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A METHOD TO COMPARE TWO COMPLEXITY FUNCTIONS USING COMPLEXITY CLASSES

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Complexitatea unui algoritm poate fi exprimată ca o funcție, numită funcție de complexitate. In acest articol studiem compararea a două funcții de complexitate folosind clase de complexitate. Dupa ce definim multimea tuturor funcțiilor de complexitate comparabile cu o funcție dată, prezentăm câteva proprietăți ale acestei mulțimi. Cele mai importante rezultate din articolul nostru sunt cateva criterii suficiente pentru ca două funcții de complexitate să fie comparabile şi câteva criterii suficiente pentru ca două funcții de complexitate să fie incomparabile.

The complexity of an algorithm can be expressed as a function, called complexity function. In this paper we study the comparison of two complexity functions using complexity classes. After defining the set of all complexity functions comparable with a given function, we give some properties of this set. The most important results of our paper are some sufficient criteria for two complexity functions to be comparable and some sufficient criteria for two complexity functions to be incomparable.

Keywords: algorithm, complexity function, complexity class, complexity functions comparison

1. Introduction

Complexity functions are used in various research fields. For example, in [1] complexity functions describe some properties of the dynamic systems, and in [2] complexity functions describe the complexity of the structure of models related to some technical systems. In this paper, complexity functions are used for measuring the complexity of algorithms.

The complexity of an algorithm can be expressed using a complexity function, i.e., a positive real valued function defined on the set of positive integers. In many cases such functions have complicated expressions and using these functions is a difficult task. For this reason, computer scientists often express the complexity of an algorithm using complexity classes, a simpler way of expressing the complexity of an algorithm, but a less exact one. Some basic

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properties of the complexity classes are presented in almost any paper or book that contains some elements of algorithms complexity theory. See, for example [3], [4], [5].

In this paper we study the complexity functions using the elementary theory of functions and sets. Other approaches use more advanced mathematical theories: for example, in [6], the authors use the nonsymmetric Hausdorff distance for studying the complexity functions; in [7] the authors introduce a new quasi-metric on the dual p-complexity space for studying the complexity distances between algorithms. Nevertheless, our approach is powerful enough to help us to obtain several interesting results.

As one can observe, when comparing algorithms, in fact we compare complexity functions, or at least complexity classes. A possible use case of algorithms comparison is when someone wants to develop a very efficient algorithm for solving a given problem, see for example [8]. Another use case is when someone is interested in complexity analysis in heterogeneous systems, see for example [9].

An interesting idea is presented in [4]: the authors only draw an analogy between the comparison of the complexity functions using complexity classes and the comparison of real numbers. Their immediate conclusion was that every two real numbers can be compared, but not every two complexity functions can be compared.

Starting from the results presented in [10], [11], this paper studies the comparison of two complexity functions using complexity classes. After we define the set of all complexity functions comparable with a given function, we give some properties of this set. We also present some interesting properties of the complexity classes. The main contributions of this paper are some sufficient criteria for two complexity functions to be comparable and some sufficient criteria for two complexity functions to be incomparable.

The paper is organized as follows. Section 2 contains the definitions used for the rest of the paper. Section 3 presents some properties of the complexity classes. Section 4 contains the main results of our paper. Finally, in Section 5, we present the conclusions of the paper.

2. Definitions

We will denote by R_+ the set of all positive real numbers and by N_+ the set of all positive integers. We will consider the function $g : N_+ \to R_+$ to be an arbitrary fixed complexity function. Consider the following complexity classes (see [4], [5]):

$$\Theta(g(n)) = \{f : N_+ \to R_+ \mid \exists c_1, c_2 \in R_+, \exists n_0 \in N_+ \textit{ such that } c_1 \cdot g(n) \le f(n) \le c_2 \cdot g(n), \forall n \ge n_0\} \tag{1}$$

$$O(g(n)) = \{f : N_+ \to R_+ \mid \exists c \in R_+, \exists n_0 \in N_+ \textit{ such that } f(n) \le c \cdot g(n), \forall n \ge n_0\} \tag{2}$$

$$\Omega(g(n)) = \{f : N_+ \to R_+ \mid \exists c \in R_+, \exists n_0 \in N_+ \textit{ such that } c \cdot g(n) \le f(n), \forall n \ge n_0\} \tag{3}$$

$$o(g(n)) = \{f : N_+ \to R_+ \mid \forall c \in R_+, \exists n_0 \in N_+ \textit{ such that } f(n) < c \cdot g(n), \forall n \ge n_0\} \tag{4}$$

$$\omega(g(n)) = \{f : N_+ \to R_+ \mid \forall c \in R_+, \exists n_0 \in N_+ \textit{ such that } c \cdot g(n) < f(n), \forall n \ge n_0\} \tag{5}$$

Definition 1. Let $f : N_+ \to R_+$ be a complexity function. The function $f(n)$ is *comparable* with the function $g(n)$ if

$$f(n) \in \Theta(g(n)) \cup O(g(n)) \cup \Omega(g(n)) \cup o(g(n)) \cup \omega(g(n)) \tag{6}$$

We say that the function $f(n)$ is *incomparable* with the function $g(n)$ if $f(n)$ is not comparable with $g(n)$. We denote by $C(g(n))$ the set of all the complexity functions comparable with the function $g(n)$.

Remark 1. We have the following identity:

$$C(g(n)) = \Theta(g(n)) \cup O(g(n)) \cup \Omega(g(n)) \cup o(g(n)) \cup \omega(g(n)) \tag{7}$$

Definition 2. We define the following complexity classes:

$$o\Theta(g(n)) = O(g(n)) \setminus (o(g(n)) \cup \Theta(g(n))) \tag{8}$$

$$\Theta\omega(g(n)) = \Omega(g(n)) \setminus (\Theta(g(n)) \cup \omega(g(n))) \tag{9}$$

3. Some properties of the complexity classes

This section shows some properties of the complexity classes defined in the previous section.

*Proposition 1.*We have the following properties:
a) $\Theta(g(n)) \neq \emptyset$, $O(g(n)) \neq \emptyset$, $\Omega(g(n)) \neq \emptyset$
b) $o(g(n)) \neq \emptyset$, $\omega(g(n)) \neq \emptyset$
c) $o\Theta(g(n)) \neq \emptyset$, $\Theta\omega(g(n)) \neq \emptyset$.

Proof. a) These results follow from the following observations:

$$g(n) \in \Theta(g(n)),\ g(n) \in O(g(n)),\ g(n) \in \Omega(g(n))\,. \tag{10}$$

b) It can be proved, using (4) and (5), that $g(n)/n \in o(g(n))$ and $n \cdot g(n) \in \omega(g(n))$.

c) Let us show that $o\Theta(g(n)) \neq \emptyset$. Consider two complexity functions: $f_1(n) \in \Theta(g(n))$ and $f_2(n) \in o(g(n))$. We define the following complexity function:

$$f : N_+ \to R_+, \quad f(n) = \begin{cases} f_1(n), & n = 2 \cdot k + 1 \\ f_2(n), & n = 2 \cdot k \end{cases} \tag{11}$$

The function $f(n)$ is defined by $f_1(n)$ for n odd number, and by $f_2(n)$ for n even number.

Next, we prove that $f(n) \in o\Theta(g(n))$. We have to show that $f(n) \in O(g(n))$, $f(n) \notin \Theta(g(n))$, and $f(n) \notin o(g(n))$.

From $f_1(n) \in \Theta(g(n))$ we have:

$$\begin{aligned} &\exists c_1^{'}, c_2^{'} \in R_+, \exists n_0^{'} \in N_+ \ \textit{such that} \\ &c_1^{'} \cdot g(n) \le f_1(n) \le c_2^{'} \cdot g(n), \forall n \ge n_0^{'} \end{aligned} \tag{12}$$

From $f_2(n) \in o(g(n))$ we have:

$$\forall c^{''} \in R_+, \exists n_0^{''} \in N_+ \ \textit{such that} \ f_2(n) < c^{''} \cdot g(n), \forall n \ge n_0^{''} \tag{13}$$

From the definition of $f(n)$ it follows that

$$\begin{aligned} &\exists c_1^{'}, c_2^{'} \in R_+, \exists n_0^{'} \in N_+ \ \textit{such that} \\ &c_1^{'} \cdot g(n) \le f(n) \le c_2^{'} \cdot g(n), \forall n \ge n_0^{'}, n = 2 \cdot k + 1 \end{aligned} \tag{14}$$

$$\begin{aligned}&\forall c'' \in R_+, \exists n_0'' \in N_+ \text{ such that}\\ &f(n) < c'' \cdot g(n), \forall n \ge n_0'', n = 2 \cdot k\end{aligned} \tag{15}$$

and consequently

$$\begin{aligned}&\exists c_2' \in R_+, \exists n_0' \in N_+ \text{ such that}\\ &f(n) \le c_2' \cdot g(n), \forall n \ge n_0', n = 2 \cdot k + 1\end{aligned} \tag{16}$$

$$\begin{aligned}&\exists c'' \in R_+, \exists n_0'' \in N_+ \text{ such that}\\ &f(n) \le c'' \cdot g(n), \forall n \ge n_0'', n = 2 \cdot k\end{aligned} \tag{17}$$

Let be $c = \max\{c_2', c''\}$ and let be $n_0 = \max\{n_0', n_0''\}$. It follows that

$$f(n) \le c \cdot g(n), \forall n \ge n_0 \tag{18}$$

so we have $f(n) \in O(g(n))$.

Next, we assume that $f(n) \in \Theta(g(n))$. We have:

$$\begin{aligned}&\exists c_1, c_2 \in R_+, \exists n_0 \in N_+ \text{ such that}\\ &c_1 \cdot g(n) \le f(n) \le c_2 \cdot g(n), \forall n \ge n_0\end{aligned} \tag{19}$$

From the definition of $f(n)$ we have

$$\forall c'' \in R_+, \exists n_0'' \in N_+ \text{ such that } f(n) < c'' \cdot g(n), \forall n \ge n_0'', n = 2 \cdot k \tag{20}$$

Let be $\overline{n}_0 = \max\{n_0, n_0''\}$. For $c'' = c_1$ we have

$$c_1 \cdot g(n) \le f(n), \forall n \ge \overline{n}_0 \text{ and } f(n) < c_1 \cdot g(n), \forall n \ge \overline{n}_0, n = 2 \cdot k \tag{21}$$

Consequently $f(n) \notin \Theta(g(n))$.

Using the same idea, it can be proved that $f(n) \notin o(g(n))$.

So, we have $f(n) \in O(g(n))$, $f(n) \notin \Theta(g(n))$, and $f(n) \notin o(g(n))$. It follows that $f(n) \in O(g(n)) \setminus (o(g(n)) \cup \Theta(g(n)))$ that is $f(n) \in o\Theta(g(n))$.

For proving that $\Theta\omega(g(n)) \ne \emptyset$ one can use a similar idea.

*Proposition 2.*We have the following properties:
a) $o(g(n)) \cap \omega(g(n)) = \varnothing,\ O(g(n)) \cap \Omega(g(n)) = \Theta(g(n))$
b) $o(g(n)) \cap \Omega(g(n)) = \varnothing,\ O(g(n)) \cap \omega(g(n)) = \varnothing$
c) $o(g(n)) \subseteq O(g(n))$, $\Theta(g(n)) \subseteq O(g(n))$
d) $\omega(g(n)) \subseteq \Omega(g(n))$, $\Theta(g(n)) \subseteq \Omega(g(n))$
Proof. The results can be obtained using (1), (2), (3), (4), and (5).

*Proposition 3.*We have the following properties:
a) $o(g(n)) \cap \Theta(g(n)) = \varnothing,\ o(g(n)) \cap o\Theta(g(n)) = \varnothing,\ o\Theta(g(n)) \cap \Theta(g(n)) = \varnothing.$
b) $o(g(n)) \cup o\Theta(g(n)) \cup \Theta(g(n)) = O(g(n))$.
In other words, the complexity classes $o(g(n))$, $o\Theta(g(n))$ and $\Theta(g(n))$ form a partition of the complexity class $O(g(n))$.

Proof. a) The first equality can be obtained using the definitions (1) and (4). The other two equalities are easily obtained from the definition of the complexity class $o\Theta(g(n))$.

b) From Proposition 3, we have $o(g(n)) \cup \Theta(g(n)) \subseteq O(g(n))$. Using the definition of $o\Theta(g(n))$ we have $o(g(n)) \cup o\Theta(g(n)) \cup \Theta(g(n)) = O(g(n))$.

Proposition 4. We have the following properties:
a) $\Theta(g(n)) \cap \omega(g(n)) = \varnothing,\ \Theta(g(n)) \cap \Theta\omega(g(n)) = \varnothing,\ \Theta\omega(g(n)) \cap \omega(g(n)) = \varnothing$
b) $\Theta(g(n)) \cup \Theta\omega(g(n)) \cup \omega(g(n)) = \Omega(g(n))$.
In other words, the complexity classes $\Theta(g(n))$, $\Theta\omega(g(n))$ and $\omega(g(n))$ form a partition of the complexity class $\Omega(g(n))$.

Proof. The proof follows the same idea as the proof for Proposition 4.

Proposition 5. Let be N_1 and N_2 two infinite subsets of N_+, such that N_1 and N_2 form a partition of N_+. Let be $f_1(n)$ and $f_2(n)$ two complexity functions. Let be

$$f(n) = \begin{cases} f_1(n), n \in N_1 \\ f_2(n), n \in N_2 \end{cases} \tag{22}$$

Then, we have:
a) If $f_1(n) \in \Theta(g(n))$ and $f_2(n) \in \Theta(g(n))$ then $f(n) \in \Theta(g(n))$.
b) If $f_1(n) \in O(g(n))$ and $f_2(n) \in O(g(n))$ then $f(n) \in O(g(n))$.
c) If $f_1(n) \in \Omega(g(n))$ and $f_2(n) \in \Omega(g(n))$ then $f(n) \in \Omega(g(n))$.

d) If $f_1(n) \in o(g(n))$ and $f_2(n) \in o(g(n))$ then $f(n) \in o(g(n))$.
e) If $f_1(n) \in \omega(g(n))$ and $f_2(n) \in \omega(g(n))$ then $f(n) \in \omega(g(n))$.

Proof. For proving these results, we use the definitions from (1), (2), (3), (4), and (5).

a) From $f_1(n) \in \Theta(g(n))$, $f_2(n) \in \Theta(g(n))$, and from the expression of $f(n)$ we have:

$$\begin{aligned} &\exists c_1^{'}, c_2^{'} \in R_+, \exists n_0^{'} \in N_+ \ such \ \ that \\ &c_1^{'} \cdot g(n) \le f_1(n) \le c_2^{'} \cdot g(n), \forall n \ge n_0^{'}, n \in N_1 \end{aligned} \tag{23}$$

$$\begin{aligned} &\exists c_1^{''}, c_2^{''} \in R_+, \exists n_0^{''} \in N_+ \ such \ \ that \\ &c_1^{''} \cdot g(n) \le f_2(n) \le c_2^{''} \cdot g(n), \forall n \ge n_0^{''}, n \in N_2 \end{aligned} \tag{24}$$

Let be $c_1 = \min(c_1^{'}, c_1^{''})$, $c_2 = \max(c_2^{'}, c_2^{''})$, and $n_0 = \max(n_0^{'}, n_0^{''})$. Then, we have:

$$\begin{aligned} &c_1 \cdot g(n) \le f_1(n) \le c_2 \cdot g(n), \forall n \ge n_0, n \in N_1 \\ &c_1 \cdot g(n) \le f_2(n) \le c_2 \cdot g(n), \forall n \ge n_0, n \in N_2 \end{aligned} \tag{25}$$

It follows that $f(n) \in \Theta(g(n))$.

b), c), d), e) The proofs use the same idea as the proof for a).

Proposition 6. Let be N_1 and N_2 two infinite subsets of N_+, such that N_1 and N_2 form a partition of N_+. Let be $f_1(n)$ and $f_2(n)$ two complexity functions. Let be

$$f(n) = \begin{cases} f_1(n), n \in N_1 \\ f_2(n), n \in N_2 \end{cases} \tag{26}$$

Then, we have:
a) If $f_1(n) \in \Theta(g(n))$ and $f_2(n) \notin \Theta(g(n))$ then $f(n) \notin \Theta(g(n))$.
b) If $f_1(n) \in O(g(n))$ and $f_2(n) \notin O(g(n))$ then $f(n) \notin O(g(n))$.
c) If $f_1(n) \in \Omega(g(n))$ and $f_2(n) \notin \Omega(g(n))$ then $f(n) \notin \Omega(g(n))$.
d) If $f_1(n) \in o(g(n))$ and $f_2(n) \notin o(g(n))$ then $f(n) \notin o(g(n))$.
e) If $f_1(n) \in \omega(g(n))$ and $f_2(n) \notin \omega(g(n))$ then $f(n) \notin \omega(g(n))$.

Proof. For proving these results, we use the definitions from (1), (2), (3), (4), and (5).

a) From $f_2(n) \notin \Theta(g(n))$ we have that the property

$$\begin{aligned}&\exists c_1^{"}, c_2^{"} \in R_+, \exists n_0^{"} \in N_+ \ such \ \ that \\ &c_1^{"} \cdot g(n) \leq f_2(n) \leq c_2^{"} \cdot g(n), \forall n \geq n_0^{"}, n \in N_2\end{aligned} \tag{27}$$

is false. If follows that the property

$$\begin{aligned}&\exists c_1^{"}, c_2^{"} \in R_+, \exists n_0^{"} \in N_+ \ such \ \ that \\ &c_1^{"} \cdot g(n) \leq f(n) \leq c_2^{"} \cdot g(n), \forall n \geq n_0^{"}, n \in N_2\end{aligned} \tag{28}$$

is false. Since N_2 is an infinite subset of N_+, we have $f(n) \notin \Theta(g(n))$.

b), c), d), e) The proofs use the same idea as the proof for a).

Proposition 7. Let be N_1 and N_2 two infinite subsets of N_+, such that N_1 and N_2 form a partition of N_+. Let be $f_1(n)$ and $f_2(n)$ two complexity functions. Let be

$$f(n) = \begin{cases} f_1(n), n \in N_1 \\ f_2(n), n \in N_2 \end{cases} \tag{29}$$

Then, we have:

a) If $f_1(n) \in o(g(n))$ and $f_2(n) \in \Theta(g(n))$ then $f(n) \in o\Theta(g(n))$

b) If $f_1(n) \in \Theta(g(n))$ and $f_2(n) \in \omega(g(n))$ then $f(n) \in \Theta\omega(g(n))$

Proof. For proving these results we use Proposition 5, Proposition 6, and the properties: $o(g(n)) \cap \Theta(g(n)) = \emptyset$ and $\Theta(g(n)) \cap \omega(g(n)) = \emptyset$.

a) From $f_1(n) \in o(g(n))$ and $f_2(n) \in \Theta(g(n))$ we have $f_1(n) \in o(g(n))$ and $f_2(n) \notin o(g(n))$. Consequently, $f(n) \notin o(g(n))$.

From $f_1(n) \in o(g(n))$ and $f_2(n) \in \Theta(g(n))$ we have $f_1(n) \notin \Theta(g(n))$ and $f_2(n) \in \Theta(g(n))$. Consequently, $f(n) \notin \Theta(g(n))$.

From $f_1(n) \in o(g(n))$ and $f_2(n) \in \Theta(g(n))$ we have $f_1(n) \in O(g(n))$ and $f_2(n) \in O(g(n))$. Consequently, $f(n) \in O(g(n))$.

It follows that $f(n) \in O(g(n)) \setminus (o(g(n)) \cup \Theta(g(n)))$. Consequently, $f(n) \in o\Theta(g(n))$.

b) The proof uses the same idea as the proof for a).

Proposition 8. We have the following property: $f(n) \in o\Theta(g(n))$ if and only if $g(n) \in \Theta\omega(f(n))$.

Proof. We have the following well known properties:

$$f(n) \in \Theta(g(n)) \ \textit{if and only if} \ g(n) \in \Theta(f(n)) \tag{30}$$

$$f(n) \in O(g(n)) \ \textit{if and only if} \ g(n) \in \Omega(f(n)) \tag{31}$$

$$f(n) \in o(g(n)) \ \textit{if and only if} \ g(n) \in \omega(f(n)) \tag{32}$$

From Definition 2, we have:

$$o\Theta(g(n)) = O(g(n)) \setminus (o(g(n)) \cup \Theta(g(n))) \tag{33}$$

$$\Theta\omega(g(n)) = \Omega(g(n)) \setminus (\Theta(g(n)) \cup \omega(g(n))) \tag{34}$$

Consider that $f(n) \in o\Theta(g(n))$. We show that $g(n) \in \Theta\omega(f(n))$. From the definition of $o\Theta(g(n))$ we have

$$f(n) \in O(g(n)) \setminus (o(g(n)) \cup \Theta(g(n))) \tag{35}$$

so, we have

$$f(n) \in O(g(n)), f(n) \notin o(g(n)), f(n) \notin \Theta(g(n)) \tag{36}$$

Using (30), (31), (32) it follows that $g(n) \in \Omega(f(n))$, $g(n) \notin \omega(f(n))$, $g(n) \notin \Theta(f(n))$. Consequently, $g(n) \in \Omega(f(n)) \setminus (\Theta(f(n)) \cup \omega(f(n)))$. It follows that $g(n) \in \Theta\omega(f(n))$.

The other implication can be proved using the same idea.

Proposition 9. Let be N_1 and N_2 two infinite subsets of N_+, such that N_1 and N_2 form a partition of N_+. Let be $f_1(n)$ and $f_2(n)$ two complexity functions. Let be

$$f(n) = \begin{cases} f_1(n), n \in N_1 \\ f_2(n), n \in N_2 \end{cases} \tag{37}$$

Then, we have:
a) If $f_1(n) \in o\Theta(g(n))$ and $f_2(n) \in o(g(n))$ then $f(n) \in o\Theta(g(n))$.
b) If $f_1(n) \in o\Theta(g(n))$ and $f_2(n) \in \Theta(g(n))$ then $f(n) \in o\Theta(g(n))$.
c) If $f_1(n) \in \Theta\omega(g(n))$ and $f_2(n) \in \omega(g(n))$ then $f(n) \in \Theta\omega(g(n))$.
d) If $f_1(n) \in \Theta\omega(g(n))$ and $f_2(n) \in \Theta(g(n))$ then $f(n) \in \Theta\omega(g(n))$.

Proof.

a) From $f_1(n) \in o\Theta(g(n))$ we have that $f_1(n) \in O(g(n))$, $f_1(n) \notin o(g(n))$, and $f_1(n) \notin \Theta(g(n))$. From $f_2(n) \in o(g(n))$ we have that $f_2(n) \in O(g(n))$, $f_2(n) \in o(g(n))$, and $f_2(n) \notin \Theta(g(n))$. Consequently, using Proposition 5 and Proposition 6, we have $f(n) \in O(g(n))$, $f(n) \notin o(g(n))$, and $f(n) \notin \Theta(g(n))$. It follows that $f(n) \in o\Theta(g(n))$.

b), c), d) The proofs use the same idea as the proof for a).

4. The main results

Theorem 1. Let be $f(n) \in C(g(n))$. Then $g(n) \in C(f(n))$.

Proof. We will use a well known property of the complexity classes:

$$f_1(n) \in O(f_2(n)) \ \textit{if and only if} \ f_2(n) \in \Omega(f_1(n)) \tag{38}$$

The hypothesis $f(n) \in C(g(n))$ implies that $f(n) \in O(g(n)) \cup \Omega(g(n))$. From Proposition 2, we have $O(g(n)) \cap \Omega(g(n)) = \Theta(g(n))$.

It follows that we have two possibilities: either $f(n) \in O(g(n))$ or $f(n) \in (\Omega(g(n)) \setminus \Theta(g(n)))$. If $f(n) \in O(g(n))$ then $g(n) \in \Omega(f(n))$, hence $g(n) \in C(f(n))$. If $f(n) \in (\Omega(g(n)) \setminus \Theta(g(n))) \subseteq \Omega(g(n))$ then $g(n) \in O(f(n))$, hence $g(n) \in C(f(n))$.

Theorem 2. The complexity classes $o(g(n))$, $o\Theta(g(n))$, $\Theta(g(n))$, $\Theta\omega(g(n))$ and $\omega(g(n))$ form a partition of the set $C(g(n))$, that is:

a) $C(g(n)) = o(g(n)) \cup o\Theta(g(n)) \cup \Theta(g(n)) \cup \Theta\omega(g(n)) \cup \omega(g(n))$

b) The complexity classes $o(g(n))$, $o\Theta(g(n))$, $\Theta(g(n))$, $\Theta\omega(g(n))$ and $\omega(g(n))$ are pairwise disjoint.

Proof. a) For proving this result we use Remark 1, Proposition 3, and Proposition 4. From

$$C(g(n)) = \Theta(g(n)) \cup O(g(n)) \cup \Omega(g(n)) \cup o(g(n)) \cup \omega(g(n)) \tag{39}$$

we have

$$C(g(n)) = \Theta(g(n)) \cup o(g(n)) \cup o\Theta(g(n)) \cup \Theta(g(n)) \cup \\ \cup \Theta(g(n)) \cup \Theta\omega(g(n)) \cup \omega(g(n)) \cup o(g(n)) \cup \omega(g(n)) \quad (40)$$

It follows that

$$C(g(n)) = o(g(n)) \cup o\Theta(g(n)) \cup \Theta(g(n)) \cup \Theta\omega(g(n)) \cup \omega(g(n)) \quad (41)$$

b) From Proposition 3, it follows that $o(g(n))$, $o\Theta(g(n))$ and $\Theta(g(n))$ are pairwise disjoint. From Proposition 4, if follows that $\Theta(g(n))$, $\Theta\omega(g(n))$ and $\omega(g(n))$ are pairwise disjoint.

From Proposition 2, if follows that $o(g(n))$ and $\omega(g(n))$ are disjoint. Using Proposition 2, we have that $o(g(n)) \cap \Omega(g(n)) = \emptyset$, hence $o(g(n))$ and $\Theta\omega(g(n))$ are disjoint. Using again Proposition 2, have $O(g(n)) \cap \omega(g(n)) = \emptyset$, hence $o\Theta(g(n))$ and $\omega(g(n))$ are disjoint.

From Proposition 2 we have that $O(g(n)) \cap \Omega(g(n)) = \Theta(g(n))$. We know that $o\Theta(g(n)) \subseteq O(g(n))$ and $\Theta\omega(g(n)) \subseteq \Omega(g(n))$. We also know that $o\Theta(g(n))$ and $\Theta(g(n))$ are disjoint and $\Theta(g(n))$ and $\Theta\omega(g(n))$ are disjoint. It follows that $o\Theta(g(n))$ and $\Theta\omega(g(n))$ are disjoint.

Consequently $o(g(n))$, $o\Theta(g(n))$, $\Theta(g(n))$, $\Theta\omega(g(n))$ and $\omega(g(n))$ are pairwise disjoint.

Theorem 3. Let be $f_1(n) \in o(g(n))$ and $f_2(n) \in \omega(g(n))$ two complexity functions. Then $f_1(n) \in C(f_2(n))$ and $f_2(n) \in C(f_1(n))$.

Proof. From $f_1(n) \in o(g(n))$, we have:

$$\forall c^{'} \in R_{+}, \exists n_0^{'} \in N_{+} \ such \ \ that \ f_1(n) < c^{'} \cdot g(n), \forall n \geq n_0^{'} \quad (42)$$

From $f_2(n) \in \omega(g(n))$, we have:

$$\forall c^{"} \in R_{+}, \exists n_0^{"} \in N_{+} \ such \ \ that \ \ c^{"} \cdot g(n) < f_2(n), \forall n \geq n_0^{"} \quad (43)$$

Let be $c \in R_{+}$; for $c^{'} = c^{"} = c$ there exist $n_0^{'}$ and $n_0^{"}$ with the above properties. Let be $n_0 = \max\{n_0^{'}, n_0^{"}\}$. Then, we have:

$$f_1(n) < c \cdot g(n) \ \ and \ \ c \cdot g(n) < f_2(n), \forall n \geq n_0 \tag{44}$$

that is:

$$f_1(n) < c \cdot g(n) < f_2(n), \forall n \geq n_0 \tag{45}$$

It follows that:

$$\exists \overline{c} = 1 \in R_+, \exists \overline{n}_0 = n_0 \in N_+ \ such \ \ that \ f_1(n) \leq \overline{c} \cdot f_2(n), \forall n \geq \overline{n}_0 \tag{46}$$

hence $f_1(n) \in O(f_2(n))$. From here, we have $f_1(n) \in C(f_2(n))$. Next, using Theorem 1, it follows that $f_2(n) \in C(f_1(n))$.

Remark 3. If $f(n) \in C(g(n))$ we say that $f(n)$ and $g(n)$ are comparable. Note that, from Theorem 1, if $f(n) \in C(g(n))$ then $g(n) \in C(f(n))$.

Theorem 4. We have the following properties:
a) Let be $f_1(n) \in o(g(n))$ and $f_2(n) \in \Theta(g(n))$. Then $f_1(n) \in C(f_2(n))$.
b) Let be $f_1(n) \in o(g(n))$ and $f_2(n) \in \Theta\omega(g(n))$. Then $f_1(n) \in C(f_2(n))$.

Proof. We prove that if $f_1(n) \in o(g(n))$ and $f_2(n) \in \Omega(g(n))$ then $f_1(n) \in C(f_2(n))$. From $f_1(n) \in o(g(n))$ and $f_2(n) \in \Omega(g(n))$ we have:

$$\forall c' \in R_+, \exists n_0' \in N_+ \ such \ \ that \ f_1(n) < c' \cdot g(n), \forall n \geq n_0' \tag{47}$$

$$\exists c'' \in R_+, \exists n_0'' \in N_+ \ such \ \ that \ \ c'' \cdot g(n) \leq f_2(n), \forall n \geq n_0'' \tag{48}$$

If we choose $c' = c''$, then we have:

$$f_1(n) < c'' \cdot g(n) \leq f_2(n), \forall n \geq \max(n_0', n_0'') \tag{49}$$

Next, we have

$$\begin{aligned} &\exists c = 1 \in R_+, \exists n_0 = \max(n_0', n_0'') \in N_+ \ such \ \ that \\ &f_1(n) < c \cdot f_2(n), \forall n \geq n_0 \end{aligned} \tag{50}$$

Consequently, $f_1(n) \in O(f_2(n))$. It follows that $f_1(n) \in C(f_2(n))$.

Using Proposition 4, we have:

$$\Theta(g(n)) \subseteq \Omega(g(n)) \tag{51}$$

$$\Theta\omega(g(n)) \subseteq \Omega(g(n)) \tag{52}$$

a) From $f_1(n) \in o(g(n))$ and $f_2(n) \in \Theta(g(n))$ we have that $f_1(n) \in o(g(n))$ and $f_2(n) \in \Omega(g(n))$. It follows that $f_1(n) \in C(f_2(n))$.

b) From $f_1(n) \in o(g(n))$ and $f_2(n) \in \Theta\omega(g(n))$ we have that $f_1(n) \in o(g(n))$ and $f_2(n) \in \Omega(g(n))$. It follows that $f_1(n) \in C(f_2(n))$.

Theorem 5. We have the following properties:
a) Let be $f_1(n) \in \omega(g(n))$ and $f_2(n) \in \Theta(g(n))$. Then $f_1(n) \in C(f_2(n))$.
b) Let be $f_1(n) \in \omega(g(n))$ and $f_2(n) \in o\Theta(g(n))$. Then $f_1(n) \in C(f_2(n))$.

Proof. The proof follows the same idea as the proof for the Theorem 4.

Theorem 6. We have the following properties:
a) Let be $f_1(n) \in o\Theta(g(n))$ and $f_2(n) \in \Theta(g(n))$. Then $f_1(n) \in C(f_2(n))$.
b) Let be $f_1(n) \in \Theta(g(n))$ and $f_2(n) \in \Theta\omega(g(n))$. Then $f_1(n) \in C(f_2(n))$.
c) Let be $f_1(n) \in o\Theta(g(n))$ and $f_2(n) \in \Theta\omega(g(n))$. Then $f_1(n) \in C(f_2(n))$.

Proof. Using the same idea used in the proof of Theorem 4, one can prove that if $f_1(n) \in O(g(n))$ and $f_2(n) \in \Omega(g(n))$ then $f_1(n) \in C(f_2(n))$.
a) We have $o\Theta(g(n)) \subseteq O(g(n))$ and $\Theta(g(n)) \subseteq \Omega(g(n))$. Consequently, we have $f_1(n) \in O(g(n))$ and $f_2(n) \in \Omega(g(n))$. It follows that $f_1(n) \in C(f_2(n))$.
b), c) The proofs follow the same idea used for the proof of a).

Theorem 7. We have the following properties:
a) There exists $f_1(n) \in o(g(n))$, $f_2(n) \in o\Theta(g(n))$ such that $f_1(n) \notin C(f_2(n))$
b) There exists $f_1(n) \in \Theta\omega(g(n))$, $f_2(n) \in \omega(g(n))$ such that $f_1(n) \notin C(f_2(n))$

Proof.

a) Let be

$$f_1(n) = g(n)/n\,,\ \ f_2(n) = \begin{cases} g(n)/n^2\,, & n = 2\cdot k+1 \\ g(n)\,, & n = 2\cdot k \end{cases} \tag{53}$$

It is easy to see that $g(n)/n \in o(g(n))$, $g(n)/n^2 \in o(g(n))$, and $g(n) \in \Theta(g(n))$. We have $f_1(n) \in o(g(n))$. Using Proposition 7, we have that $f_2(n) \in o\Theta(g(n))$.

One can observe that $f_1(n) \in \omega(g(n)/n^2)$ and $f_1(n) \in o(g(n))$. In addition, the set of odd naturals and the set of even naturals are infinite sets. It follows that $f_1(n) \notin C(f_2(n))$.

b) The proof uses the same idea as the proof for a).

Theorem 8. We have the following properties:

a) Let be $g(n) \in o(f_1(n))$ and $g(n) \in \omega(f_2(n))$. Then $f_1(n) \in C(f_2(n))$.

b) Let be $g(n) \in o(f_1(n))$ and $g(n) \in \Theta(f_2(n))$. Then $f_1(n) \in C(f_2(n))$.

c) Let be $g(n) \in o(f_1(n))$ and $g(n) \in \Theta\omega(f_2(n))$. Then $f_1(n) \in C(f_2(n))$.

d) Let be $g(n) \in \omega(f_1(n))$ and $g(n) \in \Theta(f_2(n))$. Then $f_1(n) \in C(f_2(n))$.

e) Let be $g(n) \in \omega(f_1(n))$ and $g(n) \in o\Theta(f_2(n))$. Then $f_1(n) \in C(f_2(n))$.

Proof. We will use Proposition 8, formulas (31), (32), (35), Theorem 3, Theorem 4, and Theorem 5.

a) From $g(n) \in o(f_1(n))$ and $g(n) \in \omega(f_2(n))$ we have that $f_1(n) \in \omega(g(n))$ and $f_2(n) \in o(g(n))$. It follows that $f_1(n) \in C(f_2(n))$.

b), c), d), e) The proofs use the same idea as the proof for a).

Theorem 9. Let be N_1 and N_2 two infinite subsets of N_+, such that N_1 and N_2 form a partition of N_+. Let be $f_1(n)$ and $f_2(n)$ two complexity functions. Let be

$$f(n) = \begin{cases} f_1(n), n \in N_1 \\ f_2(n), n \in N_2 \end{cases} \tag{54}$$

Then, we have:

a) If $f_1(n) \in o\Theta(g(n))$ and $f_2(n) \in o(g(n))$ then $f(n) \in C(g(n))$.

b) If $f_1(n) \in o\Theta(g(n))$ and $f_2(n) \in \Theta(g(n))$ then $f(n) \in C(g(n))$.

c) If $f_1(n) \in o(g(n))$ and $f_2(n) \in \Theta(g(n))$ then $f(n) \in C(g(n))$.

d) If $f_1(n) \in \Theta\omega(g(n))$ and $f_2(n) \in \omega(g(n))$ then $f(n) \in C(g(n))$.

e) If $f_1(n) \in \Theta\omega(g(n))$ and $f_2(n) \in \Theta(g(n))$ then $f(n) \in C(g(n))$.

f) If $f_1(n) \in \omega(g(n))$ and $f_2(n) \in \Theta(g(n))$ then $f(n) \in C(g(n))$.

Proof. For proving the theorem, we use Proposition 9 and Proposition 5.

a) From $f_1(n) \in o\Theta(g(n))$ and $f_2(n) \in o(g(n))$ we have that $f(n) \in o\Theta(g(n))$. Consequently, $f(n) \in O(g(n))$. It follows that $f(n) \in C(g(n))$.

c) From $f_1(n) \in o(g(n))$ and $f_2(n) \in \Theta(g(n))$ we have that $f_1(n) \in O(g(n))$ and $f_2(n) \in O(g(n))$. Consequently, $f(n) \in O(g(n))$. It follows that $f(n) \in C(g(n))$.

b), d), e) The proofs use the same idea as the proof for a).

f) The proof uses the same idea as the proof for c).

Theorem 10. Let be N_1 and N_2 two infinite subsets of N_+, such that N_1 and N_2 form a partition of N_+. Let be $f_1(n)$ and $f_2(n)$ two complexity functions. Let be

$$f(n) = \begin{cases} f_1(n), n \in N_1 \\ f_2(n), n \in N_2 \end{cases}. \tag{55}$$

Then, we have:

a) If $f_1(n) \in o(g(n))$ and $f_2(n) \in \omega(g(n))$ then $f(n) \notin C(g(n))$.

b) If $f_1(n) \in o(g(n))$ and $f_2(n) \in \Theta\omega(g(n))$ then $f(n) \notin C(g(n))$.

c) If $f_1(n) \in o\Theta(g(n))$ and $f_2(n) \in \omega(g(n))$ then $f(n) \notin C(g(n))$.

d) If $f_1(n) \in o\Theta(g(n))$ and $f_2(n) \in \Theta\omega(g(n))$ then $f(n) \notin C(g(n))$.

Proof. For proving that $f(n) \in C(g(n))$, we need to find a complexity class that contains both $f_1(n)$ and $f_2(n)$. We will show that this is impossible.

Using Remark 1, we have:

$$C(g(n)) = \Theta(g(n)) \cup O(g(n)) \cup \Omega(g(n)) \cup o(g(n)) \cup \omega(g(n)) \tag{56}$$

The largest two complexity classes are $O(g(n))$ and $\Omega(g(n))$. So we can use the form of $C(g(n))$ discussed in Remark 2: $C(g(n)) = O(g(n)) \cup \Omega(g(n))$.

a) From $f_1(n) \in o(g(n))$ we have $f_1(n) \in O(g(n))$ and $f_1(n) \notin \Omega(g(n))$. From $f_2(n) \in \omega(g(n))$ we have $f_2(n) \notin O(g(n))$ and $f_2(n) \in \Omega(g(n))$. It follows that $f(n) \notin C(g(n))$.

b), c), d) The proofs use the same idea as the proof for a).

6. Conclusion

In this paper we presented some interesting results related to the comparison of two complexity functions using complexity classes. These results are important in practice because when we compare two complexity functions, in fact, we compare two algorithms complexities. Using the results from this paper, some algorithms can be designed to tell us if two functions are comparable or to tell us if two functions are incomparable.

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