increasing. This fact was proved by Chu et al. [12], when they obtained in case of f given by (2.1)

$$x \frac{f^{\prime}(x)}{f(x)}=\frac{x(x-1) \psi(x)-x \ln \Gamma(x)}{(x-1)^{2}}.$$

Here, ψ is the digamma function, i.e., the logarithmic derivative of the gamma function,

$$\psi(x)=(\ln \Gamma(x))^{\prime}=\frac{\Gamma^{\prime}(x)}{\Gamma(x)}.$$

For further details, see e.g., [1, p. 258]. After some computations, it follows

$$\left(x \frac{f^{\prime}(x)}{f(x)}\right)^{\prime}=\frac{g(x)}{(x-1)^{3}},$$

where

$$g(x)=x(x-1)^{2} \psi^{\prime}(x)-\left(x^{2}-1\right) \psi(x)+(x+1) \ln \Gamma(x) \tag{2.3}$$

Chu et al. [12] made some efforts to prove that the function g in (2.3) is positive.

We give here a direct proof of this fact.

In this sense, we use the following estimates for $\ln \Gamma(x)$, ψ, and $\psi^{\prime}$:

$$\begin{aligned}
\ln \Gamma(x) & >\left(x-\frac{1}{2}\right) \ln x-x+\frac{1}{2} \ln 2 \pi+\frac{1}{12 x}-\frac{1}{360 x^{3}}, \qquad (2.4)\\
\psi(x) & <\ln x-\frac{1}{2 x}-\frac{1}{12 x^{2}}+\frac{1}{120 x^{4}}, \\
\psi^{\prime}(x) & >\frac{1}{x}+\frac{1}{2 x^{2}}+\frac{1}{6 x^{3}}.
\end{aligned}$$

These inequalities are truncated asymptotic series for these functions (e.g. [1, p. 257, 259, 260], respectively). Inequalities (2.4) are true, thanks to a basic result of Alzer [6, Theorem 8]. More exactly, the truncated asymptotic series associated to the Gamma function,

$$F_{n}(x)=\ln \Gamma(x)-\left(x-\frac{1}{2}\right) \ln x+x-\frac{1}{2} \ln 2 \pi-\sum_{i=1}^{2 n} \frac{B_{2 i}}{2 i(2 i-1) x^{2 i-1}} \tag{2.5}$$

and

$$G_n(x) = -\ln\Gamma(x) + \left(x - \frac{1}{2}\right)\ln x - x + \frac{1}{2}\ln 2\pi + \sum_{i=1}^{2n+1} \frac{B_{2i}}{2i(2i-1)x^{2i-1}} \tag{2.6}$$

are completely monotonic, for every integer $n \geq 1$. Here, B_j-s are the Bernoulli numbers, defined by the relations

$$\frac{t}{e^t - 1} = \sum_{j=0}^{\infty} B_j \frac{t^j}{j!}, \quad |t| < 2\pi.$$

The first Bernoulli numbers are:

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = -\frac{1}{30} \ldots.$$

For more properties, see [1, p. 804].

A real valued function w is completely monotonic if w is indefinite derivable on its domain D and for every $n \in \mathbb{N}$, we have $(-1)^n w^{(n)}(x) \geq 0$, for every $x \in D$.

Now we can see that estimates for $\ln\Gamma(x)$ and its derivatives ψ, ψ' can be obtained from (2.5)-(2.6), using the complete monotonicity of functions F_n and G_n. The inequalities resulting from (2.5)-(2.6) are of great help in establishing approximations for almost all functions related to the gamma function. See, e.g., [10], [11], [19]-[29].

The polygamma functions $\psi^{(n)}(x)$, $n = 0, 1, 2, 3, \ldots$ are the derivatives of $\psi(x)$. The functions ψ, ψ', ψ'', ... are also called the di-, tri-, and tetra-gamma functions, ..., respectively. By succesively differentiate the completely monotonic functions F_n and G_n the following asymptotic series for $\psi^{(n)}(x)$ can be obtained:

$$\psi^{(n)}(x) \sim (-1)^{n-1}\left[\frac{(n-1)!}{x^n} + \frac{n!}{2x^{n+1}} + \sum_{j=1}^{\infty} B_{2j}\frac{(2j+n-1)!}{(2j)!x^{2j+n}}\right], \quad x \to \infty.$$

See, e.g., [1, p. 260]. By using the inequalities (2.4) (consequences of (2.5)-(2.6)), we get $g(x) > h(x)$, where

$$\begin{aligned} h(x) &= x(x-1)^2\left(\frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3}\right) \\ &\quad - \left(x^2 - 1\right)\left(\ln x - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4}\right) \\ &\quad + (x+1)\left(\left(x - \frac{1}{2}\right)\ln x - x + \frac{1}{2}\ln 2\pi + \frac{1}{12x} - \frac{1}{360x^3}\right). \end{aligned}$$

It remains to show that $h(x) > 0$. First, remark that

$$h''(x) = \frac{P(x)}{30x^6},$$

where $P(x) = 5 - x + 13x^2 - 15x^3 - 15x^4 + 15x^5$. As

$$P(x) = 15(x-1)^5 + 60(x-1)^4 + 75(x-1)^3 + 28(x-1)^2 - 5(x-1) + 2,$$

it results that $P(x) > 0$.

Thus $h'' > 0$, so h' is strictly increasing.

We have $h'(1.1) = 4.2218... \times 10^{-3} > 0$, and using the monotonicity of h', it results that $h' > 0$, on $(1.1, \infty)$. Further, h is strictly increasing.

But $h(2) = 2.3099... \times 10^{-2} > 0$, so $h > 0$ on $(2, \infty)$ and we are done: $g > 0$ and $\left(x\frac{f'(x)}{f(x)}\right)' > 0$. That is f is geometrically convex on $(2, \infty)$.

3. **The Minc-Sathre inequality**

Let us consider the family of inequalities:

$$\frac{((n+1)!)^{\frac{1}{n+1}}}{(n!)^{\frac{1}{n}}} < \left(\frac{n+2}{n+1}\right)^{\frac{4n+a}{4n+4}}, \quad n = 1, 2, 3, \ldots, \tag{3.1}$$

where a is a real parameter. Chu et al. proved (3.1) in [12, rel. 1.8] in case $a = 3$.

We prove in this paper an improvement of (3.1). More precisely, we show that (3.1) is also true for $a = 0$.

Theorem 3.1. *For every integer $n \geq 18$, we have*

$$\frac{((n+1)!)^{\frac{1}{n+1}}}{(n!)^{\frac{1}{n}}} < \left(\frac{n+2}{n+1}\right)^{\frac{n}{n+1}}. \tag{3.2}$$

(this inequality is reversed for $n = 1, 2, 3, \ldots, 17$). Moreover, inequality (3.2) holds true even for the real values $n = x \in [18, \infty)$.

Proof. We mention that the inequality (3.2) can be directly verified for $1 \leq n \leq 17$, using a computer software for symbolic comutation. We prove that (3.2) holds true for every real number $n = x \geq 18$.

In this sense, let us consider the real variable $x \geq 18$. We use the following inequality:

$$\frac{(\Gamma(x+2))^{\frac{1}{x+1}}}{(\Gamma(x+1))^{\frac{1}{x}}} < \frac{2x+1}{2x-1}\left(\sqrt{2\pi x}\exp\left(\frac{2}{3x} - \frac{1}{4x^2}\right)\right)^{\frac{-1}{x(x+1)}}, \quad x \geq 1,$$

which is the left-hand side of an inequality stated by Chen and Mortici [9, rel. 57]. Now it suffices to show that:

$$\frac{2x+1}{2x-1}\left(\sqrt{2\pi x}\exp\left(\frac{2}{3x} - \frac{1}{4x^2}\right)\right)^{\frac{-1}{x(x+1)}} < \left(\frac{x+2}{x+1}\right)^{\frac{x}{x+1}},$$

or, by raising to the $x(x+1)$-th power:

$$\left(\frac{2x+1}{2x-1}\right)^{x(x+1)}\left(\sqrt{2\pi x}\exp\left(\frac{2}{3x}-\frac{1}{4x^2}\right)\right)^{-1}<\left(\frac{x+2}{x+1}\right)^{x^2}.$$

By taking the logarithm, we have to prove that $z(x)<0$, where

$$z(x)=x(x+1)\ln\frac{2x+1}{2x-1}-\left(\frac{1}{2}\ln 2\pi x+\frac{2}{3x}-\frac{1}{4x^2}\right)-x^2\ln\frac{x+2}{x+1}.$$

We have

$$z'''(x)=\frac{Q(x)}{x^5(2x+1)^3(2x-1)^3(x+2)^3(x+1)^3},$$

where

$$\begin{aligned}Q(x) &= -64x^{14}+544x^{13}+1744x^{12}-168x^{11}-8820x^{10}\\ &\quad -22\,332x^9-26\,779x^8-10\,565x^7+7343x^6\\ &\quad +6801x^5-804x^4-2058x^3-316x^2+184x+48.\end{aligned}$$

This $Q(x)$ is negative for $x\geq 11$, since it can be represented as a polynomial of 14th degree, in powers of $(x-11)\geq 0$, with all coefficients negative:

$$Q(x)=-64(x-11)^{14}-9312(x-11)^{13}-625\,168(x-11)^{12}-\cdots.$$

It follows that $z'''<0$. Hence z' is strictly concave, and taking into account that $\lim_{x\to\infty}z'(x)=0$, we deduce that $z'<0$.
Finally, z is strictly decreasing, with $z(18)=-1.4528\ldots\times 10^{-2}<0$. It results that $z(x)<0$, for every $x\in(18,\infty)$. □

Remark 3.1. Some computations in this work were performed using the Maple software.

4. **Conclusions**

In this paper we provided a new proof for the fact that the function $\Gamma(x)^{\frac{1}{x-1}}$ is geometrically convex. As an application, we obtained an improvement of Minc-Sathre inequality.

REFERENCES

[1] *M. Abramowitz, and I. A. Stegun*, (Eds), Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables, Dover, New York, 1965.

[2] *H. Alzer*, On an inequality of H. Minc and L. Sathre, J. Math. Anal. Appl., **179** (1993) 396-402.

[3] *H. Alzer*, On some inequalities involving $(n!)^{1/n}$, I, Rocky Mt. J. Math., **24** (3) (1994), 867-873.

[4] *H. Alzer*, On some inequalities involving $(n!)^{1/n}$, II, Period. Math. Hung., **28** (3) (1994), 229-233.

[5] *H. Alzer*, Inequalities for the volume of the unit ball in $\mathbb{R}^n$, J. Math. Anal. Appl., **252** (2000), 353-363.

[6] H. Alzer, On some inequalities for the gamma and psi functions, Math. Comp., **66** (217) (1997), 373-389.

[7] *J. Böhm, and E. Hertel*, Polyedergeometrie in n-dimensionalen Räumen Konstanter Kräummung, Birkhäuser, Basel, 1981.

[8] *L. M. Brégman*, Some properties of nonnegative matrices and their permanents, Sov. Math. Dokl. **14** (1973), 945-949.

[9] *C.-P. Chen, and C. Mortici*, Asymptotic expansions and inequalities relating to the gamma function, Appl. Anal. Discrete Math., **16** (2022), 379-399.

[10] *C.-P. Chen*, Sharp inequalities involving $(n!)^{1/n}$, Mediterr. J. Math., **15** (177), (2018).

[11] *C.-P. Chen, N. Elezovic, L. Vukšić*, Asymptotic formulae associated with the Wallis power function and digamma function, J. Classical Anal., **2** (2013), 151-166.

[12] *Y. Chu, X. Zhang, and Z. Zhang*, The geometric convexity of a function involving gamma function with applications, Commun. Korean Math. Soc., **25** (2010), 373-383.

[13] *C. E. Finol, and M. Wojtowicz*, Multiplicative properties of real functions with applications to classical functions, Aequationes Math., **59** (2000), 134-149.

[14] *M. Marcus, and H. Minc*, Permanents, Amer. Math. Monthly, **72** (6) (1965), 577-591.

[15] *J. Matkowski*, L^p-like paranorms, Selected topics in functional equations and iteration theory, (Graz, 1991), 103-138, Grazer Math. Ber., 316, Karl-Franzens-Univ. Graz, Graz, 1992.

[16] *H. Minc, and L. Sathre*, Some inequalities involving $(r!)^{1/r}$, Proc. Edinburgh Math. Soc. **14** (1964/65), 41-46.

[17] *H. Minc*, Upper bound for permanents of $(0,1)$-matrices, Bull. Amer. Math. Soc., **69** (1963), 789-791.

[18] *P. Montel*, Sur les functions convexes et les fonctions sousharmoniques, J. Math. Pures Appl., **9** (7), (1928), 29-60.

[19] *C. Mortici*, Estimates of the function and quotient by Minc-Sathre, Appl. Math. Comp., **253** (2015), 52-60.

[20] *C. Mortici*, A lower bound for a quotient of roots of factorials, Bull. Acad. St. Rep. Mold. Mat., **61** (3) (2009), 64-69.

[21] *C. Mortici*, Monotonicity properties of the volume of the unit ball in $\mathbb{R}^n$, Optim. Lett., **4** (2010), 457-464.

[22] *C. Mortici*, A double inequality involving $\Gamma(r)^{1/r}$ via the multiplication formula of Gauss, J. Adv. Math. Stud., **2** (2) (2009), 57-62.

[23] *C. Mortici, and H. M. Srivastava*, Estimates for the arctangent function related to Shafer's inequality, Colloq. Math., **136** (2) 2014, 263-270.

[24] *C. Mortici*, Product approximations via asymptotic integration, Amer. Math. Monthly, **117** (5) (2010), 434-441.

[25] *C. Mortici, V.G. Cristea, and D. Lu*, Completely monotonic functions and inequalities associated to some ratio of gamma function, Appl. Math. Comput., **240** (2014), 168-174.

[26] *C. Mortici*, A new fast asymptotic series for the gamma function, Ramanujan J., **38** (3) (2015), 549-559.

[27] *C. Mortici*, The proof of Muqattash-Yahdi conjecture, Math. Comp. Modelling, **51** (9-10) (2010), 1154-1159.

[28] *C. Mortici*, Fast convergences towards Euler-Mascheroni constant, Comput. Appl. Math., **29** (3) (2010), 479-491.

[29] *C. Mortici*, On Ramanujan's large argument formula for the Gamma function, Ramanujan J., **26** (2) (2011), 185-192.