U.P.B. Sci. Bull., Series A, Vol. 86, Iss. 3, 2024 ISSN 1223-7027

ON FREE GAMMA MODULES

Mehmet Soner Pehlivan[1], Okan Arslan[2], Hatice Kandamar[3]

Let M be a Γ-ring in the sense of Barnes and N be a $M\Gamma$-module. In this study, we define linear independent subset of N and spanning subset of N and obtain various characterizations of a free gamma module analogous to the corresponding results in usual module theory.

Keywords: Gamma rings, Unitary modules, Strong unity, Free modules
MSC2020: 13C05, 16D40, 16Y80

1. Introduction

The theory of gamma rings has been introduced by Nobusawa as a generalization of rings by defining triple products on two abelian groups [10]. Barnes weakened the definition of gamma ring given by Nobusawa by omitting one of the triple products [3] as follows:

Let M and Γ be two abelian groups. M is said to be a **Γ-ring** (in the sense of Barnes) if there exists ternary multiplication $M \times \Gamma \times M \to M$ satisfying below conditions for all $a, b, c \in M$, $\alpha, \beta \in \Gamma$:

(i) $(a+b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha+\beta)c = a\alpha c + a\beta c$, $a\alpha(b+c) = a\alpha b + a\alpha c$
(ii) $(a\alpha b)\beta c = a\alpha(b\beta c)$

In addition, if there exist $\delta \in \Gamma$ and $e \in M$ such that $a\delta e = e\delta a = a$ for all $a \in M$, then a pair (δ, e) is called **strong unity** of M.

Many studies have been carried out on the structure of the gamma rings, with the definition given by Barnes [9, 8, 6].

The notion of gamma modules was first introduced by Kyuno [7]. Later, Ameri and Sadeghi extended the definition given by Kyuno as follows [2]:

Let M be a Γ-ring and N be an abelian group. N is called **left $M\Gamma$-module** if the mapping $M \times \Gamma \times N \to N$ satisfies the following conditions for all $n, n_1, n_2 \in N$; $m, m_1, m_2 \in M$ ve $\alpha, \alpha_1, \alpha_2, \beta \in \Gamma$:

(i) $m\alpha(n_1 + n_2) = m\alpha n_1 + m\alpha n_2$
$(m_1 + m_2)\alpha n = m_1\alpha n + m_2\alpha n$
$m(\alpha_1 + \alpha_2)n = m\alpha_1 n + m\alpha_2 n$
(ii) $(m_1\alpha m_2)\beta n = m_1\alpha(m_2\beta n)$

A **right $M\Gamma$-module** can be defined similarly. An abelian group N is called **$M\Gamma$-module** if N is both left and right $M\Gamma$-module.

We define a unitary left $M\Gamma$-module N as follows: Let N be a left $M\Gamma$-module and the Γ-ring M has a strong unity (δ, e). N is called **unitary left $M\Gamma$-module** if $e\delta n = n$ for all $n \in N$.

[1]Postdoctoral Research Assistant, Adnan Menderes University, Aydın, Turkey.
[2]Asst. Prof., Adnan Menderes University, Aydın, Turkey, e-mail: oarslan@adu.edu.tr (Corresponding Author).
[3]Professor, Adnan Menderes University, Aydın, Turkey.

Example 1.1. *Let* $M = \left\{ \begin{bmatrix} a & 0 & a \\ 0 & b & 0 \end{bmatrix} \mid a, b \in \mathbb{Q} \right\}$ *and* $\Gamma = \left\{ \begin{bmatrix} x & 0 \\ 0 & y \\ x & 0 \end{bmatrix} \mid c, d \in \mathbb{Q} \right\}$. *If* $m = \begin{bmatrix} a & 0 & a \\ 0 & b & 0 \end{bmatrix}$, $m' = \begin{bmatrix} c & 0 & c \\ 0 & d & 0 \end{bmatrix} \in M$ *and* $\alpha = \begin{bmatrix} x & 0 \\ 0 & y \\ x & 0 \end{bmatrix} \in \Gamma$, *then*

$$m\alpha m' = \begin{bmatrix} 2axc & 0 & 2axc \\ 0 & byd & 0 \end{bmatrix} \in M.$$

Therefore, we define ternary product $M \times \Gamma \times M \to M$ *as usual matrix multiplication. Then, it is easy to see that below equalities hold for all* $m, m_1, m_2 \in M$ *and* $\alpha, \beta \in \Gamma$*;*

(i) $(m_1 + m_2)\,\alpha m = m_1 \alpha m + m_2 \alpha m$

$m_1\,(\alpha + \beta)\,m_2 = m_1 \alpha m_2 + m_1 \beta m_2$

$m\alpha\,(m_1 + m_2) = m\alpha m_1 + m\alpha m_2$

(ii) $(m\alpha m_1)\,\beta m_2 = m\alpha\,(m_1 \beta m_2)$

Let $e = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix} \in M$ *and* $\delta = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \in \Gamma$. *Then, we have* $m\delta e = e\delta m = m$ *for all* $m \in M$*. Hence,* M *is a* Γ*-ring with a strong unity* (e, δ).

Now, let $N = \left\{ \begin{bmatrix} n_{11} & n_{12} & n_{13} & n_{14} \\ n_{21} & n_{22} & n_{23} & n_{24} \end{bmatrix} \mid n_{ij} \in \mathbb{Q},\ i = 1, 2,\ j = 1, 2, 3, 4 \right\}$.

If $n = \begin{bmatrix} n_{11} & n_{12} & n_{13} & n_{14} \\ n_{21} & n_{22} & n_{23} & n_{24} \end{bmatrix} \in N$, *then one can show that* $m_1 \alpha n \in N$*. Thus, we define the product* $M \times \Gamma \times N \to N$ *as usual matrix multiplication. Also, it is easy to see that below equalities hold for all* $m, m_1 \in M$, $\alpha, \beta \in \Gamma$, $n, n_1, n_2 \in N$*;*

(i) $m\alpha(n_1 + n_2) = m\alpha n_1 + m\alpha n_2$

$(m + m_1)\alpha n = m\alpha n + m_1 \alpha n$

$m(\alpha + \beta)n = m\alpha n + m\beta n$

(ii) $(m\alpha m_1)\beta n = m\alpha(m_1 \beta n)$

In addition, the equality $e\delta n = n$ *provides for all* $n \in N$*. Therefore,* N *is a unitary left* $M\Gamma$*-module.*

Paul and Uddin introduced the concept of free gamma module by defining the linearly Γ-independence of any subset of a gamma module [11]. However, according to their definition, some Γ-modules do not have any linearly Γ-independent subsets; see Example 1.2 (i). Abbas et al., who worked on free gamma modules in 2018, reconsidered the definition of a linearly independent set in gamma modules [1]. Nevertheless, their definition is inconsistent, as there are examples of subsets which are both Γ-independent and Γ-dependent; see Example 1.2 (ii).

Example 1.2. *Define* N, M *and* Γ *as in the Example 1.1.*

(i) *The* $M\Gamma$*-module* N *has no linearly* Γ*-independent subset.*

(ii) *Let* $X = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\}$ *be a subset of* N*. Then,* X *is both* $M\Gamma$*-linearly independent and* $M\Gamma$*-linearly dependent.*

In this paper, we show that the direct product of $M\Gamma$-modules is again $M\Gamma$-module and the direct sum of $M\Gamma$-modules is $M\Gamma$-submodule of this direct product. We redefined and gave examples to linear independence and basis in gamma modules to overcome the problems that we encountered on the definitions of linear independency and generating set which are shown in the Example 1.2. We also show that a free gamma module can be written as the internal direct sum of its cyclic $M\Gamma$-submodules. Moreover, we prove that each of the cyclic $M\Gamma$-submodules are isomorphic to the $M\Gamma$-module L which is known as the left operator ring of gamma ring. We also prove that free gamma modules satisfies the universal property.

In this study, "$M\Gamma$-module" will be used instead of "left $M\Gamma$-module" unless otherwise specified.

2. Some preliminaries on gamma rings

Definition 2.1. [5] *Let M be a Γ-ring, $a \in M$ and $\gamma \in \Gamma$. The map $[a,\gamma] : M \to M$ defined as $[a,\gamma]\, m = a\gamma m$ for every $m \in M$ is a group endomorphism. Let L be the set of all finite sums of the elements $[a_i,\gamma_i]$ where $a_i \in M$ and $\gamma_i \in \Gamma$. Then L is a subring of the endomorphism ring of M which is called the* ***left operator ring*** *of M. Multiplication on L is defined as*

$$\sum_i [a_i,\gamma_i] \sum_j [b_j,\beta_j] = \sum_{i,j} [a_i\gamma_i b_j,\beta_j] .$$

Definition 2.2. [7] *Let N be a $M\Gamma$-module and Q be a nonempty subset of N. Q is called a* ***$M\Gamma$-submodule*** *if $q_1 - q_2 \in Q$ and $m\gamma q \in Q$ for all $m \in M$, $q, q_1, q_2 \in Q$ and $\gamma \in \Gamma$.*

Definition 2.3. [7] *Let M be a Γ-ring. The map $\varphi : N \to Q$ for the $M\Gamma$-modules N and Q is called* ***$M\Gamma$-module homomorphism*** *if it satisfies the below conditions for all $x, y \in N$, $m \in M$ and $\gamma \in \Gamma$:*

(i) $\varphi(x + y) = \varphi(x) + \varphi(y)$.
(ii) $\varphi(m\gamma x) = m\gamma\varphi(x)$.

In addition, if φ is bijective, then it is called ***$M\Gamma$-module isomorphism****. The* ***kernel*** *of the $M\Gamma$-module homomorphism φ is defined as the set $\{x \in N \mid \varphi(x) = 0\}$, denoted by $Ker\varphi$ and the* ***image*** *of the $M\Gamma$-module homomorphism φ is defined as the set $\{y \in Q \mid y = \varphi(x),\ x \in N\}$, denoted by $Im\varphi$.*

Remark 2.1. *Let $\varphi : N \to Q$ be a $M\Gamma$-module homomorphism. Then, it is clear that the subsets $Ker\varphi$ and $Im\varphi$ are $M\Gamma$-submodules of N and Q respectively. If N_1 and N_2 are $M\Gamma$-submodules of N, then $N_1 + N_2$ and $N_1 \cap N_2$ are also $M\Gamma$-submodules of N.*

Definition 2.4. *Let N be a $M\Gamma$-module and X be a subset of N. The $M\Gamma$-submodule* ***generated by the subset*** *X is defined as the intersection of all $M\Gamma$-submodules of N containing X and denoted by (X). If X is finite, then the $M\Gamma$-submodule generated by the subset X is called* ***finitely generated*** *$M\Gamma$-module. If $X = \{n\}$, then the $M\Gamma$-submodule generated by the subset is X is called* ***cyclic*** *$M\Gamma$-submodule and denoted by (n).*

Definition 2.5. *Let $\{N_i \mid i \in I\}$ be a family of $M\Gamma$-modules indexed by a nonempty set I. The* ***direct product*** *of $M\Gamma$-modules N_i is the set of all functions $f : I \to \bigcup_{i\in I} N_i$ such that $f(i) \in N_i$ for all $i \in I$ and denoted by $\prod_{i\in I} N_i$. We denote the functions $f \in \prod_{i\in I} N_i$ by $\{a_i\}$ where $a_i = f(i)$ for each $i \in I$. The* ***direct sum*** *of $M\Gamma$-modules N_i is the set of all $f \in \prod_{i\in I} N_i$ such that $f(i) = 0_{N_i}$ for all but a finite number of $i \in I$ and denoted by $\sum_{i\in I} N_i$.*

Definition 2.6. *A $M\Gamma$-module N is called the* ***internal direct sum*** *of a family of its $M\Gamma$-submodules $\{N_i \mid i \in I\}$ and denoted by $\sum_{i\in I} N_i$ if the following conditions are satisfied:*

(i) *N is written as the sum of the family $\{N_i \mid i \in I\}$.*
(ii) *$N_k \cap N_k^* = (0_N)$ for every $k \in I$ where N_k^* is the sum of the family $\{N_i \mid i \neq k\}$.*

In order to define a gamma free module it is required that the concepts of linear independency and generating set. Due to the deficiencies of these concepts which we previously mentioned, we have redefined the concepts of linear independency and generating set as follows.

Definition 2.7. *Let N be a $M\Gamma$-module.*

(i) *The set of elements $x_1, \dots, x_n \in N$ is said to be* ***linearly independent*** *if the equality $\sum_{i=1}^{n} m_i\alpha_i x_i = 0_N$ yields $m_i\alpha_i M = (0_M)$ where $m_i \in M$ and $\alpha_i \in \Gamma$ for every $i = 1, \dots, n$.*

(ii) *Any subset X of N is said to be* ***linearly independent set*** *if all finite subsets of X are linearly independent.*

(iii) *N is said to be* ***spanned*** *by a subset S of N if every element of N may be written as a finite linear combination $\sum_{i=1}^{n} m_i\gamma_i s_i$ for $m_i \in M$, $\gamma_i \in \Gamma$ and $s_i \in S$. The elements of the subset S are called* ***generators*** *of N.*

Definition 2.8. *A linearly independent subset of a $M\Gamma$-module N that spans N is called a* ***basis*** *of N. If N is unitary $M\Gamma$-module and N has a nonempty basis X, then N is called* ***free*** *$M\Gamma$-module on the set X.*

Example 2.1. *Let N be the $M\Gamma$-module given Example 1.1. It can be shown that the subset X of N consisting of the elements*

$$x_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, x_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, x_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, x_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

is linearly independent. Moreover, each $n = \begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix} \in N$ can be written as

$$n = m_1\gamma_1 x_1 + m_2\gamma_2 x_2 + m_3\gamma_3 x_3 + m_4\gamma_4 x_4$$

for $m_1 = m_2 = m_3 = m_4 = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix} \in M$ and $\gamma_1 = \begin{bmatrix} a & 0 \\ 0 & f \\ a & 0 \end{bmatrix}$, $\gamma_2 = \begin{bmatrix} b & 0 \\ 0 & g \\ b & 0 \end{bmatrix}$, $\gamma_3 = \begin{bmatrix} c & 0 \\ 0 & h \\ c & 0 \end{bmatrix}$, $\gamma_4 = \begin{bmatrix} d & 0 \\ 0 & e \\ d & 0 \end{bmatrix} \in \Gamma$. This shows that the set X spans the $M\Gamma$-module N. Hence, N is free $M\Gamma$-module on the set X.

3. Results

Proposition 3.1. *Let N be a $M\Gamma$-module. Then the following conditions hold for all $m \in M$, $\alpha \in \Gamma$, $n \in N$ and $k \in \mathbb{Z}$.*

(i) *$0_M\alpha n = 0_N$, $m0_\Gamma n = 0_N$ and $m\alpha 0_N = 0_N$.*

(ii) *$(-m)\alpha n = m(-\alpha)n = m\alpha(-n) = -(m\alpha n)$.*

(iii) *$(km)\alpha n = m(k\alpha)n = m\alpha(kn) = k(m\alpha n)$.*

Proof. We show the first part of these equations. Other equations can be verified similarly by using the definition $M\Gamma$-module.

(i) Since $0_M\alpha n = (0_M + 0_M)\,\alpha n = 0_M\alpha n + 0_M\alpha n$, we get $0_N = 0_M\alpha n$ by adding the additive inverse of $0_M\alpha n$ to both sides of the equation.

(ii) Since $m\alpha n + (-m)\alpha n = (m + (-m))\,\alpha n = 0_M\alpha n = 0_N$, we have $(-m)\alpha n = -m\alpha n$.

(iii) $(km)\alpha n = (m + m + \cdots + m)\,\alpha n = m\alpha n + m\alpha n + \cdots + m\alpha n = k(m\alpha n)$.

□

Proposition 3.2. *Let $\varphi : N \to Q$ be a $M\Gamma$-module homomorphism. Then the following conditions hold.*

(i) *φ is injective if and only if $Ker\varphi = (0)$.*

(ii) *φ is a $M\Gamma$-module isomorphism if and only if there is a $M\Gamma$-module homomorphism $\phi : Q \to N$ such that $\phi\varphi = 1_N$ and $\varphi\phi = 1_Q$.*

Proof. (i) Let φ is an injective $M\Gamma$-module homomorphism and $n \in \mathrm{Ker}\varphi$. Then, $\varphi(n) = 0 = \varphi(0)$ since φ is an additive group homomorphism. Thus, $n = 0$ by hypothesis. Conversely, if $\mathrm{Ker}\varphi = (0)$ and $\varphi(n_1) = \varphi(n_2)$, then $n_1 - n_2 \in \mathrm{Ker}\varphi = (0)$ by hypothesis. Hence, φ is injective.

(ii) Let φ be a $M\Gamma$-module isomorphism. We know that φ is bijecive if and only if there is a map $\phi : Q \to N$ such that $\phi\varphi = I_N$ and $\varphi\phi = I_Q$. Therefore, we only show ϕ is a $M\Gamma$-module homomorphism. Then,

$$\begin{aligned}\phi(q_1 + q_2) &= \phi\left(\left(\varphi\phi\right)(q_1) + \left(\varphi\phi\right)(q_2)\right) = \phi\left(\varphi\left(\phi(q_1) + \phi(q_2)\right)\right)\\ &= \left(\phi\varphi\right)\left(\phi(q_1) + \phi(q_2)\right) = \phi(q_1) + \phi(q_2)\end{aligned}$$

and

$$\begin{aligned}\phi\left(m\gamma q\right) &= \phi\left(m\gamma\left(\varphi\phi\right)(q)\right) = \phi\left(m\gamma\varphi\left(\phi(q)\right)\right) = \phi\left(\varphi\left(m\gamma\phi(q)\right)\right)\\ &= \left(\phi\varphi\right)\left(m\gamma\phi(q)\right) = m\gamma\phi(q)\end{aligned}$$

for all $q, q_1, q_2 \in Q$, $m \in M$ and $\gamma \in \Gamma$. The converse is an immediate consequence of the fact that the characterization of bijectivity given above.

□

Proposition 3.3. *The left operator ring L of the $\Gamma-$ring M is a $M\Gamma$-module.*

Proof. We show that the map $M \times \Gamma \times L \to L$ defined by $m\gamma \sum_{i=1}^{t}[m_i, \gamma_i] = \sum_{i=1}^{t}[m\gamma m_i, \gamma_i]$ is well defined. The other conditions can be shown easily. Let $m = m'$, $\gamma = \gamma'$, $\sum_{i=1}^{t}[m_i, \gamma_i] = \sum_{j=1}^{k}[u_j, \beta_j]$. Then,

$$\begin{aligned}\sum_{i=1}^{t}[m\gamma m_i, \gamma_i]x &= \sum_{i=1}^{t} m\gamma m_i\gamma_i x = m\gamma \sum_{i=1}^{t} m_i\gamma_i x\\ &= m'\gamma' \sum_{j=1}^{k} u_j\beta_j x = \sum_{j=1}^{k} m'\gamma' u_j\beta_j x\\ &= \sum_{j=1}^{k}[m'\gamma' u_j, \beta_j]x\end{aligned}$$

for all $x \in M$. This shows that the mapping $M \times \Gamma \times L \to L$ is well-defined. □

Proposition 3.4. *Let N be a $M\Gamma$-module, X be any subset of N and $n \in N$.*

(i) $M\Gamma n := \left\{\sum_{i=1}^{t} m_i\gamma_i n \mid t \in \mathbb{Z}^+,\ m_i \in M,\ \gamma_i \in \Gamma\right\}$ *is a $M\Gamma$-submodule of N.*

(ii) *The cyclic $M\Gamma$-submodule generated by n equals to*

$$\left\{kn + \sum_{i=1}^{t} m_i\gamma_i n \mid t \in \mathbb{Z}^+,\ k \in \mathbb{Z},\ m_i \in M,\ \gamma_i \in \Gamma\right\}.$$

If N is unitary $M\Gamma$-module, then $(n) = M\Gamma n$.

(iii) *The $M\Gamma$-submodule generated by X is*

$$\left\{ \sum_{i=1}^{s} k_i x_i + \sum_{j=1}^{t} m_j \gamma_j x'_j \mid s,t \in \mathbb{Z}^+,\ k_i \in \mathbb{Z},\ x_i, x'_j \in X,\ m_j \in M,\ \gamma_j \in \Gamma \right\}.$$

If N is unitary $M\Gamma$-module, then

$$(X) = M\Gamma X = \left\{ \sum_{i=1}^{s} m_i \gamma_i x_i \mid s \in \mathbb{Z}^+,\ x_i \in X,\ m_i \in M,\ \gamma_i \in \Gamma \right\}.$$

Proof. (i) Let $\sum\limits_i m_i\gamma_i n,\ \sum\limits_j m_j^{'}\gamma_j^{'} n \in M\Gamma n$. Then, one can easily show that $\sum\limits_i m_i\gamma_i n - \sum\limits_j m_j^{'}\gamma_j^{'} n \in M\Gamma n$ and $m\gamma \sum\limits_i m_i\gamma_i n = \sum\limits_i (m\gamma m_i)\gamma_i n \in M\Gamma n$ for all $m \in M$, $\gamma \in \Gamma$. Thus, $M\Gamma n$ is a $M\Gamma$-submodule of N.

(ii) It is clear that the subset

$$K := \left\{ kn + \sum_{i=1}^{t} m_i\gamma_i n \mid t \in \mathbb{Z}^+,\ k \in \mathbb{Z},\ m_i \in M,\ \gamma_i \in \Gamma \right\}$$

of N is a $M\Gamma$-submodule and $n \in K$. Thus, (n) is a subset of K. Conversely, since (n) is a $M\Gamma$-submodule of N, it is obvious that K is a subset of (n).

Let the Γ-ring M has a strong unity (δ, e). If N is a unitary $M\Gamma$-module, then $n = e\delta n$ and

$$kn + \sum_{i=1}^{t} m_i\gamma_i n = k(e\delta n) + \sum_{i=1}^{t} m_i\gamma_i n.$$

Hence, we have $(n) \subseteq M\Gamma n$.

(iii) It is easy to see that the set X is a subset of

$$S := \left\{ \sum_{i=1}^{s} k_i x_i + \sum_{j=1}^{t} m_j \gamma_j x'_j \mid s,t \in \mathbb{Z}^+,\ k_i \in \mathbb{Z},\ x_i, x'_j \in X,\ m_j \in M,\ \gamma_j \in \Gamma \right\}$$

and $S \subseteq N$. Let us show that S is a $M\Gamma$-submodule of N. Clearly, S is an abelian subgroup of N. Let $m \in M$, $\gamma \in \Gamma$ and $\sum\limits_{i=1}^{s} k_i x_i + \sum\limits_{j=1}^{t} m_j\gamma_j x'_j \in S$. Since

$$m\gamma \left(\sum_{i=1}^{s} k_i x_i + \sum_{j=1}^{t} m_j \gamma_j x'_j \right) = \sum_{i=1}^{s} k_i m\gamma x_i + \sum_{j=1}^{t} (m\gamma m_j)\gamma_j x_j \in S,$$

S is a $M\Gamma$-submodule of N. Therefore, $(X) \subseteq S$. Conversely, any element $\sum\limits_{i=1}^{s} k_i x_i + \sum\limits_{j=1}^{t} m_j\gamma_j x'_j$ of S is contained in (X) since $x_i \in X \subseteq (X)$ and (X) is a $M\Gamma$-submodule of N. Consequently, we have $S = (X)$.

Finally, let (δ, e) be a strong unity of the Γ-ring M and N be a unitary $M\Gamma$-module. Then, we get $M\Gamma X \subseteq (X)$. Since $x_i = e\delta x_i$ for all $x_i \in X$, we have

$$\sum_{i=1}^{s} k_i x_i + \sum_{j=1}^{t} m_j \gamma_j x'_j = \sum_{i=1}^{s} k_i (e\delta x_i) + \sum_{j=1}^{t} m_j \gamma_j x'_j.$$

This shows that $(X) \subseteq M\Gamma X$. As a result, we obtain $(X) = M\Gamma X$. □

Theorem 3.1. *Let $\{N_i \mid i \in I\}$ be a family of $M\Gamma$-modules.*

(i) *The direct product $\prod_{i\in I} N_i$ is a $M\Gamma$-module.*

(ii) *Direct sum $\sum_{i\in I} N_i$ is a $M\Gamma$-submodule of direct product $\prod_{i\in I} N_i$.*

(iii) *For each $k \in I$, the canonical projections $\pi_k : \prod_{i\in I} N_i \to N_k$ is a $M\Gamma$-module epimorphism.*

(iv) *For each $k \in I$, the canonical injections $\iota_k : N_k \to \sum_{i\in I} N_i$ is a $M\Gamma$-module monomorphism.*

Proof. (i) The direct product $\prod_{i\in I} N_i$ is an abelian group by [4]. Furthermore, it can be shown that the mapping $M \times \Gamma \times \prod_{i\in I} N_i \to \prod_{i\in I} N_i$ given by $m\gamma\{a_i\} = \{m\gamma a_i\}$ satisfies the $M\Gamma$-module conditions. Therefore, $\prod_{i\in I} N_i$ is a $M\Gamma$-module.

(ii) The direct sum $\sum_{i\in I} N_i$ is a subgroup of direct product $\prod_{i\in I} N_i$ by [4]. Let $m \in M$, $\gamma \in \Gamma$ and $\{a_i\} \in \sum_{i\in I} N_i$. Here $a_i = 0$ for all but a finite number of $i \in I$. Then, we have $m\gamma\{a_i\} = \{m\gamma a_i\} = 0$ for all but a finite number of $i \in I$. Hence, $m\gamma\{a_i\} \in \sum_{i\in I} N_i$. So the direct sum $\sum_{i\in I} N_i$ is a $M\Gamma$-submodule of the direct product $\prod_{i\in I} N_i$.

(iii) For each $k \in I$, the map $\pi_k : \prod_{i\in I} N_i \to N_k$ given by $\{a_i\} \mapsto a_k$ is an epimorphism of groups by [4]. Also, since

$$\pi_k (m\gamma \{a_i\}) = \pi_k (\{m\gamma a_i\}) = m\gamma a_k = m\gamma\pi_k (\{a_i\})$$

for every $m \in M$ and $\gamma \in \Gamma$, the map π_k is an $M\Gamma$-module homomorphism for each $k \in I$.

(iv) The map $\iota_k : N_k \to \sum_{i\in I} N_i$ given by $a \mapsto \{a_i\}_k := \begin{cases} a, & i = k \\ 0, & i \neq k \end{cases}$ is a monomorphism of groups for each $k \in I$ by [4]. Also, for each $m \in M$, $\gamma \in \Gamma$ since $\iota_k (m\gamma a) = \{(m\gamma a)_i\}_k = m\gamma\iota_k \{a_i\}_k = m\gamma\iota_k (a)$, the map ι_k becomes a $M\Gamma$-module homomorphism. □

Before giving the equivalent conditions of a $M\Gamma$-module N to be free, we give a lemma that can be easily proven by the definition of unitary $M\Gamma$-module.

Lemma 3.1. *Let M be a Γ-ring with a strong unity and N be a unitary $M\Gamma$-module. If $m\gamma M = (0)$ for each $m \in M$ and $\gamma \in \Gamma$, then $m\gamma N = (0)$.*

Proof. Let (δ, e) be any strong unity of the Γ-ring M and $m\gamma M = (0)$. Then, $m\gamma e = 0$ since $e \in M$. Therefore, $0 = (m\gamma e)\delta n = m\gamma(e\delta n) = m\gamma n$ for all $n \in N$. Hence, we get $m\gamma N = (0)$. □

Theorem 3.2. *Let M be a Γ-ring with a strong unity, L be the left operator ring of M and F be a unitary $M\Gamma$-module. Then, the following conditions are equivalent:*

(i) *F has a nonempty basis.*

(ii) *F can be written as the internal direct sum of its cyclic $M\Gamma$-submodules, each of which is isomorphic to $M\Gamma$-module L.*

(iii) *F is $M\Gamma$-module isomorphic to a direct sum of copies of the left $M\Gamma$-module L.*

(iv) *There exists a nonempty set X and a function $i : X \to F$ with the following property: given any unitary $M\Gamma$-module N and function $f : X \to N$, there exists a unique $M\Gamma$-module homomorphism $\bar{f} : F \to N$ such that $\bar{f}i = f$.*

Proof. $(i) \Rightarrow (ii)$ Let X be a nonempty basis of $M\Gamma$-module F and $x \in X$. By Proposition 3.3 and Proposition 3.4 (i), L and $M\Gamma x$ are both $M\Gamma$-modules. The map $f : L \to M\Gamma x$, given by $\sum_i [m_i, \gamma_i] \mapsto \sum_i m_i\gamma_i x$ is a $M\Gamma$-module epimorphism. Moreover, if $\sum_{i=1}^{t} [m_i, \gamma_i] \in \mathrm{Ker} f$, then $\sum_{i=1}^{t} m_i\gamma_i x = 0$. Since $\{x\} \subset X$ is linearly independent, the equation $m_i\gamma_i M = (0)$ is satisfied, thus $[m_i, \gamma_i] = 0$ from the definition of the left operator ring L for each i. Hence, $\sum_{i=1}^{t} [m_i, \gamma_i] = 0$ is provided and the map is a monomorphism. Consequently, $M\Gamma x \simeq L$ as $M\Gamma$-module.

Now, since X spans the $M\Gamma$-module F, every $u \in F$ can be written such that $u = m_1\gamma_1 x_1 + \cdots + m_n\gamma_n x_n$ for $m_i \in M$, $\gamma_i \in \Gamma$ ve $x_i \in X$. Therefore, the $M\Gamma$-module F is written as the sum of the family $\{M\Gamma x \mid x \in X\}$. Also, let $v \in \left(\sum_{x\in X} M\Gamma x\right) \bigcap M\Gamma x^*$ for $x \neq x^*$. So, there exist $m_i, k_i \in M$, $\gamma_i, \beta_i \in \Gamma$ such that $v = \sum_{x\in X} \left(\sum_i m_i\gamma_i x\right) = \sum_i k_i\beta_i x^*$. In this case,

$$\sum_i m_i\gamma_i x_1 + \cdots + \sum_i m_i\gamma_i x_n - \sum_i k_i\beta_i x^* = 0$$

is obtained. This equation can be written as

$$\left(\sum_i m_i\gamma_i e\right) \delta x_1 + \cdots + \left(\sum_i m_i\gamma_i e\right) \delta x_n - \left(\sum_i k_i\beta_i e\right) \delta x^* = 0$$

with a strong unity (δ, e) of the Γ-ring M. Since X is a linearly independent set, the equations $\left(\sum_i m_i\gamma_i e\right) \delta M = \left(\sum_i k_i\beta_i e\right) \delta M = (0)$ and $\sum_i k_i\beta_i M = (0)$ are provided. Then by Lemma 3.1, $\sum_i k_i\beta_i F = (0)$ is obtained. Since $x^* \in F$, we have $v = \sum_i k_i\beta_i x^* = 0$. Hence, $\left(\sum_{x\in X} M\Gamma x\right) \bigcap M\Gamma x^* = (0)$. This implies that $F = \sum_{x\in X} M\Gamma x$.

$(ii) \Rightarrow (iii)$ Assume that F is the internal direct sum of the family of submodules $\{M\Gamma x \mid x \in X\}$. Thus, for each $x \in X$, $F \simeq \sum_{x\in X} M\Gamma x$ and $M\Gamma x \simeq L$. Therefore, F becomes isomorphic to a direct sum of $M\Gamma$-modules L.

$(iii) \Rightarrow (i)$ Suppose that (δ, e) is a strong unity of the Γ-ring M and $F \simeq \sum_{x\in X} L$ as $M\Gamma$-module. Take an element θ_{x_i} of the $M\Gamma$-module $\sum_{x\in X} L$ where the x_i-th component is $[e, \delta]$ and the other components are $0_L = [0_M, \delta]$. If $\sum_{i=1}^{k} m_i\beta_i\theta_{x_i} = 0$ then $m_i\beta_i e = 0$ since $[0_M, \delta] = m_i\beta_i[e, \delta] = [m_i\beta_i e, \delta]$ for each i. Therefore, $m_i\beta_i M = 0$ equality is provided for each i. Thus, the subset $\{\theta_{x_i} \mid x_i \in X\}$ of the $M\Gamma$-module L is linearly independent. In addition, since

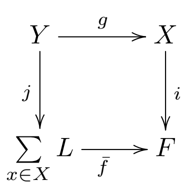
$$\left\{\left(\sum_j [m_j, \gamma_j]\right)_i\right\} = \sum_i \left(\sum_j [m_j, \gamma_j]\right) \theta_{x_i}$$

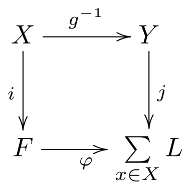
for every $\left\{\left(\sum_j [m_j, \gamma_j]\right)_i\right\} \in \sum_{x\in X} L$, $\sum_{x\in X} L$ is spanned by subset $\{\theta_{x_i} \mid x_i \in X\}$. Thus the set $\{\theta_{x_i} \mid x_i \in X\}$ is a basis of $\sum_{x\in X} L$. As a result, since $F \simeq \sum_{x\in X} L$, $M\Gamma$-module F also has a basis.

$(i) \Rightarrow (iv)$ Let X be a basis of F and $i : X \to F$ be the inclusion map. Suppose we are given a map $f : X \to N$. Since the set X is a basis of F, each element u of F can be written as $u = \sum_i m_i \gamma_i a_i$ for $m_i \in M$, $\gamma_i \in \Gamma$ and $a_i \in X$. Since the set X is linearly independent, the map $\bar{f} : F \to N$ given by $\bar{f}(u) = \bar{f}(\sum_i m_i \gamma_i a_i) = \sum_i m_i \gamma_i f(a_i)$ is well-defined and $M\Gamma$-module homomorphism such that $\bar{f}i = f$. Finally, we must show that $\bar{f}$ is unique. Suppose $g : F \to N$ is an $M\Gamma$-module homomorphism such that $gi = f$. Since $g(a) = g(i(a)) = f(a)$ for every $a \in X$, one can show that $g = \bar{f}$.

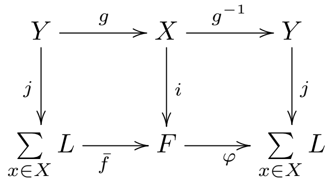
$(iv) \Rightarrow (iii)$ Suppose that (iv) is provided. The set $Y = \{\theta_{x_i} \mid x_i \in X\}$ is a basis of the $M\Gamma$-module $\sum_{x \in X} L$ by the proof of (iii)⇒(i). By hypothesis, there is unique $M\Gamma$-module homomorphism $\bar{f} : \sum_{x \in X} L, \to F$ for the function $j : Y \to \sum_{x \in X} L$ and the strong unitary $M\Gamma$-module F such that $\bar{f}j = f$.

Now, let us show that $\bar{f}$ is a $M\Gamma$-module isomorphism. Since $|X| = |Y|$, there is a bijection map $g : Y \to X$. Consider the composition map $ig : Y \to F$ for the inclusion map $i : X \to F$. By the proof of $(iii) \Rightarrow (i) \Rightarrow (iv)$, the $M\Gamma$-module $\sum_{x \in X} L$ satisfies the hypothesis. Thus the diagrams given below are commutative:





Denoting identity map defined on any set A by 1_A and combining these two diagrams, we get the diagram given below.



Therefore, we obtain $(\varphi\bar{f})j = j(g^{-1}g) = j1_Y = j$. Thus by $1_{\sum_{x \in X} L} j = j$ and the uniqueness property of the $\bar{f}$, we have $\varphi\bar{f} = 1_{\sum_{x \in X} L}$. Similarly, one can show that $\bar{f}\varphi = 1_F$. Therefore, F is isomorphic to $\sum_{x \in X} L$. □

Corollary 3.1. *Let M be a Γ-ring with a strong unity. Then, every unitary $M\Gamma$-module N is a homomorphic image of a free $M\Gamma$-module F. If N is finitely generated, then F can be chosen to be finitely generated.*

Proof. Let X be a set of generators of N and F be the free $M\Gamma$-module on the set X. Then the inclusion map $i : X \to N$ induces a $M\Gamma$-module homomorphism $\bar{f} : F \to N$ such that $X \subseteq \mathrm{Im}\bar{f}$ from the Theorem 3.2 (iv). Since X generates N, there exist $m_i \in M$ and $\gamma_i \in \Gamma$ such that $n = \sum_i m_i \gamma_i x_i$ for every $n \in N$. Hence, we get $\mathrm{Im}\bar{f} = N$ since $\bar{f}$ is a $M\Gamma$-module homomorphism. □

4. **Conclusion**

In this study, the free gamma module structure is considered. The definition of a free gamma module has been given in the literature, but some deficiencies/errors have been detected in the definitions of linear independence and generating sets. According to their definitions, an example of a free gamma module cannot be given or a set can be both linearly dependent and linearly independent. To overcome these problems, in this paper, we redefine linear independence, generator set and some basic concepts related to gamma modules and then give a characterization of a free gamma module. In obtaining this characterization, we consider the unitary gamma module defined on a gamma ring with strong unity, which have not been considered before in the literature.

REFERENCES

[1] *M. Abbas, H. Hassan and H. Abbas*, On ΓR-free gamma modules, International Mathematical Forum, **13** (2018), 99–106.

[2] *R. Ameri and R. Sadeghi*, Gamma modules, Ratio Mathematica, **20** (2010), No. 1, 127–147.

[3] *W. Barnes*, On the Γ-rings of Nobusawa, Pacific Journal of Mathematics, **18** (1966), No. 3, 411–422.

[4] *T. W. Hungerford*, Algebra, Springer, New York, 2012.

[5] *S. Kyuno*, Gamma rings, Hadronic Press, Palm Habor, 1991.

[6] *S. Kyuno*, On the radicals of Gamma-rings, Osaka J. Math., **12** (1975), No. 1, 639–645.

[7] *S. Kyuno*, On the semi-simple gamma rings, Tohoku Mathematical Journal, **29** (1977), No. 2, 217–225.

[8] *S. Kyuno*, On prime gamma rings, Pacific Journal of Mathematics, **75** (1978), No. 1, 185–190.

[9] *J. Luh*, On the theory of simple Gamma-rings, Michigan Mathematical Journal, **16** (1969), No. 1, 65–75.

[10] *N. Nobusawa*, On a generalization of the ring theory, Osaka J. Math., **1** (1964), 81–89.

[11] *M. S. Uddin and A. C. Paul*, Free ΓM-modules, Asian Journal of Information Technology, **5** (2006), No. 11, 1262–1268.