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BINARY RELATIONS – ADDENDA 2

(SECTIONS, COMPOSABILITIES)

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Prima completare din această parte a lucrării se referă la secţiunea generalizată a unei relaţii binare – în legătură cu unele proprietăţi relativ la relaţii şi operaţii de algebră Booleană şi categoriale generalizate, respectiv cu restricţiile şi relaţia indusă a unei relaţii binare. Ultima completare este dedicată compozabilităţilor – condiţii echivalente de s-compozabilitate[1] şi noţiunea mai tare de compozabilitate(dualizată) cu unificarea unor proprietăţi cunoscute şi de tip funcţional; rezultatele principale sunt relativ la o ierarhie a autocompozabilităţii recurente (dualizată) – utilă în diferite abordări ale automatelor nedeterministe.

The first addendum from this part of the paper refers to the generalized section of a binary relation - in connection with some properties relative to Boolean algebra relations and operations and generalized categorical operations, respectively with the restrictions and the induced relation of a binary relation. The last addendum is dedicated to the composabilities – equivalent conditions of w-composability [1] and the stronger notion of (dualized) composability with the unification of some known and functional properties; the main results are relative to a hierarchy of the (dualized) recurrent self-composability - useful in different approaches of non- deterministic automata.

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1. Introduction

The generalizations relative to categorical operations, restrictions and inducement in arbitrary sets from the first part of the paper [1] are continued in the first addendum from this part of the paper with the section of a binary relation relative to a arbitrary set; the section of the relation $R \in \mathcal{Rel}(A, B)$ relative to the arbitrary set X – for short the X-section of R is $R(X) = \{b \in B \,/\, \exists x \in X, (x, b) \in R\}$ $\in \mathcal{P}(B)$. We retrieve the special sections such as codomain, right-segment that is determined by un element - codom(R) = $R(A)$, $R\langle x\rangle = R(\{x\}) \in$ $\mathcal{P}(B)$,

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respectively domain, left-segment that is determined by un element - dom(R) $= R^{-1}(B)$, $R^{-1}\langle y\rangle = R^{-1}(\{y\}) \in$ $P(A)$. In addition we supplement the notion of field with that of subfield where field(R) = dom(R)$\cup$codom(R), subfield(R) = dom(R)$\cap$ codom(R). Really the taking of the section it is done onto the set of the parts of the source-set – more exactly it is done onto the parts of the domain because $R(X)$ = $R(X\cap A)$ = $R(X\cap$dom(R)) and hence for $R \neq \varnothing$ we have $R(X) \neq \varnothing$ iff $X\cap$ dom(R)$\neq \varnothing$; particularly we have codom(R) = R(dom(R)) and $R\langle x\rangle \neq \varnothing$ iff $x \in$ dom(R) – and analogously for the sections of the inverse R^{-1}. Consequently for the taking of the generalized section we retrieve the properties relative to the Boolean algebra relations and operations with sets and relations and generalized composition – which is supplemented with other generalized categorical operations(relations inducement, product). In addition the segments as special sections preserve intersection and allow among other things the description of the kernel and of the r"-morphism (see [1]).

However, the main results are relative to the sections of the restrictions and of the induced relation of a binary relation – with the getting of some extensions as restrictions and induced relations.

In the second addendum from this part of the paper the equivalent conditions of w-composability are given – which was defined in the first part of the paper [1] and the stronger notion of (dualized) composability is defined - with the unification of some known and functional properties and an addendum to the domain(and dually codomain) characterization theorem [2]; in addition it is relevant to remark the example concerning a characterization of the subtotal relations – in point of regularity inclusively. However, the main results are relative to a hierarchy of the (dualized) recurrent self-composability, the reduction of "recurrent self-bicomposability" to self-bicomposability, self-composabilities vs. recurrent w-self-composability and the univocality case.

The taking of the section onto the set of the parts of the source-set was categorical studied in [3], [4]; the above addenda may be categorical validated in categories with intersections and unions of "objects" [5] – analogously with the restrictions and the inducement in (sub)objects [1], [6].

We close up with an example concerning generalized sections relative to subtotal relations and subdiagonal relations (see example 1.1 from [1]).

Example 1.1. Let X, Y be arbitrary sets and let x, y be arbitrary elements. For $\omega_{U,V} \in Rel_{st}(A, B)$ – the set of subtotal relations in A, B relative to the set $Rel(A, B)$ of binary relations between A, B we have $\ker \omega_{U,V} = (\omega_{U,V})^{-1}\, \omega_{U,V} = \omega_{V,U}\, \omega_{U,V} = \omega_U$, $co\ker \omega_{U,V} = \omega_{U,V}(\omega_{U,V})^{-1} = \omega_{U,V}\omega_{V,U} = \omega_V$, $\omega_{U,V}(X) = \varnothing$ for $U \cap X = \varnothing$, respectively $\omega_{U,V}(X) = V$ for $U \cap X \neq \varnothing$ and $(\omega_{U,V})^{-1}(Y) =$

$\omega_{V,U}(Y)=\varnothing$ for $V\cap Y=\varnothing$, respectively $(\omega_{U,V})^{-1}(Y)=U$ for $V\cap Y\neq\varnothing$ - particularly codom($\omega_{U,V}$) = dom(coker $\omega_{U,V}$) = codom(coker $\omega_{U,V}$) = V, $\omega_{U,V}\langle x\rangle=\varnothing$ for $x\notin U$, $\omega_{U,V}\langle x\rangle=V$ for $x\in U$, dom($\omega_{U,V}$) = codom($\omega_{V,U}$) = dom(ker $\omega_{U,V}$) = codom(ker $\omega_{U,V}$) = U, $(\omega_{U,V})^{-1}\langle y\rangle=\omega_{V,U}\langle y\rangle=\varnothing$ for $y\notin V$, $(\omega_{U,V})^{-1}\langle y\rangle = U$ for $y\in V$, field($\omega_{U,V}$) = $U\cup V$, subfield($\omega_{U,V}$) = $U\cap V$ and analogously in homogeneous case *Rel*(A) = *Rel*(A, A). In addition for $R\in$ *Rel*(A, B) we have $R\omega_{U,V}=\omega_{U,R(V)}$ because $(R\omega_{U,V})\langle x\rangle=R(\omega_{U,V}\langle x\rangle)=R(V)=\omega_{U,R(V)}\langle x\rangle$ for $x\in U$, respectively $(R\omega_{U,V})\langle x\rangle = \omega_{U,R(V)}\langle x\rangle=\varnothing$ for $x\notin U$ -and analogously $\omega_{U,V}R=\omega_{R^{-1}(U),V}$; particularly we have $R\omega_{dom(R)}=\omega_{codom(R)}R=\omega_{dom(R),codom(R)}$. For $\Delta_U\in$ *Rel*$_{\mathrm{sd}}$(A) - the set of subdiagonal relation in A relative to *Rel*(A) we have $\Delta_U(X)=(\Delta_U)^{-1}(X)=U\cap X$ - particularly dom(Δ_U) = codom(Δ_U) = field(Δ_U) = subfield(Δ_U) = U, $\Delta_U\langle x\rangle=(\Delta_U)^{-1}\langle x\rangle=\varnothing$ for $x\notin U$, respectively $\Delta_U\langle x\rangle=(\Delta_U)^{-1}\langle x\rangle=\{x\}$ for $x\in U$.

2. Sections

Observation 2.1.i (Boolean algebra relation and operation) The taking of the sections μ: *P*(A) → *P*(B), $X\mapsto R_0(X)$, $R_0\in$ *Rel*(A, B) and ρ: *Rel*(A, B) → *P*(B), $R\mapsto R(X_0)$ are order morphisms and they only are the superior semilattice morphisms; consequently the taking of the corresponding generalized sections preserve the inclusion and the union(as Boolean algebra relation and operation). In addition, we have $C_{codom(R_0)}R(X)\subseteq R(CX)$ where $X\in$ *P*(A), $C_{codom(R_0)}R(X)\subseteq CR(X)$, respectively $CR(X_0)\subseteq(CR)(X_0)$ where $X_0\cap A\neq\varnothing$ (see [7]).

ii (generalized composition) In the case of the taking of the generalized section and of the generalized composition the behaviour towards composition it is preserved too – for $R\in$ *Rel*(A, B), $S\in$ *Rel*(C, D) and the set X we have (SR)(X) = S(R(X)) (see [7]).

iii (inclusion preserving vs. equality preserving) The inclusion preserving by the taking of the sections – and by restrictions and inducement in sets [1] and by generalized categorical operations and the corresponding equality preserving are connected (according to elementary axioms of the system ZFC, see [7]).

Theorem 2.1 (inclusion, intersection) Let be the relations R, $S\in$ *Rel*(A, B) and the following statements:

ir) $S \subseteq R$,

sec) for each X, $S(X) \subseteq R(X)$,

seg)for each x, $S\langle x\rangle \subseteq R\langle x\rangle$.

We have the equivalences ir$\Leftrightarrow$sec$\Leftrightarrow$seg (inclusion preserving and reflecting), respectively the equality $(R \cap S)\langle x\rangle = R\langle x\rangle \cap S\langle x\rangle$ (intersection preserving).

Proof. We have the implications ir$\Rightarrow$sec$\Rightarrow$seg (by definition and according to the observation 2.1.i); consequently, it suffices to prove the implication seg$\Rightarrow$ir – which results from $(x, y) \in S$ iff $y \in S\langle x\rangle$ implies $y \in R\langle x\rangle$ iff $(x, y) \in R$. The equality follows from $y \in R\langle x\rangle \cap S\langle x\rangle \neq \varnothing$ iff $(x, y) \in R \cap S$ iff $y \in (R \cap S)\langle x\rangle \neq \varnothing$; in the empty case the equality is obtained by reductio ad absurdum – according to the inclusion $(R \cap S)\langle x\rangle \subseteq R\langle x\rangle \cap S\langle x\rangle$ (see observation 2.1.i).

Theorem 2.2 (inducement, product) Let $R \in \mathit{Rel}(A, B)$, $S \in \mathit{Rel}(C, D)$ be relations and let U, X, Y, Z be arbitrary sets; we have $(R, S)(X) \subseteq R(X) \times S(X)$, $(R, S)^{-1}(Y \times Z) = R^{-1}(Y) \cap S^{-1}(Z)$, respectively $(R \times S)(U \times X) = R(U) \times S(X)$ and analogously for the sections of the inverse.

Proof. For inducement (and analogously for product) we have $(b, d) \in (R, S)(X) \neq \varnothing$ iff there exists $x \in X$ such that $(x, (b, d)) \in (R, S)$ iff there exists $x \in X$, $(x, b) \in R$ and $(x, d) \in S$ implies there exists $x \in X$, $(x, b) \in R$ and there exists $x \in X$, $(x, d) \in S$ iff $(b, d) \in R(X) \times S(X) \neq \varnothing$, respectively $x \in (R, S)^{-1}(Y \times Z) \neq \varnothing$ iff there exists $(y, z) \in Y \times Z$, $(x, (y, z)) \in (R, S)$ iff there are $y \in Y$, $z \in Z$, $(x, y) \in R$ and $(x, z) \in S$ iff there exists $y \in Y$, $(x, y) \in R$ and there exists $z \in Z$, $(x, z) \in S$ iff $x \in R^{-1}(Y) \cap S^{-1}(Z) \neq \varnothing$.

Corollary 2.1 ((co)dom) We have codom(R, S)$\subseteq$codom(R)$\times$codom(S), dom(R, S) = dom(R)$\cap$dom(S), respectively (co)dom($R \times S$) = (co)dom(R) $\times$ (co)dom(S).

Observation 2.2 (descriptions with segments) Let be $R \in \mathit{Rel}(A, B)$ and the associated power relation ${}^{p}R \in \mathit{Rel}(P(A), P(B))$ which is defined by $(X, Y) \in {}^{p}R$ iff for each $a \in X$, $b \in Y$, $(a, b) \in R$ and $\{(\varnothing, Y), (X, \varnothing), (\varnothing, \varnothing)\} \subset {}^{p}R$ where $X \in P^{*}(A)$, $Y \in P^{*}(B)$ (see [7]).

i (kernel) We have ker R = $\{(a, a') \in \mathrm{dom}(R)^2 / R\langle a\rangle \cap R\langle a'\rangle \neq \varnothing\}$ and dually - coker R = $\{(b, b') \in \mathrm{codom}(R)^2 / R^{-1}\langle b\rangle \cap R^{-1}\langle b'\rangle \neq \varnothing\}$ (see [1]); in the special case R right-univocal relation (R partial function) the condition becomes the known equality $R(a)$ = $R(a')$ – and analogously for cokernel with R left-univocal and $R^{-1}(b) = R^{-1}(b')$.

ii (composite) In addition let be $S \in \mathtt{Rel}(C,\ D)$; we have $SR = \{(a,\ d) \in A \times D /\ R\langle a\rangle \cap S^{-1}\langle d\rangle \neq \varnothing\}$.

iii (r"-morphism) In the inhomogeneous case relative to the inhomogeneous relational structures $(A,\ A',\ R_{A,\ A'})$, $(B,\ B',\ R_{B,\ B'})$ $(F,\ F') \in \mathtt{Rel}(A,\ B) \times \mathtt{Rel}(A',\ B')$ is (inhomogeneous) r"-morphism iff for each $a \in A,\ a' \in A'$, $(a,\ a') \in R_{A,\ A'}$ implies $(F\langle a\rangle,\ F'\langle a'\rangle) \in^{p} R_{B,\ B'}$; in the homogeneous case the ordered pair $(F,\ F')$ becomes the single-element set $\{F\}$ which is noted F (see [1]) – and if F is partial function, then the corresponding power relation coincides with the associated relation.

Theorem 2.3. Relative to the relation $R \in \mathtt{Rel}(A,\ B)$ and the arbitrary sets U, V, X, Y we consider the restriction $R|_U$, the corestriction $_V|R$ and the induced relation $R_{U,V}$. i (sections of the restrictions and of the induced relation) We have $R|_U(X) = R(U \cap X)$, $_V|R(X) = R(X) \cap V$, $(R|_U)^{-1}(Y) = R^{-1}(Y) \cap U$, $(_V|R)^{-1}(Y) = R^{-1}(V \cap Y)$, $R_{U,V}(X) = R(U \cap X) \cap V$, $(R_{U,V})^{-1}(Y) = R^{-1}(V \cap Y) \cap U$.

ii (equalities, inclusions) We have the equalities and the inclusions $R|_U = R_{U,\ R(U)} \subseteq {}_{R(U)}|R$, $_V|R = R_{R^{-1}(V),\ V} \subseteq R|_{R^{-1}(V)}$, $R_{U,V} = R_{U,\ R(U)} \cap R_{R^{-1}(V),\ V} \subseteq R_{R^{-1}(V),\ R(U)}$.

Proof. i. We have successively (according to example 1.1 and [1]) $R|_U(X) = (R\Delta_U)(X) = R(\Delta_U(X)) = R(U \cap X)$, $_V|R(X) = (\Delta_V R)(X) = \Delta_V(R(X)) = R(X) \cap V$, $(R|_U)^{-1}(Y) = {}_U|R^{-1}(Y) = R^{-1}(Y) \cap U$, $(_V|R)^{-1}(Y) = R^{-1}|_V(Y) = R^{-1}(V \cap Y)$, $R_{U,V}(X) = (_V|R)|_U(X) = {}_V|R(U \cap X) = R(U \cap X) \cap V$, $(R_{U,V})^{-1}(Y) = (R^{-1})_{V,U}(Y) = R^{-1}(V \cap Y) \cap U$.

ii. We have successively (according to i, observation 2.1 and [1]) $R|_U(X) = R(U \cap X) \subseteq R(U) \cap R(X) = {}_{R(U)}|R(X)$, $R|_U \subseteq {}_{R(U)}|R$, $R_{U,\ R(U)} = R|_U \cap {}_{R(U)}|R = R|_U$, $_V|R = (R^{-1}|_V)^{-1} \subseteq (_{R^{-1}(V)}|R^{-1})^{-1} = R|_{R^{-1}(V)}$, $R_{R^{-1}(V),\ V} = R|_{R^{-1}(V)} \cap {}_V|R = {}_V|R$, $R_{U,V} = R|_U \cap {}_V|R = R_{U,\ R(U)} \cap R_{R^{-1}(V),\ V} \subseteq {}_{R(U)}|R \cap R_{R^{-1}(V)} = R_{R^{-1}(V),\ R(U)}$.

Corollary 2.2.i ((co)dom) We have dom($R|_U$) = dom(R)$\cap U$, codom($R|_U$) = $R(U)$, dom($_V|R$) = $R^{-1}(V)$, codom($_V|R$) = codom(R)$\cap V$, dom($R_{U,V}$) = $R^{-1}(V) \cap U$, codom($R_{U,V}$) = $R(U) \cap V$.

ii (extensions) We have $R = R|_{dom(R)} = _{codom(R)}| R = R_{dom(R), codom(R)}$, $S \subseteq R$ iff $S \subseteq R_{dom(S), codom(S)}$, $\omega_{dom(R), codom(R)}$=min{ $\omega_{U,V} \in$ $\mathcal{R}el_{st}(A, B)$/ $R \subseteq \omega_{U,V}$ } because $R|_{dom(R)} = R\Delta_{dom(R)} = R$, $S = S_{dom(S), codom(S)} \subseteq R_{dom(S), codom(S)}$, $R \subseteq \omega_{A, B}$ implies $R = R_{dom(R), codom(R)} \subseteq (\omega_{A, B})_{dom(R), codom(R)} = \omega_{dom(R), codom(R)}$, $R \subseteq \omega_{U,V}$ implies dom(R)$\subseteq$dom($\omega_{U,V}$)=U, codom(R)$\subseteq$codom($\omega_{U,V}$)=V iff $\omega_{dom(R), codom(R)} \subseteq \omega_{U,V}$ (see observation 2.1 and [1]).

iii (null or homogeneous relation) We have $R = \varnothing$ iff dom(R) = $\varnothing$ iff codom(R) = $\varnothing$, respectively $R \in$ $\mathcal{R}el(A)$ iff field(R)$\in$ $\mathcal{P}(A)$ (see point ii).

3. Composabilities

Definition 3.1 (composabilities) The relations $R \in \mathcal{R}el(A, B)$, $S \in \mathcal{R}el(C, D)$ are composable (R is left-composable with S or S is right-composable with R) if each pair of R is composable with at least one pair of S, i.e. codom(R)$\subseteq$dom(S); dually, R, S are cocomposable if dom(S) $\subseteq$codom(R) and they are bicomposable if they are both composable and cocomposable, i.e. codom(R) = dom(S). Particularly, R is self-composable if codom(R)$\subseteq$dom(R); dually, R is self-cocomposable if dom(R) $\subseteq$codom(R) and it is self-bicomposable if it is both self-composable and self-cocomposable, i.e. codom(R) = dom(R).

Observation 3.1.i (necessary and sufficient conditions) R, S are w-composable ($SR \neq \varnothing$ - see [1] and corollary 2.2.iii) iff dom(SR) $\neq \varnothing$ iff codom(SR) $\neq \varnothing$ iff codom(R) $\cap$ dom(S) $\neq \varnothing$. In the categorical case B = C R, S are (co)composable if S(R) is left(right)-total relation. Particularly, R is w-self-composable iff dom(R^2)$\neq \varnothing$ iff codom(R^2)$\neq \varnothing$ iff subfield(R) $\neq \varnothing$; in the homogeneous case $R \in$ $\mathcal{R}el(A)$ is self-composable if R is left-total relation – and dually.

ii (hierarchy) Non-banally we have (see point i) R, S (co)composable imply R, S w-composable – and analogously for self-composability.

iii (inclusions, conditioned equalities) We have (see [8], [7]) dom(SR) $\subseteq$ dom(R), codom(SR)$\subseteq$codom(S), respectively R, S composable implies dom(SR) = dom(R), R, S cocomposable implies codom(SR) = codom(S), R, S bicomposable implies dom(SR) = dom(R), codom(SR) = codom(S) – and analogously for self-composability.

iv ((co)kernel) The relations R, R^{-1}, respectively R^{-1}, R are bicomposable; consequently we have(see point iii and [1])dom(R) = dom(ker R) = codom(ker R), respectively codom(R) = dom(coker R) = codom(coker R), hence ker R, respectively coker R are self-bicomposable.

v (addendum to domain characterization theorem) The domain characterization theorem places the equivalence $X \subseteq$ dom(R) iff $X \subseteq R^{-1}(R(X))$, where $R \in \mathit{Rel}(A, B)$, $X \in P(A)$; for codomain the dual is valid too (see [2]). An addendum of this theorem (in the version with generalized sections, see [7]) principally consists in the inclusion ker $R(X) \subseteq$ dom(R) and the equality $\Delta_{dom(R)} = \max\{\Delta_X \in \mathit{Rel}_{sd}(A) / \Delta_X \subseteq \ker R\}$ – and dually for codomain (ker R is dom(R)-reflexive, see point iv and [1]); this addendum permits a new proof of the theorem - $X \subseteq$ dom(R) imply $X = \Delta_{dom(R)}(X) \subseteq$ ker $R(X)$, respectively $X \subseteq$ ker $R(X) \subseteq$ dom(R) (see example 1.1).

Theorem 3.1 (complements) Let $R \in \mathit{Rel}(A, B)$, $S \in \mathit{Rel}(C, D)$ be relations. i (restrictiveness vs. composability) R is right-composable with the corestriction $_{dom(R)}|S$ and left-cocomposable with the restriction $S|_{codom(R)}$; consequently R is right-composable and left-cocomposable with the induced relation $S_{codom(R),\, dom(R)}$.

ii (partial reflectiveness) In addition let be $R' \in \mathit{Rel}(A, B)$, $S' \in \mathit{Rel}(C, D)$; R', S composable, $SR' \subseteq SR$ implies dom(R') $\subseteq$ dom(R), respectively R, S' cocomposable, $S'R \subseteq SR$ implies codom(S') $\subseteq$ codom(S).

iii (the univocality case) In the special case R right-univocal relation (R partial function) we have R, S composable iff dom(SR) = dom(R); dually, in the case S left-univocal relation we have R, S cocomposable iff codom(SR) = codom(S). In the case R right-univocal relation and S left-univocal relation we have R, S bicomposable iff dom(SR) = dom(R), codom(SR) = codom(S).

Proof. i. We have (see theorem 2.1, corollary 2.2 and [1]) codom($S_{codom(R),\, dom(R)}$) $\subseteq$ codom($_{dom(R)}|S$) = codom(S) $\cap$ dom(R) $\subseteq$ dom(R), dom($S_{codom(R),\, dom(R)}$) $\subseteq$ dom($S|_{codom(R)}$) = dom(S) $\cap$ codom(R) $\subseteq$ codom(R).

ii. We have (see theorem 2.1 and observation 3.1.iii) dom(R') = dom(SR') $\subseteq$ dom(SR) $\subseteq$ dom(R), respectively codom(S') = codom($S'R$) $\subseteq$ codom(SR) $\subseteq$ codom(S).

iii. $R \neq \varnothing$ or dually $S \neq \varnothing$ implies $SR \neq \varnothing$ or equivalently codom(R) $\cap$ dom(S) $\neq \varnothing$, i.e. R, S w-composable (necessary composability condition, see observation 3.1, points i, ii); we have codom(R) = R(dom(R)) = R(dom(SR)) = $R((SR)^{-1}($ codom(SR)) = ((coker R)S^{-1})(codom(SR)) $\subseteq$((coker R)S^{-1}) (codom(S)) = coker $R(S^{-1}$(codom(S)) = coker R(dom(S)) = $\Delta_{codom(R)}$(dom(S)) = codom(R) $\cap$ dom(S), codom(R) $\subseteq$ dom(S) and dually dom(S) = S^{-1}(codom(S)) = S^{-1}(codom(SR)) = $S^{-1}((SR)$ (dom(SR)) = ((ker S)R)(dom(SR)) $\subseteq$ ((ker S)R) (dom(R)) = ker S(R(dom(R)) = ker S(codom(R)) = $\Delta_{dom(S)}$(codom(R)) = dom(S) $\cap$

codom(R), dom(S) $\subseteq$ codom(R) (see [9]). Reciprocal implications are unconditionally (by observation 3.1.iii). The case R right-univocal relation and S left-univocal relation follows from the above two cases.

Example 3.1. Let be the sets A, B with $|A|$, $|B|$ >2 (in Card). i (equalities, inclusions) For $R\in \mathtt{Rel}(A, B)$ we have the equalities dom(R) = dom($RR^{-1}R$) (= D), codom(R) = codom($RR^{-1}R$) (= C) and the inclusions $R\subseteq RR^{-1}R\subseteq \omega_{D,C}$.

The first equality is a result of the bicomposability of R, coker R, i.e. dom(R) = dom((coker R)R) = dom($RR^{-1}R$) – and analogously the second equality relative to the bicomposability of ker R, R (see observation 3.1, points iv, iii); the inclusions follow from the D-reflexivity of ker R [1], i.e. $\Delta_D \subseteq$ ker R implies R = $R\Delta_D \subseteq R$ker R = $RR^{-1}R$, respectively from the anterior equalities and corollary 2.2.ii.

ii (characterization of the subtotal relations set) We have (see example 1.1 and [1]) $\mathtt{Rel}_{\mathrm{st}}(A, B)$ = $\{R\in \mathtt{Rel}(A, B)/ R=\omega_{dom(R), codom(R)}\}$; in addition for $\mathtt{Preu}(A, B)$ = $\mathtt{iReg}(A, B)$ = $\{R\in \mathtt{Rel}(A, B)/ R= RR^{-1}R\}$, $\mathtt{K}_{\mathrm{st}}(A, B)$ = $\{R\in \mathtt{Rel}(A, B)/\ker R = \omega_{dom(R)}\}$, $\mathtt{Cok}_{\mathrm{st}}(A, B)$ = $\{R\in \mathtt{Rel}(A, B)/\mathrm{coker}\ R = \omega_{codom(R)}\}$ – the sets of binary relations between A, B respectively preunivocal (or i-regulated) relations, relations with subtotal kernel, relations with subtotal cokernel we have $\mathtt{Rel}_{\mathrm{st}}(A, B)$ = $\mathtt{Preu}(A, B) \cap \mathtt{K}_{\mathrm{st}}(A, B)$ = $\mathtt{Preu}(A, B) \cap \mathtt{Cok}_{\mathrm{st}}(A, B)$ which hold true by double inclusion (according to i and example 1.1 – see also [1]).

iii (strict inclusions) For $\mathtt{Reg}(A, B) = \{R\in \mathtt{Rel}(A, B)/\exists R' \in \mathtt{Rel}(B, A), R= RR'R\} \subset \mathtt{Rel}(A, B)$ – the regulated relations set between A, B in the unregulated category $\mathtt{Rel}$ in which the dysfunctional relations (see [10]) were named also preunivocal relations (or i-regulated relations for unification of terminology [1], [9]) we have the strict inclusions $\mathtt{Rel}_{\mathrm{st}}(A, B)\subset \mathtt{K}_{\mathrm{st}}(A, B) \cap \mathtt{Cok}_{\mathrm{st}}(A, B) \subset \mathtt{Rel}(A, B)$, $\mathtt{iReg}(A, B) \subset \mathtt{Reg}(A, B)$ because $\mathtt{S}$ = $\{\dot{S} = S\setminus\{(a, b)\}/ S\in \mathtt{Rel}_{\mathrm{st}}(A, B), |\mathrm{dom}(S)|, |\mathrm{codom}(S)|>1\}$, $\mathtt{R}$ = $\{\dot{R} = R\setminus\{(a, b)\}/ R\in (\mathtt{K}_{\mathrm{st}}(A, B) \cap \mathtt{Cok}_{\mathrm{st}}(A, B))\setminus \mathtt{Rel}_{\mathrm{st}}(A, B), \exists b', \forall x\in \mathrm{dom}(R), b'\in R\langle x\rangle\setminus\{b\}\neq\varnothing, R^{-1}\langle b'\rangle = \mathrm{dom}(R), \exists a', \forall y\in \mathrm{codom}(R), a'\in R^{-1}\langle y\rangle\setminus\{a\}\neq\varnothing, R\langle a'\rangle = \mathrm{codom}(R)\}\subset(\mathtt{K}_{\mathrm{st}}(A, B) \cap \mathtt{Cok}_{\mathrm{st}}(A, B))\setminus \mathtt{Rel}_{\mathrm{st}}(A, B)$, $\mathtt{S} \subset \mathtt{Reg}(A, B)\setminus \mathtt{iReg}(A, B)$, $\mathtt{R}\subset \mathtt{Rel}(A, B)\setminus \mathtt{iReg}(A, B)$.

For T = $\dot{S}\in \mathtt{S}$, D = dom(S), $\dot{D}= D\setminus\{a\}\neq\varnothing$, C = codom(S), $\dot{C}= C\setminus\{b\}\neq\varnothing$ we have (see example 1.1, observation 2.1, theorem 2.1 and [1]) dom(T) = D, codom(T) = C, ker $T\langle x\rangle = (T^{-1}T)\langle x\rangle = T^{-1}(T\langle x\rangle) \supseteq T^{-1}\langle b'\rangle = D=\omega_D\langle x\rangle = \omega_{dom(T)}\langle x\rangle$ with $b'\in\dot{C}$, coker $T\langle y\rangle = (TT^{-1})\langle y\rangle = T(T^{-1}\langle y\rangle) \supseteq$

$T\langle a'\rangle = C = \omega_C\langle y\rangle = \omega_{codom(T)}\langle y\rangle$ with $a' \in \dot{D}$, $TT^{-1}T = \omega_{D,\,C} = S \supset T$ and T' = $\omega_{\dot{C},\,\{a\}} \cup \omega_{\{b\},\,\dot{D}}$, $T'T = \omega_{\dot{C},\,\{a\}}T \cup \omega_{\{b\},\,\dot{D}}T = \omega_{T^{-1}(\dot{C}),\,\{a\}} \cup \omega_{T^{-1}\langle b\rangle,\,\dot{D}} = \omega_{D,\,\{a\}} \cup \omega_{\dot{D}}$, $TT'T = T(\omega_{D,\,\{a\}} \cup \omega_{\dot{D}}) = T\omega_{D,\,\{a\}} \cup T\omega_{\dot{D}} = \omega_{D,\,T\langle a\rangle} \cup \omega_{\dot{D},\,T(\dot{D})} = \omega_{D,\,\dot{C}} \cup \omega_{\dot{D}.\,C} = T$, respectively. Analogously for $T = \dot{R} = R \setminus P \in$ R, D = dom(R), C = codom(R) we have dom(T) = D, ker $T\langle x\rangle = T^{-1}(T\langle x\rangle) \supseteq T^{-1}\langle b'\rangle = (R^{-1} \setminus P^{-1})\langle b'\rangle = R^{-1}\langle b'\rangle$ = D = $\omega_D\langle x\rangle$ = $\omega_{dom(T)}\langle x\rangle$, codom($T$) = C, coker $T\langle y\rangle = T(T^{-1}\langle y\rangle) \supseteq$ $T\langle a'\rangle$ = $R\langle a'\rangle$ = $C = \omega_C\langle y\rangle = \omega_{codom(T)}\langle y\rangle$, $TT^{-1}T = \omega_{D,\,C} \supset R \supset T$.

Definition 3.2 (recurrent self-composabilities) For $m, n \in IN^*$ ($m = n = 1$ implicitly) the relation $R \in$ Rel(A, B) is m, n-self-composable if R^m, R^n are composable; dually R is m, n-self-cocomposable if R^m, R^n are cocomposable.

Theorem 3.2 (hierarchies) Let $R \in$ Rel(A, B) be a relation. i (recurrent self-composability) For each $m, n \in IN^*$ we have the implications:

R m, n-self-composable implies R $m+1$, n-self-composable (1);

R m, $n+1$-self-composable implies R m, n-self-composable (2).

For $m \le n$ the implication (2) becomes equivalence; for $m > n$ (2) becomes conditioned equivalence by dom(R) = … = dom(R^m).

ii (recurrent self-cocomposability) For each $m, n \in IN^*$ we have the implications:

R m, n-self-cocomposable implies R m, $n+1$-self-cocomposable (3);

R $m+1$, n-self-cocomposable implies R m, n-self-cocomposable (4).

For $n \le m$ the implication (4) becomes equivalence; for $n > m$ (4) becomes conditioned equivalence by codom(R) = … = codom(R^n).

Proof. i. The implications (1) and (2) are true because codom(R^{m+1})$\subseteq$ codom(R^m)$\subseteq$dom(R^n), respectively codom(R^m)$\subseteq$dom(R^{n+1})$\subseteq$dom(R^n) (see observation 3.1.iii); the reciprocal implication follows from codom(R^m)$\subseteq$ dom(R^n)$\subseteq$dom(R^m) = dom(R^{m+n})$\subseteq$dom(R^{n+1}) – with the mention that for $m \ge n$ we have the equality dom(R^n) = dom(R^m) (instead of inclusion).

ii. The dual implications (3) and (4) are true because dom(R^{n+1})$\subseteq$ dom(R^n)$\subseteq$codom(R^m), respectively dom(R^n)$\subseteq$codom(R^{m+1})$\subseteq$codom(R^m);the reciprocal implication follows from dom(R^n)$\subseteq$codom(R^m)$\subseteq$codom(R^n) = codom(R^{m+n})$\subseteq$codom(R^{m+1}) - with the mention that for $n \ge m$ we have the equality codom(R^m) = codom(R^n) (instead of inclusion).

Corollary 3.1 (self-bicomposability vs. recurrent self-bicomposability) For each $m, n \in IN^*$ and "*R* is *m*, *n*-self-bicomposable if R is both *m*, *n*-self-compo - sable and *m*, *n*-self-cocomposable, i.e. R^m, R^n are bicomposable." we have the equivalence:

R is *m*, *n*-self-bicomposable iff *R* is self-bicomposable.

Proof. The equivalence follows from the equivalences *R* *m*, *n*-self-bi - composable iff *R* $m+1$, *n*-self-bicomposable iff *R* *m*, $n+1$-self-bicomposable.

Observation 3.2.i (self-composabilities vs. recurrent w-self-composability) The implications (1), (2) are composable, i.e. "*R* $m, n+1$-self-composable implies *R* *m*, *n*-self-composable implies *R* $m+1$, *n*-self-composable" where for $m \le n$ the first implication becomes equivalence; consequently for each $m, n \in IN^*$ we have the implications "R self-composable implies *R* *m*, *n*-self-composable implies *R* $(m+n)$-w-self-composable" where for each $p \in IN^* \setminus \{1\}$, $i \in \{1, \ldots, p-1\}$ R is p-w-self-composable iff codom(R^i) $\cap$ dom(R^{p-i}) $\neq \varnothing$ [1]. It follows "for each $m, n \in IN^*$ R self-composable implies codom(R^m)$\cap$ dom(R^n)$\neq \varnothing$"; the above implication is stronger than the implication "R self-composable implies subfield(*R*) $\neq \varnothing$" because for each $m, n \in IN^*$ codom(R^m)$\cap$dom(R^n)$\neq \varnothing$ implies subfield(*R*) $\neq \varnothing$. In addition the sequence dom(*R*) = dom(R^2) is extended indefinitely, i.e. we have dom(*R*)=dom(R^2) $= \ldots =$ dom(R^{m+n}) (see observation 3.1, points ii, iii).

Dually it follows "for each $m, n \in IN^*$ R self-cocomposable implies codom(R^m) $\cap$dom(R^n)$\neq \varnothing$" – which is stronger than the implication "R self-cocomposable implies subfield(*R*) $\neq \varnothing$" and the indefinite extension of the sequence codom(*R*)= codom(R^2), i.e. codom(*R*) = codom(R^2) $= \ldots =$ codom(R^{m+n}).

ii (the univocality case) In the special case *R* right-univocal relation (*R* partial function) the sequence dom(*R*) = dom(R^2) is extended indefinitely with R self-composable – and dually; in the case *R* biunivocal relation it follows the conjunction of the respective sequences with R self-bicomposable (see theorem 3.1.iii).

4. Conclusions

The first addendum of this part of the paper refers to the generalized section – some properties such as intersection preserving for segments and the sections for relations inducement and product (see the theorems 2.1, 2.2); these complete the properties of the generalized section relative to Boolean algebra relations and operations with sets and relations and generalized composition (which were presented in **Introduction** – see also [7]); however, the main results are relative to the sections of the restrictions and of the induced relation of a binary relation – with the getting of some extensions as restrictions and induced relations (see theorem 2.3 and corollary 2.2, points i, ii).

The last addendum is dedicated to the composabilities – equivalent conditions of w-composability [1] and the stronger notion of (dualized) composability with the unification of some known and functional properties and an addendum to the domain(and dually codomain) characterization theorem (see [2], respectively observation 3.1 and theorem 3.1); in addition it is relevant to remark a characterization of the subtotal relations – in point of regularity inclusively (see example 3.1). However, the main results are relative to a hierarchy of the (dualized) recurrent self-composability, the reduction of "recurrent self-bicomposability" to self-bicomposability, self-composabilities vs. recurrent w-self-composability and the univocality case (see theorem 3.2, corollary 3.1 and observation 3.2).

The above addenda may be categorical validated in categories with intersections and unions of "objects" [5] – analogously with the restrictions and the inducement in (sub)objects [1], [6]; the last addendum may be useful in different approaches of non- deterministic automata.

R E F E R E N C E S

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