MITTAG-LEFFLER-HYERS-ULAM-RASSIAS STABILITY OF DETERMINISTIC SEMILINEAR FRACTIONAL VOLTERRA INTEGRAL EQUATION AND OF STOCHASTIC SYSTEMS BY BROWNIAN MOTION

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Abstract. In this paper, we define and investigate Mittag-Leffler-Hyers-Ulam and Mittag-Leffler-Hyers-Ulam-Rassias stability of deterministic semilinear fractional Volterra integral equation. Also, we prove that this equation is stable with respect to the Chebyshev and Bielecki norms. The stability of stochastic systems driven by Brownian motion has also been studied.

Keywords: Mittag-Leffler-Hyers-Ulam stability; Mittag-Leffler-Hyers-Ulam-Rassias stability; deterministic Volterra integral equation; Chebyshev norm; Bielecki norm; Asymptotic stability.

1. **Introduction**

The stability theory for functional equations started with a problem related to the stability of group homomorphism that was considered by Ulam in 1940 ([14]). The first answer to the question of Ulam was given by Hyers in 1941 in the case of Banach spaces in [4]. Thereafter, this type of stability is called the Hyers-Ulam stability. In 1978, Th. M. Rassias [11] generalized the Hyers Theorem by considering the stability problem with unbounded Cauchy differences. In fact, he introduced a new type of stability which is called the Hyers-Ulam-Rassias stability.

Alsina and Ger were the first authors who investigated the Hyers-Ulam stability of a differential equation [1]. Recently some authors ([7], [5], [6], [12], [15] and [16]) extended the Ulam stability problem from an integer-order differential equation to a fractional-order differential equation.

Integral equations of various types play an important role in many branches of functional analysis and its applications; for example in physics, economics and other fields. Also, the fractional differential equations are useful tools in the modelling of many physical phenomena and processes in economics, chemistry, aerodynamics, etc. (We refer the reader to [8, 9, 10, 13] for more details). There are different types of fractional integral equations. In

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[2], the authors by defining all types of Mittag-Leffler-Hyers-Ulam stability of a fractional integral equation proved that every mapping of this type can be somehow approximated by an exact solution of the considered equation.

In this paper we present similar definitions to that of [2] and prove the stability results for the following deterministic semilinear fractional Volterra integral equation

$$X(t) = \xi_t + \tfrac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}(AX(s) + h(X(s)))ds, \ t \geq 0, \beta \in (0,1), A < 0.$$

The functions $h : \mathbb{R} \longrightarrow \mathbb{R}$ and $\xi_t : \mathbb{R}^+ \longrightarrow \mathbb{R}$ are both measurable.

2. Mittag-Leffler-Hyers-Ulam stability

In this section, we will study Mittag-Leffler-Hyers-Ulam stability of the following deterministic semilinear fractional Volterra integral equation

$$X(t) = \xi_t + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}(AX(s) + h(X(s)))ds, \quad t \geq 0, \tag{1}$$

where $\beta \in (0,1), A < 0$. The functions $h : \mathbb{R} \longrightarrow \mathbb{R}$ and $\xi_t : \mathbb{R}^+ \longrightarrow \mathbb{R}$ are both measurable.

Definition 2.1. *equation* (1) *is Mittag-Leffler-Hyers-Ulam stable if there exists a real number $c > 0$ such that, for each $\varepsilon > 0$ and for each solution $X(t)$ of the inequality*

$$|X(t) - \xi_t - \tfrac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}(AX(s) + h(X(s)))ds| \leq \varepsilon E_\beta(t^\beta),$$

there exists a unique solution $X_0(t)$ of equation (1) *satisfying the following inequality:*

$$|X(t) - X_0(t)| \leq c\varepsilon E_\beta(t^\beta).$$

Before the main Theorem we have following Theorem:

Theorem 2.1. *Let (X, d) be a generalized complete metric space. Assume that $\Lambda : X \longrightarrow X$ is a strictly contractive operator with the Lipschitz constant $L < 1$. If there exists a nonnegative integer k such that $d(\Lambda^{k+1}x, \Lambda^k x) < \infty$ for some $x \in X$, then:*

(*a*) *The sequence $\Lambda^n x$ convergence to a fixed point x^* of Λ;*

(*b*) *x^* is the unique fixed point of Λ in*

$$X^* = \{y \in X | d(\Lambda^k x, y) < \infty\};$$

(*c*) *If $y \in X^*$, then*

$$d(y, x^*) \leq \frac{1}{1-L} d(\Lambda y, y).$$

Theorem 2.2. *Suppose that $\beta \in (0,1), A < 0$ and $h : \mathbb{R} \longrightarrow \mathbb{R}, \xi_t : \mathbb{R}^+ \longrightarrow \mathbb{R}$ are two measurable functions and L be a positive constant with $0 < |A| + L < 1$, such that for function h we have $|h(X_1(s)) - h(X_2(s))| \leq L|X_1(s) - X_2(s)|$, for $s \in [0,t]$ and for each $\varepsilon > 0$*

$$|X(t) - \xi_t - \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}(AX(s) + h(X(s)))ds| \leq \varepsilon E_\beta(t^\beta), \tag{2}$$

then the equation (1) *is Mittag-Leffler-Hyers-Ulam stable.*

Proof. Let us consider the space of continuous functions

$$Z = \{X : [a,b] \longrightarrow \mathbb{R} \mid X \quad is \quad continuous\}.$$

For defining the generalized metric, we apply the similar definition of theorem 3.1 of [7]:

$$d(X_1, X_2) = \inf\{K \in [0, \infty] \mid |X_1(t) - X_2(t)| \leq K\varepsilon E_\beta(t^\beta), \quad t \in [a, b]\}, \tag{3}$$

for $\varepsilon > 0$, it is known that (Z, d) is a generalized complete metric space, so for all $X \in Z$ and $t \in [a, b]$ we define an operator $\Lambda : Z \longrightarrow Z$ by

$$(\Lambda X)(t) = \xi_t + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}(AX(s) + h(X(s)))ds, \tag{4}$$

so ΛX is continuous and this ensures that Λ is a well defined operator. Now from the definition of Λ in (4) we have

$|(\Lambda X_1)(t) - (\Lambda X_2)(t)| \leq \frac{(|A|+L)}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}|X_1(s) - X_2(s)|ds$

$\leq (|A| + L)K\varepsilon \sum_{n=0}^{\infty} \frac{t^{n\beta}}{\Gamma(n\beta+1)} = (|A| + L)K\varepsilon E_\beta(t^\beta),$

for all $t \in [a, b]$; that is $d(\Lambda X_1, \Lambda X_2) \leq (|A| + L)K\varepsilon E_\beta(t^\beta)$. Hence, we can conclude that $d(\Lambda X_1, \Lambda X_2) \leq (|A| + L)d(X_1, X_2)$ for any $X_1, X_2 \in Z$, and since $0 < (|A| + L) < 1$ the strictly continuous property is verified.

Let us take $Y_0 \in Z$. By continuity of Y_0 and ΛY_0, it follows that there exists a constant $0 < K_1 < \infty$ such that

$$|(\Lambda Y_0)(t) - Y_0(t)| = |\xi_t + \tfrac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}(AY_0(s) + h(Y_0(s)))ds - Y_0(t)| \leq K_1 E_\beta(t^\beta),$$

for all $t \in [a, b]$, since Y_0 is bounded on $[a, b]$ and $min_{t\in[a,b]} E_\beta(t^\beta) > 0$, thus, (3) implies that $d(\Lambda Y_0, Y_0) < \infty$. Therefore, according to theorem 2.1, there exists a continuous function $X_0 : [a, b] \longrightarrow \mathbb{R}$ such that $\Lambda^n Y_0 \longrightarrow X_0$ in (Z, d) as $n \to \infty$ and $\Lambda X_0 = X_0$; that is, X_0 satisfies the equation (1) for every $t \in [a, b]$. We will now prove that

$$\{X \in Z | d(Y_0, X) < \infty\} = Z.$$

For any $X \in Z$, since X and Y_0 are bounded in $[a, b]$ and $\min_{t\in[a,b]} E_\beta(t^\beta) > 0$, so there exists a constant $0 < C_X < \infty$ such that

$$|Y_0(t) - X(t)| \leq C_X E_\beta(t^\beta),$$

for any $t \in [a, b]$. Hence, we have $d(Y_0, X) < \infty$ for all $X \in Z$; that is, $\{X \in Z | d(Y_0, X) < \infty\} = Z$. Hence, in view of theorem 2.1, we conclude that X_0 is the unique continuous function which satisfies the equation (1). On the other hand, from (2) it follows that $d(X, \Lambda X) \leq \varepsilon E_\beta(t^\beta)$. Finally, theorem 2.1 together with the above inequality imply that

$$d(X, X_0) \leq \tfrac{1}{1-(|A|+L)} d(\Lambda X, X) \leq \tfrac{1}{1-(|A|+L)} \varepsilon E_\beta(t^\beta).$$

This means that the equation (1) is Mittag-Leffler-Hyers-Ulam stable. □

Example 2.1. *Let $\xi_t = sin(\frac{1}{t})$, $\beta = \frac{1}{2}$, $A = -\frac{1}{3}$ and $L = \frac{1}{3}$. Given a polynomial $p_x(s)$, we assume $h(X(s)) = p_x(s) + L[X(s) + X(sins)]$, that satisfies*

$$|X(t) - sin(\tfrac{1}{t}) - \tfrac{1}{\sqrt{\pi}} \int_0^t (t-s)^{-\frac{1}{2}}(-\tfrac{1}{3}X(s) + p_x(s) + \tfrac{1}{3}[X(s) + X(sins)])ds$$
$$\leq \varepsilon E_{\frac{1}{2}}(t^{\frac{1}{2}}).$$

According theorem 2.2, there exists a unique $X_0(t)$ such that

$$X_0(t) = sin(\frac{1}{t}) + \frac{1}{\sqrt{\pi}} \int_0^t (t-s)^{-\frac{1}{2}} (-\frac{1}{3}X(s) + p_x(s) + \frac{1}{3}[X(s) + X(sins)])ds,$$

and

$$|X(t) - X_0(t)| \leq \frac{1}{1-(|-\frac{1}{3}|+\frac{1}{3})} \varepsilon E_{\frac{1}{2}}(t^{\frac{1}{2}}) = 3\varepsilon E_{\frac{1}{2}}(t^{\frac{1}{2}}).$$

Next, we use the Chebyshev norm $\| \, . \, \|_c$ to obtain the above similar result for equation (1).

Theorem 2.3. *Suppose that* $\beta \in (0,1), A < 0$ *and* $h : \mathbb{R} \longrightarrow \mathbb{R}, \xi_t : \mathbb{R}^+ \longrightarrow \mathbb{R}$ *are two measurable functions and* L *be a positive constant with*

$$0 < (|A| + L)E_\beta(t) < 1,$$

such that for function h *we have* $|h(X_1(s)) - h(X_2(s))| \leq L|X_1(s) - X_2(s)|$ *for* $s \in [0,t]$. *Then the equation* (1) *is Mittag-Leffler-Hyers-Ulam stable via the Chebyshev norm.*

Proof. Just like the discussion in theorem 3.1, we prove that Λ defined in (4) is a contraction map on Z with respect to the Chebyshev norm. We have:

$$|(\Lambda X_1)(t) - (\Lambda X_2)(t)| \leq \frac{(|A|+L)}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} |X_1(s) - X_2(s)| ds$$

$$= (|A| + L) \parallel X_1 - X_2 \parallel_c \frac{t^\beta}{\Gamma(\beta+1)} \leq (|A| + L) \parallel X_1 - X_2 \parallel_c .E_\beta(t),$$

for all $t \in [a,b]$. Hence, we can conclude that

$$d(\Lambda X_1, \Lambda X_2) \leq (|A| + L)E_\beta(t) d(X_1, X_2),$$

for any $X_1, X_2 \in Z$. Now, since $0 < (|A| + L)E_\beta(t) < 1$; the strictly continuous property is verified. By a similar argument to that of theorem 2.2 we have

$$d(X, X_0) \leq \frac{1}{1-(|A|+L)E_\beta(t)} d(\Lambda X, X) \leq \frac{1}{1-(|A|+L)E_\beta(t)} \varepsilon E_\beta(t^\beta),$$

which means that equation (1) is Mittag-Leffler-Hyers-Ulam stable via the Chebyshev norm. □

In the following theorem we have used the Bielecki norm

$$\parallel g \parallel_B := \max_{t\in[a,b]} |g(t)| e^{-\theta t} \ , \ \theta > 0, \ a, b \in \mathbb{R};$$

to obtain the similar theorem 3.1 for the fundamental equation (1) via the Bielecki norm.

Theorem 2.4. *Suppose that* $\beta \in (0,1), A < 0$ *and* $h : \mathbb{R} \longrightarrow \mathbb{R}, \ \xi_t : \mathbb{R}^+ \longrightarrow \mathbb{R}$ *are two measurable functions and* L *be a positive constant with*

$$0 < \frac{(|A|+L).t^\beta.e^{\theta t}}{\Gamma(\beta).\sqrt{2(2\beta-1)\theta}} < 1,$$

such that for function h *we have* $|h(X_1(s)) - h(X_2(s))| \leq L|X_1(s) - X_2(s)|$ *for* $s \in [0,t]$. *Then the equation* (1) *is Mittag-Leffler-Hyers-Ulam stable via the Bielecki norm.*

Proof. Just like the discussion in theorem 2.2, we prove that equation (1) is Mittag-Leffler-Hyers-Ulam stable via the Bielecki norm. □

3. Mittag-Leffler-Hyers-Ulam-Rassias stability

At first we introduce the concept of Mittag-Leffler-Hyers-Ulam-Rassias stability and then we prove that the equation (1) has the Mittag-Leffler-Hyers-Ulam-Rassias stability.

Definition 3.1. *Equation* (1) *is Mittag-Leffler-Hyers-Ulam-Rassias stable if there exists a real number* $c > 0$ *such that for each* $\varepsilon > 0$ *and for each solution* X *of the inequality*

$$|X(t) - \xi_t - \tfrac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}(AX(s) + h(X(s)))ds| \leq \varepsilon\varphi(t)E_\beta(t^\beta),$$

there exists a unique solution X_0 *of equation* (1) *satisfying the following inequality:*

$$|X(t) - X_0(t)| \leq c\varepsilon\varphi(t)E_\beta(t^\beta),$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ *is a continuous function.*

Theorem 3.1. *Suppose that* $\beta \in (0, 1), A < 0$ *and* $h : \mathbb{R} \longrightarrow \mathbb{R}$, $\xi_t : \mathbb{R}^+ \longrightarrow \mathbb{R}$ *are two measurable functions. Set* $M = \frac{1}{\Gamma(\beta)}\left(\frac{1-p}{\beta-p}\right)^{1-p} t^{\beta-p}$ *with* $0 < p < \beta$ *and let* L, B *be positive constants with* $0 < (|A|+L)MB < 1$ *such that for function* h *we have* $|h(X_1(s))-h(X_2(s))| \leq L|X_1(s) - X_2(s)|$ *for* $s \in [0, t]$ *and for each* $\varepsilon > 0$ *we have*

$$|X(t) - \xi_t - \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}(AX(s) + h(X(s)))ds| \leq \varepsilon\varphi(t)E_\beta(t^\beta), \tag{5}$$

where $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ *is a* $L^{\frac{1}{p}}$*-integrable function such that for all* $t \in \mathbb{R}$ *satisfies*

$$(\textstyle\int_0^t (\varphi(s))^{\frac{1}{p}} ds)^p \leq B\varphi(t).$$

Then the equation (1) *is Mittag-Leffler-Hyers-Ulam-Rassias stable.*

Proof. Let us consider the space of continuous functions Z like in theorem 2.2, endowed with the generalized metric defined by

$$d(X_1, X_2) = \inf\{K \in [0, \infty] \mid |X_1(t) - X_2(t)| \leq K\varepsilon\varphi(t), t \in [a, b]\}, \tag{6}$$

for $\varepsilon > 0$. It is known that (Z, d) is a generalized complete metric space. We define function Λ just like (4), so we have

$$|(\Lambda X_1)(t) - (\Lambda X_2)(t)| \leq \tfrac{(|A|+L)K\varepsilon}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}\varphi(s)ds$$

$$\leq \tfrac{(|A|+L)K\varepsilon}{\Gamma(\beta)} \left(\tfrac{1-p}{\beta-p}\right)^{1-p} t^{\beta-p}.B\varphi(t) = (|A| + L)MBK\varepsilon\varphi(t),$$

for all $t \in [a, b]$. Hence, we can conclude that $d(\Lambda X_1, \Lambda X_2) \leq (|A|+L)MBd(X_1, X_2)$ for any $X_1, X_1 \in Z$, and since $0 < (|A| + L)MB < 1$, the strictly continuous property is verified. Just like the discussion in theorem 2.2, we have

$$d(X, X_0) \leq \tfrac{1}{1-(|A|+L)MB} d(\Lambda X, X) \leq \tfrac{1}{1-(|A|+L)MB} \varepsilon\varphi(t)E_\beta(t^\beta).$$

This means that the equation (1) is Mittag-Leffler-Hyers-Ulam stable. □

Example 3.1. *Let* $\xi_t = sin(\frac{1}{t}), B = 1, M = \frac{1}{\Gamma(\beta)}(\frac{1-p}{\beta-p})^{1-p} t^{\beta-p}$ *with* $0 < p < \beta$. *Choose* $L > 0$ *such that* $L < \min\{1, M^{-1} - |A|^{-1}\}$. *Given a polynomial* $p_x(s)$, *we assume* $h(X(s)) = p_x(s) + L[X(s) + X(sins)]$ *that satisfies*

$$|X(t) - sin(\tfrac{1}{t}) - \tfrac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}(AX(s) + p_x(s) + L[X(s) + X(sins)])ds|$$
$$\leq \varepsilon e^{-\beta t} E_\beta(t^\beta),$$

for all $t \in [a,b]$ *.If we set* $\varphi(t) = e^{-\beta t}$ *we obtain*

$$\left|\left(\int_0^t (e^{-\beta s})^{\frac{1}{p}} ds\right)^p\right| = \left(\frac{p}{\beta} - \frac{p}{\beta} e^{-\frac{\beta t}{p}}\right)^p \le e^{-\beta t},$$

for all $t \in [a.b]$*. According theorem 3.1 there exists a unique* $X_0(t)$ *such that*
$X_0(t) = sin(\frac{1}{t}) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}(AX(s) + p_x(s) + L[X(s) + X(sins)])ds,$
and

$$|X(t) - X_0(t)| \le \frac{1}{1-(|A|+L)M} \varepsilon e^{-\beta t} E_\beta(t^\beta).$$

Here we introduce the Young integral, which is an integral with respect to Holder continuous functions.

Definition 3.2. *For* $T > 0, \gamma \in (0,1)$*,let* $C_1^\gamma([0,T];\mathbb{R})$ *be the set of* γ*-Holder continuous functions* $g : [0,T] \to \mathbb{R}$ *of one variable such that the seminorm*

$$\| g \|_{\gamma,[0,T]} := sup_{r \neq t} r, t \in [0,t] \frac{| g_t - g_r |}{| t - r |^\gamma},$$

is finite. Also by $\| g \|_{\infty,[0,T]}$ *we denote the supremum norm of* g*.*

Definition 3.3. *The noise is an additive solution for linear stochastic differential equations and has the form* $\int_0 \sigma(s) dW_s^H$*. Here*

$$W_t^H = \int_0^t (t-s)^{H-1/2} dW_s, \ H \in (1/2, 1),$$

where W *is a Brownian motion and* σ *is a deterministic function such that*

$$\int_0^\infty \sigma^2(s) e^{2\lambda s} ds < \infty,$$

for some $\lambda > 0$*.*

Now we have following results about the semilinear fractional Volterra integral equation with additive noise:

$$X(t) = \xi_t + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} AX(s) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} d\theta_s, \ t \ge 0, \tag{7}$$

where the initial condition $\xi = \{\xi_t; t \ge 0\}$ is measurable and bounded on compact sets, $\beta \in (0,1)$, $A \in \mathbb{R}$, $\alpha \in (1,2)$ and $\theta = \{\theta_s, s \ge 0\}$ is a γ- Holder continuous function with $\gamma \in (0,1)$.

The second integral in (7) is a Young one and it is well-defined if $\alpha - 1 + \gamma > 1$, because $s \mapsto (t-s)^{\alpha-1}$ is $(\alpha-1)$- Holder continuous on $[0,T]$.

Corollary 3.1. *Suppose that* $\xi = \{\xi_t; t \ge 0\}$ *is measurable and bounded on compact sets,* $\beta \in (0,1)$*,* $A \in \mathbb{R}$*,* $\alpha \in (1,2)$*,* $h : \mathbb{R} \longrightarrow \mathbb{R}$ *is a measurable function,* $\theta = \{\theta_s, s \ge 0\}$ *is a* γ*-Holder continuous function with* $\gamma \in (0,1)$*. Also suppose that for each* $\varepsilon > 0$

$$|X(t) - \xi_t - \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}(AX(s) + h(X(s))) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} d\theta_s| \le \varepsilon E_\beta(t^\beta).$$

Then the equation (7) *is Mittag-Leffler-Hyers-Ulam stable.*

Corollary 3.2. *With above assumptions if we have*

$$|X(t) - \xi_t - \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}(AX(s) + h(X(s))) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} d\theta_s| \le \varepsilon \varphi(t) E_\beta(t^\beta).$$

Then the equation (7) *is Mittag-Leffler-Hyers-Ulam-Rassias stable.*

4. **Asymptotic stability**

The concept of the asymptotic stability of a solution $X(t)$ of equation (1) was considered in [3]. Now we consider this concept by the following definition:

Definition 4.1. *Let $B(x,r)$ denotes the closed ball centered at x with radius $r, (r > 0)$ the symbol B_r stands the ball $B(0,r)$. The equation (1) is asymptotic stable if for any $\varepsilon > 0$, there exist $T > 0$ and $r > 0$ such that, if $X(t), Y(t) \in B_r$ and $X(t)$, $Y(t)$ are solutions of equation (1), then $|X(t) - Y(t)| \leq \varepsilon$ for all $t \geq T$.*

Theorem 4.1. *Suppose that $\beta \in (0,1), A < 0$ and $h : \mathbb{R} \longrightarrow \mathbb{R}, \xi_t : \mathbb{R}^+ \longrightarrow \mathbb{R}$ are two measurable functions and there exists $0 < L < 1$ such that*

$$|h(X(s)) - h(Y(s))| \leq L|X(s) - Y(s)|, \ 0 < s < t,$$

then the equation (1) is asymptotic stable.

Proof. Put $r = \frac{\beta\Gamma(\beta)}{2(|A|+L).t^\beta}\varepsilon$. Suppose that $X(t)$, $Y(t)$ are solutions of equation (1) such that $X(t), Y(t) \in B_r$, so $|X(t) - Y(t)| < 2r$. Now we have

$$|X(t) - Y(t)| \leq \frac{|A|.2r + L.2r}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} ds = \frac{2r(|A| + L)}{\Gamma(\beta)} . \frac{t^\beta}{\beta} = \varepsilon.$$

$\square$

Example 4.1. *Let $\xi_t = sin(\frac{1}{t})$, $\beta = \frac{1}{2}$, $A = -\frac{1}{3}$ and $L = \frac{1}{3}$. Given a polynomial $p_x(s)$, we assume $h(X(s)) = p_x(s) + L[X(s) + X(sins)]$, that satisfies*

$$|h(X(s)) - h(Y(s))| \leq \tfrac{1}{3}|X(s) - Y(s)|,$$

so for $r = \frac{3}{8}\sqrt{\frac{\pi}{t}}\ \varepsilon$ we have $|X(t) - Y(t)| \leq \varepsilon$.

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