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ON ORDERED HYPERSTRUCTURES

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In this paper we study a (semi)hypergroup $(H,\circ)$ besides a binary relation $\leq$, where $\leq$ is a partial preorder or a partial order such that satisfies the monotone condition. This structure is called a partially preordered (ordered) (semi)hypergroup. Also, we consider some well-known hypergroups and define a binary relation on them such that to become partially preordered (ordered) hypergroups. Finally, we associate a semihypergroup to a Γ-semigroup and prove some properties.

Keywords: Semihypergroup; Γ-semigroup; Hypergroup; Po-hypergroup; Po-semihypergroup

1. Introduction

The hyperstructure theory was born in 1934, when Marty introduced the notion of a hypergroup [11]. Algebraic hyperstructures are a generalization of classical algebraic structures. In a classical algebraic structure the composition of two elements is an element, while in an algebraic hyperstructure the composition of two elements is a non-empty set. More exactly, let H be a non-empty set. Then the map $\circ : H\times H \to P^*(H)$ is called a hyperoperation, where $P^*(H)$ is the family of non-empty subsets of H. The couple $(H,\circ)$ is called a hypergroupoid.

In the above definition, if A and B are two non-empty subsets of H and $x\in H$, then we define

$$A\circ B=\bigcup_{a\in A,b\in B} a\circ b;\ \ x\circ A=\{x\}\circ A \ \ and \ \ A\circ x = A\circ\{x\}.$$

An element $e\in H$ is called a unit element of hypergroupoid $(H,\circ)$ if $x\in e\circ x\cap x\circ e$ for every $x\in H$.

A hypergroupoid $(H,\circ)$ is called a semihypergroup if for every $x,y,z\in H$, we have $x\circ(y\circ z)=(x\circ y)\circ z$ and is called a quasihypergroup if for every $x\in H$, $x\circ H=H=H\circ x$. This condition is called the reproduction axiom. The couple $(H,\circ)$ is called a hypergroup if it is a semihypergroup and a quasihypergroup.

Since then, hundreds of papers and several books have been written on this topic, see [6 and 15]. A recent book on hyperstructures [5] points out on their applications in cryptography, codes, automata, probability, geometry, lattices,

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binary relations, graphs and hypergraphs. Another book [7] is devoted especially to the study of hyperring theory. Several kinds of hyperrings are introduced and analyzed. The volume ends with an outline of applications in chemistry and physics, analyzing several special kinds of hyperstructures: e-hyperstructures and transposition hypergroups. The theory of suitable modified hyperstructures can serve as a mathematical background in the field of quantum communication systems.

The concept of ordering hypergroups introduced by Chvalina [1] as a special class of hypergroups and studied by many authors, see [2, 3, 4, 8, 9 and 10].

The term "poset" is short for "partially ordered set", that is, a set whose elements are ordered but not all pairs of elements are required to be comparable in the order. A partial order is a binary relation R on a set X which satisfies conditions reflexivity, antisymmetry and transitivity. Sometimes we need to weaken the definition of partial order. We say that a partial preorder is a relation which satisfies conditions reflexivity and transitivity.

An algebraic system $(G,.,\leq)$ is called a partially preordered (ordered) groupoid if $(G,.)$ is a groupoid and $(G,\leq)$ is a partially preordered (ordered) set which satisfies monotone condition as follows: if $x \leq y$, then $a.x \leq a.y$ and $x.a \leq y.a$ for every $x, y, a \in G$.

A term "po-groupoid" is used for partially ordered groupoid. A po-groupoid $(G,.,\leq)$ is a po-(semi)group if $(G,.)$ is a (semi)group.

The notion of Γ-semigroups was introduced by Sen in [13]. Let S and Γ be non-empty sets. Then S is called a Γ-semigroup if there exists a mapping $S \times \Gamma \times S \to S$, written (a,γ,b) by $a\gamma b$, such that satisfies the identities $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a,b,c \in S$ and $\alpha,\beta \in \Gamma$. Let S be an arbitrary semigroup and Γ be a non-empty set. Define a mapping $S \times \Gamma \times S \to S$ by $a\alpha b = ab$ for all $a,b \in S$ and $\alpha \in \Gamma$. It is easy to see that S is a Γ-semigroup. Thus a semigroup can be considered as a Γ-semigroup.

2. Partially ordered semihypergroups

In this section we introduce the concept of partially preordered (ordered) semihypergroups and prove some results. First, we recall some preliminaries of semihypergroups.

Let $(S,\circ)$ be a semihypergroup and I be a subset of S. Then I is called a left (right) hyperideal if $S \circ I \subseteq I (I \circ S \subseteq I)$, and I is called an ideal of S if it is a left and a right ideal.

Strongly regular relations have an important role in the theory of hyperstructures. Starting from a (semi)hypergroup and using a strongly regular relation we can constract a (semi)group on the quotient set.

Let $(S,\circ)$ be a semihypergroup and R be an equivalence relation on S. If A and B are non-empty subsets of S, then $A\overline{R}B$ means that for all $a \in A$, there exists $b \in B$ such that aRb and for all $b' \in B$, there exists $a' \in A$ such that $a'Rb'$. Also, $A\overline{\overline{R}}B$ means that for all $a \in A$ and $b \in B$, we have aRb.

Definition 2.1 The equivalence relation R is called

(1) regular on the right (on the left) if for all $x \in S$, from aRb, it follows that $(a \circ x)\overline{R}(b \circ x)((x \circ a)\overline{R}(x \circ b)$ respectively);

(2) strongly regular on the right (on the left) if for all $x \in S$, from aRb, it follows that $(a \circ x)\overline{\overline{R}}(b \circ x)((x \circ a)\overline{\overline{R}}(x \circ b)$ respectively);

(3) R is called regular (strongly regular) if it is regular (strongly regular) on the right and on the left.

Theorem 2.2 [6] Let $(S,\circ)$ be a (semi)hypergroup and R be an equivalence relation on S.

(1) If R is regular, then S/R is a (semi)hypergroup, with respect to the following hyperoperation: $\overline{x} \otimes \overline{y} = \{\overline{z} \mid z \in x \circ y\}$.

(2) If the above hyperoperation is well defined on S/R, then R is regular.

Theorem 2.3 [6] Let $(S,\circ)$ be a (semi)hypergroup and R be an equivalence relation on S.

(1) If R is strongly regular, then S/R is a (semi)group, with respect to the following operation: $\overline{x} \otimes \overline{y} = \{\overline{z} \mid z \in x \circ y\}$.

(2) If the above operation is well defined on S/R, then R is strongly regular.

Definition 2.4 An algebraic hyperstructure $(H,\circ,\leq)$ is called a partially preordered (ordered) semihypergroup, if $(H,\circ)$ is a semihypergroup and $\leq$ is a partial preorder (order) relation on H such that the monotone condition holds as follows:

$$x \leq y \Rightarrow a \circ x \leq a \circ y \quad \textit{for all} \quad x, y, a \in S,$$

where, if A and B are non-empty subsets of H, then we say that $A \leq B$ if for every $a \in A$ there exists $b \in B$ such that $a \leq b$.

Obviously, every po-semigroup is a po-semihypergroup. In the following we give some other examples of po-semihypergroups.

Example 1 Let $(X,\leq)$ be a poset and $\emptyset \neq Q \subseteq X$. If for every $x, y \in X$, we define $x \circ y = Q$, then $(X,\circ,\leq)$ is a po-semihypergroup.

Example 2 Let $(S,.,\leq)$ be a po-semigroup. If for every $x, y \in S$, we define

$x \circ y = \{x^i : i \in \mathbb{N}\}$, then $(S, \circ, \leq)$ is a po-semihypergroup.

Example 3 Let $(S, ., \leq)$ be a po-semigroup. If for every $x, y \in S$, we define $x \circ y = < x, y >$, where $< x, y >$ is the ideal of S generated by $\{x, y\}$, then $(S, \circ, \leq)$ is a po-semihypergroup.

Definition 2.5 A non-empty subset I of a po-semihypergroup $(S, \circ, \leq)$ is called a left (right) ideal of S if the following conditions hold:

(1) $S \circ I \subseteq I (I \circ S \subseteq I)$;

(2) If $a \in I$ and $b \leq a$, then $b \in I$ for every $b \in S$.

I is called an ideal of S if it is a left and a right ideal.

If $(S, \circ, \leq)$ is a po-semihypergroup and $A \subseteq S$, then $(A]$ is the subset of S defined as follows:

$$(A] = \{t \in S : t \leq a, \textit{ for some } a \in A\}.$$

Let A be a non-empty subset of S. Then the left, right and two-sided ideals of S generated by A are denoted by $< A >_l$, $< A >_r$ and $< A >$, respectively. It is easy to see that

$$\begin{aligned} < A > &= (A] \cup (S \circ A] \cup (A \circ S] \cup (S \circ A \circ S]; \\ < A >_l &= (A] \cup (S \circ A]; \\ < A >_r &= (A] \cup (A \circ S]. \end{aligned}$$

Example 4 Consider Example 1. If A is a non-empty subset of X containing Q, then $(A]$ is an ideal of po-semihypergroup $(S, \circ, \leq)$.

Example 5 In the Example 2, every right ideal of po-semigroup $(S, ., \leq)$ is a right ideal of po-semihypergroup $(S, \circ, \leq)$ and S is the only left ideal.

Lemma 2.6 Let $(S, \circ, \leq)$ be a po-semihypergroup. Then the following assertions hold:

(1) $A \subseteq (A]$ for every $A \subseteq S$.

(2) If $A \subseteq B$, then $(A] \subseteq (B]$ for every $A, B \subseteq S$.

(3) $(A] \circ (B] \subseteq (A \circ B]$ for every $A, B \subseteq S$.

(4) $((A]] = (A]$ for every $A \subseteq S$.

(5) If A and B are ideals of S, then $(A \circ B]$ and $A \cup B$ are ideals of S.

(6) For every $a \in S$, $(S \circ a \circ S]$ is an ideal of S.

(7) If $A, B, C \subseteq S$ such that $A \subseteq B$, then $C \circ A \subseteq C \circ B$ and $A \circ C \subseteq B \circ C$.

Proof. The proof is straightforward.

Definition 2.7 Let $(S_1, \circ_1, \leq_1)$ and $(S_2, \circ_2, \leq_2)$ be two po-semihypergroups. A map $\varphi : S_1 \to S_2$ is called a homomorphism if for all $x, y \in S_1$ we have

(1) $\varphi(x\circ_1 y)\subseteq\varphi(x)\circ_2\varphi(y)$; (2) If $x\leq_1 y$, then $\varphi(x)\leq_2\varphi(y)$;
and φ is called a good homomorphism if $\varphi(x\circ_1 y)=\varphi(x)\circ_2\varphi(y)$.

Example 6 Let $(\mathbb{N},\circ_1,|)$ and $(\mathbb{N},\circ_2,\leq)$ be two po-semihypergroups, where $x\circ_1 y=\{x^k\}$ k runs in a subset of $\mathbb{N}$, $x\circ_2 y=\{x^i : i\in\mathbb{N}\}$, $|$ is the relation of divisibility and $\leq$ is the usual order relation on $\mathbb{N}$. Then the identity map $\varphi:(\mathbb{N},\circ,|)\to(\mathbb{N},\circ,\leq)$ is a homomorphism.

Theorem 2.8 Let $(S,\circ,\leq)$ be a partially preordered semihypergroup and R be a strongly regular relation on S. Then $(S/R,\otimes,\preceq)$ is a partially preordered semigroup, with respect to the following operation:

$$\overline{x}\otimes\overline{y}=\{\overline{z}\mid z\in x\circ y\}$$

and for all $\overline{x},\overline{y}\in S/R$ a preordere relation $\preceq$ defined as follows:

$$\overline{x}\preceq\overline{y}\Leftrightarrow\forall x_1\in\overline{x}\ \exists y_1\in\overline{y}\ \textit{such that}\ x_1\leq y_1.$$

Proof. By Theorem 2.2, $(S/R,\otimes)$ is a semigroup. First, we prove that the binary relation $\preceq$ is a partial preorder on S/R. Since $x\leq x$ so $\overline{x}\preceq\overline{x}$ for every $\overline{x}\in S/R$. Thus $\preceq$ is reflexive. If $\overline{x}\preceq\overline{y}$ and $\overline{y}\preceq\overline{z}$, then for every $x_1\in\overline{x}$ there exists $y_1\in\overline{y}$ such that $x_1\leq y_1$. Since $y_1\in\overline{y}\leq\overline{z}$ there exists $z_1\in\overline{z}$ such that $y_1\leq z_1$. Hence $\overline{x}\leq\overline{z}$ thus $\preceq$ is transitive.

Suppose that $\overline{x},\overline{y},\overline{a}\in S/R$ such that $\overline{x}\preceq\overline{y}$. If $\overline{t}=\overline{a}\otimes\overline{x}$, then for every $t_1\in\overline{t}$ there exist $a_1\in\overline{a}$ and $x_1\in\overline{x}$ such that $t_1\in a_1\circ x_1$. Since $x_1\in\overline{x}\preceq\overline{y}$ there exists $y_1\in\overline{y}$ such that $x_1\leq y_1$. So $a_1\circ x_1\leq a_1\circ y_1$. Thus there exists $s_1\in a_1\circ y_1$ such that $t_1\leq s_1$. So $\overline{t}=\overline{t_1}\preceq\overline{s_1}=\overline{a}\otimes\overline{y}$. Therefore, $(S/R,\otimes,\preceq)$ is a partially preordered semigroup.

Let $(S_1,\circ_1,\leq_1)$ and $(S_2,\circ_2,\leq_2)$ be two po-semihypergroups. Then $(S_1\times S_2,\circ)$ is a semihypergroup, where the hyperoperation $\circ$ defined as follows:

$$(x_1,x_2)\circ(y_1,y_2)=(x_1\circ y_1,x_2\circ y_2).$$

The lexicographical order defined on $S_1\times S_2$ as follows: $(x_1,x_2)\leq(y_1,y_2)$ if and only if $x_1\leq_1 y_1$ or $x_1=y_1$ and $x_2\leq_2 y_2$. In the following we prove that $(S_1\times S_2,\circ,\leq)$ is a po-semihypergroup and is called the direct product of po-semihypergroups $(S_1,\circ_1,\leq_1)$ and $(S_2,\circ_2,\leq_2)$.

Theorem 2.9 Let $(S_1,\circ_1,\leq_1)$ and $(S_2,\circ_2,\leq_2)$ be two po-semihypergroups. Then $(S_1\times S_2,\circ)$ is a po-semihypergroup.

Proof. Suppose that $(x_1,x_2)\leq(y_1,y_2)$ for $(x_1,x_2),(y_1,y_2)\in S_1\times S_2$ and $(t_1,t_2)\in(a_1,a_2)\circ(x_1,x_2)$ for $(a_1,a_2)\in S_1\times S_2$. Then $t_1\in a_1\circ_1 x_1$ and $t_2\in a_2\circ_2 x_2$.

Since $(x_1, x_2) \le (y_1, y_2)$, so we have two cases:

Case (1) $x_1 \le y_1$. Then $t_1 \in a_1 \circ_1 x_1 \le_1 a_1 \circ y_1$, so there exists $s_1 \in a_1 \circ_1 y_1$ such that $t_1 \le_1 s_1$. Now, if $s_2 \in a_2 \circ_2 y_2$, then $(t_1, t_2) \le (s_1, s_2) \in (a_1, a_2) \circ (y_1, y_2)$.

Case (2) $x_1 = y_1$ and $x_2 \le_2 y_2$. Then $t_2 \in a_2 \circ_2 x_2 \le_2 a_2 \circ_2 y_2$ so there exists $s_2 \in a_2 \circ_2 y_2$ such that $t_2 \le_2 s_2$. Thus $(t_1, t_2) \le (t_1, s_2) \in (a_1, a_2) \circ (y_1, y_2)$. Therefore, $(S_1 \times S_2, \circ, \le)$ is a po-semihypergroup.

Notice that the mapping $\pi_1 : S_1 \times S_2 \to S_1, (x_1, x_2) \mapsto x_1$ is a good homomorphism, but the mapping $\pi_2 : S_1 \times S_2 \to S_2, (x_1, x_2) \mapsto x_2$ is not.

3. Partially ordered hypergroups

In this section we study the concept of partially preordered (ordered) hypergroups. Also, we consider some well-known hypergroups such as the ordering hypergroups, mentioned by Chvalina [1], and the hypergroups associated to a binary relation, mentioned by Rosenberg [12], and define a partial preorder or order on them such that become preordered hypergroups or po-hypergroups.

Definition 3.1 The po-semihypergroup $(H, \circ, \le)$ is called a po-hypergroup if $(H, \circ)$ is a hypergroup.

Example 7 Let $(X, \le)$ be a poset. If for every $x, y \in X$ we define $x \circ y = \{x, y\}$, then $(X, \circ, \le)$ is a po-hypergroup.

Example 8 Let $(X, \le)$ be a poset. If for every $x, y \in X$ we define $x \circ y = X$, then $(X, \circ, \le)$ is a po-hypergroup.

The following example gives an extensive class of partial preorderd hypergroups and po-hypergroups.

Example 9 Let $(G, ., \le)$ be a partially preordered (ordered) group and P be a non-empty subset of G. Then $(G, \circ_P, \le)$ is a preordered (ordered) hypergroup, where $\circ_P$ is the P-hyperoperation defined as: $x \circ_P y = xPy$ for every $x, y \in G$.

Since, let $x \le y$ for $x, y \in G$. Then for every $a \in G$ we should prove that $a \circ_P x \le a \circ_P y$. If $z \in a \circ_P x = aPx$, then there exists $t \in P$ such that $z = atx$. Now, since $(G, ., \le)$ is a partially preordered (ordered) group, we get $atx \le aty \in aPy = a \circ_P y$. Thus $a \circ_P x \le a \circ_p y$. Therefore, $(G, \circ_P, \le)$ is a partially preordered (ordered) hypergroup.

Next, we will construct a partially preordered (ordered) hypergroup from a topological group. A topological group is a group with a topology on it. More exactly, let $(G, .)$ be a group and (G, τ) be a topological space. Then the triple

$(G,.,\tau)$ is called a topological group if the mappings $(x,y)\to xy$ and $x\to x^{-1}$ are continuous.

A topology is said to be satisfy the axiom T_0 if for any two distinct points x and y, there is an open set containing one of them but not another.

Theorem 3.2 Let $(G,.,\tau)$ be a topological group and P be a non-empty subset of G. Then there exists a binary relation $\leq_\tau$ on G such that $(G,\circ_P,\leq_\tau)$ is a preordered hypergroup, where $\circ_P$ is the P-hyperoperation. Furthermore, if τ is T_0, then $(G,\circ_P,\leq_\tau)$ is a po-hypergroup.

Proof. For every $x,y\in G$ we define the binary relation $\leq_\tau$ defined as follows:

$$x\leq_\tau y \Leftrightarrow (x\in U\Rightarrow y\in U,\ \forall\, U\in\tau).$$

It is easy to see that $\leq_\tau$ is reflexive and transitive so $\leq_\tau$ is a partial preorder relation on G.

Now, we prove that $(G,.,\leq_\tau)$ is a preordered group. Suppose that $x\leq_\tau y$ and $a\in G$. If $ax\in U$ for $U\in\tau$, then $x\in a^{-1}U\in\tau$. Since $x\leq_\tau y$ we get $y\in a^{-1}U$. Thus $ay\in U$. Hence $ax\leq_\tau ay$. Thus $(G,.,\leq_\tau)$ is a partial preordered group and similar to Example 8, $(G,\circ_P,\leq_\tau)$ is a partially preordered hypergroup.

Supose that τ is T_0 and $x\leq_\tau y$ and $y\leq_\tau x$ for $x,y\in G$. If $x\neq y$, then there exists an open subset of G, say U, such that contains only one of x and y. If $x\in U$ and $y\notin U$, then by definition of relation $\leq_\tau$ we have x $\not\leq_\tau$ y Similarly in the case of $y\in U$ and $x\notin U$ we have y $\not\leq_\tau$ x. This is a contradiction so we conclude that $x=y$. So $\leq_\tau$ is antisymmetric thus $\leq_\tau$ is a partially ordered relation on G. Therefore, $(G,.,\leq_\tau)$ is a po-group so $(G,\circ_P,\leq_\tau)$ is a po-hypergroup.

In the following we recal an special class of hypergroups that studied by Chvalina [1] and called ordering hypergroups.

Definition 3.3 [1] A hypergroup $(H,\circ)$ is said to be a quasi-ordering hypergroup, if for every $x,y\in H$, we have $x\in x^2=x^3$ *and* $x\circ y=x^2\cup y^2$.

Moreover, if $x^2=y^2$ implies $x=y$ for every $x,y\in H$, then $(H,\circ)$ is called an ordering hypergroup.

Let $(H,\circ)$ be a quasi-ordering hypergroup. Then we define a binary relation $\leq_\circ$ on H as follows: $x\leq_\circ y\Leftrightarrow x\in y^2$, *for all* $x,y\in H$.

Lemma 3.4 Let $(H,\circ)$ be a quasi-ordering hypergroup. Then the binary relation $\leq_\circ$ is a partial preorder on H. Furthermore, if $(H,\circ)$ is an ordering hypergroup, then $\leq_\circ$ is a partial ordere on H.

Proof. Suppose that $(H,\circ)$ is a quasi-ordering hypergroup. Then for every $x\in H$, from $x\in x^2$ we have $x\leq_\circ x$ so $\leq_\circ$ is reflexive. If $x,y,z\in H$ such that $x\leq_\circ y$ and $y\leq_\circ z$, then $x\in y^2$ and $y\in z^2$. So we have $x\in y^2\subseteq z^2\circ z^2=z^2$, thus $x\leq_\circ z$. Hence $\leq_\circ$ is transitive. Therefore, $\leq_\circ$ is a partial preorder on H.

Suppose that $(H,\circ)$ is an ordering hypergroup. If $x\leq_\circ y$ and $y\leq_\circ x$ for $x,y\in H$, then $x\in y^2$ and $y\in x^2$. So $y^2\subseteq x^2\circ x^2=x^2\subseteq y^2\circ y^2=y^2$. Thus $x^2=y^2$, so by hypothesis we get $x=y$. Hence $\leq_\circ$ is an antisymmetric relation. Therefore, $\leq_\circ$ is a partial ordere on H.

Theorem 3.5 Let $(H,\circ)$ be a quasi-ordering hypergroup. Then $(H,\circ,\leq_\circ)$ is a partially preordered hypergroup. Furthermore, if $(H,\circ)$ is an ordering hypergroup, then $(H,\circ,\leq_\circ)$ is a po-hypergroup.

Proof. Suppose that $x\leq_\circ y$, for $x,y\in H$. Then for every $a\in H$ we should prove that $a\circ x\leq_\circ a\circ y$. If $t\in a\circ x=a^2\cup x^2$, then either $t\in a^2$ or $t\in x^2$. If $t\in a^2$, then $t\leq_\circ t\in a\circ y$. If $t\in x^2$, then we have $t\leq_\circ x\leq_\circ y$. So $t\leq_\circ y\in a\circ y$. Therefore, if $(H,\circ)$ is a quasi-ordering (ordering) hypergroup, then $(H,\circ,\leq_\circ)$ is a partially preordered (ordered) hypergroup.

Next, we want to find some conditions on a hypergroup such that becomes a po-hypergroup.

Theorem 3.6 Let $(H,\circ)$ be a hypergroup such that there exists an element $0\in H$ and the following conditions hold:

(1) $0\circ 0=\{0\}$; (2) $\{0,x\}\subseteq x\circ 0$ for every $x\in H$;

(3) If $x\circ 0=y\circ 0$, then $x=y$ for every $x,y\in H$.

Then there exists a binary relation $\leq$ on H such that $(H,\circ,\leq)$ is a po-hypergroup.

Proof. We define a partially ordered relation $\leq$ on H as follows:

$$x\leq y\Leftrightarrow x\in y\circ 0,\ \textit{for every}\ x,y\in H.$$

By condition (2), $x\in x\circ 0$ so $x\leq x$ for every $x\in H$. Thus $\leq$ is reflexive.

If $x\leq y$ and $y\leq x$, for $x,y\in H$, then $x\in y\circ 0$ and $y\in x\circ 0$. So we have

$$x\circ 0\subseteq (y\circ 0)\circ 0=y\circ(0\circ 0)=y\circ 0\subseteq (x\circ 0)\circ 0=x\circ(0\circ 0)=x\circ 0.$$

Thus $x\circ 0=y\circ 0$ and by condition (3), $x=y$. So $\leq$ is antisymmetric.

If $x\leq y$ and $y\leq z$ for $x,y,z\in H$, then $x\in y\circ 0$ and $y\in z\circ 0$. So $x\in y\circ 0\subseteq (z\circ 0)\circ 0=z\circ 0$. Thus $x\leq z$ and $\leq$ is transitive. Therefore, $\leq$ is a partially ordered relation on H.

Now, suppose that $x\leq y$ for $x,y\in H$. Then for every $a\in H$ we should prove that $a\circ x\leq a\circ y$. Let $t\in a\circ x$. Since $x\leq y$ so $x\in y\circ 0$. Thus

$$t\in a\circ x\subseteq a\circ(y\circ 0)=(a\circ y)\circ 0.$$

So there exists $s \in a \circ y$ such that $t \in s \circ 0$. Hence $t \leq s \in a \circ y$ and the proof is complete.

Example 10 Let $H = \{0, x, y\}$ and the hyperoperation $\circ$ defined in the following table:

$\circ$	0	x	y
0	$\{0\}$	$\{0, x\}$	H
x	$\{0, x\}$	H	H
y	H	H	H

Then $(H, \circ)$ satisfies the pervious theorem so $(H, \circ, \leq)$ is a po-hypergroup, where $\leq = \{(0,0), (0, x), (0, y), (x, x), (x, y), (y, y)\}$.

In [12], Rosenberg associated to each binary relation R on a non-empty set H is a partial hypergroupoid $H_R = (H, \circ_R)$, where for any $x, y \in H$ the hyperoperation $\circ_R$ defined as follows: $x \circ x = L_x = \{z \mid (x, z) \in R\}$ *and* $x \circ y = x \circ x \cup y \circ y$.

Let R be a binary relation on a non-empty set H. Then an element $x \in H$ is called an outer element of R if there exists $h \in H$ such that $(h, x) \notin R$.

Theorem 3.7 [12] Let R be a binary relation on a non-empty set H. Then H_R is a hypergroup if and only if

(1) R has full domain; (2) R has full range; (3) $R \subseteq R^2$;

(4) If $(a, x) \in R^2$, then $(a, x) \in R$, whenever x is an outer element of R.

Lemma 3.8 Let R be a partiallyl preordered relation on a non-empty set H. Then H_R is a hypergroup.

Proof. We verify the conditions of previous theorem. Since R is reflexive so the conditions (1) and (2) hold. From reflexivity and transitivity of R we conclude that $R^2 = R$ so the conditions (3) and (4) hold. Therefore, $(H_R, \circ_R, R)$ is a hypergroup.

Theorem 3.9 Let R be a partially ordered relation on a non-empty set H. Then $(H_R, \circ_R, R)$ is a po-hypergroup.

Proof. By previous lemma, H_R is a hypergroup. Now, we prove the monotone condition for H_R. Suppose that xRy for $x, y \in H$. We should prove that $a \circ_R xRa \circ_R y$ for every $a \in H$. Let $t \in a \circ_R x = L_a \cup L_x$. Then $t \in L_a$ or $t \in L_x$. In the former case, tRa and since $a \in a \circ_R y$ the result holds. In the later case, we have tRx since xRy so $tRy \in a \circ_R y$. Therefore, $(H_R, \circ_R, R)$ is a po-hypergroup.

4. Hyperstructures associated to Γ-semigroups

In this section we associated to each Γ-groupoid a Γ-hypergroupoid and prove some results. Let us recall some notations.

The notions of an ordered Γ-groupoid (shortly, po-Γ-groupoid) and an ordered Γ-semigroup (shortly, po-Γ-semigroup) were defined by Sen and Seth in [14]. A po-Γ-groupoid (po-Γ-semigroup) is a Γ-groupoid (Γ-semigroup) S together with an order relation $\leq$ on S such that $x \leq y$ implies $a\gamma x \leq a\gamma y$ and $x\gamma a \leq y\gamma a$, for all $x, y, a \in S$ and $\gamma \in \Gamma$.

Definition 4.1 Let S be a Γ-groupoid. Then we define the hyperoperation $\circ_\Gamma$ on S as follows:

$$x \circ_\Gamma y = x\Gamma y = \{x\gamma y \mid \gamma \in \Gamma\} \textit{ for all } x, y \in S.$$

Then $(S, \circ_\Gamma)$ is called the hypergroupoid associated to Γ-semigroup S and is denoted by S_Γ.

Lemma 4.2 If S is a Γ-semigroup, then S_Γ is a semihypergroup.

Proof. We prove the associativity condition for S_Γ. For every $x, y, z \in S_\Gamma$ we have

$$x \circ_\Gamma (y \circ_\Gamma z) = x \circ_\Gamma (y\Gamma z) = x\Gamma(y\Gamma z) = (x\Gamma y)\Gamma z = (x \circ_\Gamma y) \circ_\Gamma z.$$

So S_Γ is a semihypergroup.

Lemma 4.3 Let S be a Γ-semigroup. Then the following assertions hold:

(1) If S is a commutative Γ-semigroup, then S_Γ is a commutative semihypergroup.

(2) If S_α is a monoid with identity element e_α, for $\alpha \in \Gamma$, then e_α is a unit element of S_Γ.

(3) S is a regular Γ-semigroup if and only if S_Γ is a regular semihypergroup.

(4) The non-empty subset I of S is an ideal of S if and only if I is a hyperideal of S_Γ.

(5) S is simple if and only if S_Γ is simple.

Proof. (1) It is evident.

(2) For every $x \in S$ we have $x = x\alpha e_\alpha = e_\alpha \alpha x \in x \circ_\Gamma e_\alpha \cap e_\alpha \circ_\Gamma x$ so e_α is a unit element of S_Γ.

(3) The Γ-semigroup S is regular if and only if for every $x \in S$ there exist $y \in S$ and $\alpha, \beta \in \Gamma$ such that $x = x\alpha y\beta x$ if and only if there exists $y \in S$ such that $x \in x \circ_\Gamma y \circ_\Gamma x$ if and only if S_Γ is a regular semihypergroup.

(4) Suppose that I is an ideal of S. Then for every $a \in I$ and $s \in S$ we have $s \circ_\Gamma a = s\Gamma a \subseteq I$. Similarly $a \circ_\Gamma s \subseteq I$. So I is a hyperideal of S_Γ. Conversely, suppose that I is a hyperideal of S_Γ. Then for every $a \in I$, $s \in S$ and $\gamma \in \Gamma$ we have $s\gamma a \subseteq s \circ_\Gamma a \subseteq I$. Similarly $a\gamma s \subseteq I$. So I is an ideal of S.

(5) It gets from (4).

Proposition 4.4 If S is a Γ-group, then S_Γ is a hypergroup.
Proof. It is sufficient to check the reproduction axiom for semihypergroup S_Γ. Since S is a Γ-group so S_α is a group for every $\alpha \in \Gamma$. Then for every $x \in S_\Gamma$ we have $S = x\alpha S \subseteq x \circ_\Gamma S \subseteq S$. So $x \circ_\Gamma S = S$. Similarly, we get $S \circ_\Gamma x = S$. Thus $(S, \circ_\Gamma)$ is a hypergroup.
Proposition 4.5 Let S be a commutative Γ-semigroup and M be a minimal ideal of S. Then $(M, \circ_\Gamma)$ is a hypergroup.
Proof. For every $x \in M$ since $x \circ_\Gamma M = x\Gamma M \subseteq M$ so $x \circ_\Gamma M$ is an ideal of S contained in M. Since M is a minimal ideal of S so we have $x \circ_\Gamma M = M$. By a similar way we have $M \circ_\Gamma x = M$. Thus the reproduction axiom holds. Therefore, $(M, \circ_\Gamma)$ is a hypergroup.
Theorem 4.6 Let $(S, \leq)$ be a po-Γ-semigroup. Then $(S, \circ_\Gamma, \leq)$ is a po-semihypergroup.
Proof. Suppose that $x \leq y$ for $x, y \in S_\Gamma$. Then for every $a \in S_\Gamma$ we should prove that $a \circ_\Gamma x \leq a \circ_\Gamma y$. If $t \in a \circ_\Gamma x$, then there exists $\gamma \in \Gamma$ such that $t = a\gamma x$. Now, since S is a po-Γ-semigroup we have $t = a\gamma x \leq a\gamma y \in a \circ_\Gamma y$. It is complete the proof.
Theorem 4.7 Let S be a Γ-semigroup and R be a congruence relation on S. Then R is a regular relation on semihypergroup S_Γ. Furthermore, $(S_\Gamma/R, \otimes)$ is a semihypergroup, where the hyperoperation $\otimes$ defined on S_Γ/R as follows:

$$\overline{x} \otimes \overline{y} = \{\overline{z} : z \in x \circ_\Gamma y\} \text{ for every } \overline{x}, \overline{y} \in S_\Gamma/R.$$

Proof. Suppose that $x, y \in S_\Gamma$ such that xRy. Then for every $a \in S_\Gamma$ we should prove that $a \circ_\Gamma x\overline{R}a \circ_\Gamma y$. If $t \in a \circ_\Gamma x$, then there exists $\gamma \in \Gamma$ such that $t = a\gamma x$. Since R is a congruence relation on S we have $a\gamma xRa\gamma y \in a \circ_\Gamma y$. Similarly, if $t \in a \circ_\Gamma y$, then there exists $\gamma \in \Gamma$ such that $t = a\gamma y$. Since R is a congruence relation on S we have $a\gamma yRa\gamma x \in a \circ_\Gamma x$. Thus R is a regular relation on S_Γ. Now, by Theorem 2.2, $(S_\Gamma/R, \otimes)$ is a semihypergroup.

Let S be a Γ-semigroup and R be a congruence relation on S. Let $\dot{\Gamma} = \{\dot{\gamma} : \gamma \in \Gamma\}$. Then for every $\overline{x}, \overline{y} \in S/R$ the operation $\dot{\gamma}$ defined on S/R as follows: $\overline{x}\dot{\gamma}\overline{y} = \overline{x\gamma y}$. Suppose that $x_1 Rx$ and $y_1 Ry$. Since R is a congruence relation on S we have $x_1\gamma y_1 Rx\gamma y_1$ and $x\gamma y_1 Rx\gamma y$ for every $\gamma \in \Gamma$. So $\overline{x_1\gamma y_1} = \overline{x\gamma y}$. Thus the operation $\dot{\gamma}$ is well-defined for every $\gamma \in \Gamma$. Also, S/R

satisfies the associativity condition. Since for every $\overline{x}, \overline{y}, \overline{z} \in S/R$ and $\dot{\alpha}, \dot{\beta} \in \dot{\Gamma}$ we have

$$\overline{x}\dot{\alpha}(\overline{y}\dot{\beta}\overline{z}) = \overline{x}\dot{\alpha}(\overline{y\beta z}) = \overline{x\alpha(y\beta z)} = \overline{(x\alpha y)\beta z} = (\overline{x\alpha y})\dot{\beta}\overline{z} = (\overline{x}\dot{\alpha}\overline{y})\dot{\beta}\overline{z}.$$

Therefore, S/R is a $\dot{\Gamma}$-semigroup.

Theorem 4.8 Let S be a Γ-semigroup and R be a congruence relation on S. Then the semihypergroups $(S_\Gamma/R, \otimes)$ and $((S/R)_{\dot{\Gamma}}, \circ_{\dot{\Gamma}})$ are same.

Proof. Since R is a congruence relation on S, by pervious theorem, we have S_Γ/R is a semihypergroup. Also, by the above argument S/R is a $\dot{\Gamma}$-semigroup. Thus by Lemma 4.2, $(S/R)_{\dot{\Gamma}}$ is a semihypergroup. Also, for every $\overline{x}, \overline{y} \in S/R$ we have

$$\overline{x} \otimes \overline{y} = \{\overline{x\gamma y} : \gamma \in \Gamma\} == \{\overline{x}\dot{\gamma}\overline{y} : \dot{\gamma} \in \dot{\Gamma}\} = \overline{x} \circ_{\dot{\Gamma}} \overline{y}.$$

REFERENCES

[1] *J. Chvalina,* Commutative hypergroups in the sense of Marty and ordered Sets. Gen. Alg. and Ordered Sets, Proc. Inter. Conf., Olomouc, 19-30 (1994).

[2] *J. Chvalina and L.Chvalinová*, State hypergroups of automata. Acta Math. Inform. Univ. Ostraviensis, **4**: 105-120 (1996).

[3] *J. Chvalina and Š. Hošková*, Abelizations of weakly associative hyperstructures based on their direct squares. Acta Math. Inform. Univ. Ostraviensis, **11**: 11-23 (2003).

[4] *J. Chvalina and J. Moučka*, Hypergroups determined by orderings with regular endomorphism monoids. Ital. J. Pure Appl. Math., **16**: 227-242 (2004).

[5] *P. Corsini, V. Leoreanu-Fotea*, Applications of hyperstructures theory. Advanced in Mathematics, Kluwer Academic Publisher, (2003).

[6] *P. Corsini*, Prolegomena of hypergroup theory. Aviani Editore, (1993).

[7] *B. Davvaz, V. Leoreanu -Fotea*, Hyperring Theory and Applications. International Academic Press, USA, (2008).

[8] *Š. Hošková*, Representation of quasi-order hypergroups. Glob. J. Pure Appl. Math., **1**: 173-176 (2005).

[9] *Š. Hošková***,** Upper order hypergroups as a reflective subcategory of subquasiorder hypergroups. Ital. J. Pure Appl. Math., **20**: 215-222 (2006).

[10] *Š. Hošková and J. Chvalina*, Discrete transformation hypergroups and transformation hypergroups with phase tolerance space. Discrete Math., **308**: 4133-4143 (2008).

[11] *F. Marty*, Sur une généralization de la notion de groupe. 8th congres Math. Scandinaves, Stockholm, 45-49 (1934).

[12] *I.G. Rosenberg*, Hypergroups and join spaces determined by relations. Italian J. Pure Appl. Math., **4**: 93-101 (1998).

[13] *M. K.Sen and N. K. Saha*, On Γ-semigroup I. Bull. Calcutta Math. Soc., **78**: 180–186 (1986).

[14] *M. K.Sen and A.Seth*, On po-Γ-semigroups. Bull. Calcutta Math. Soc., **85**: 445–450 (1993).

[15] *T. Vougiouklis*, Hyperstructures and their representations. Hadronic Press, Inc, 115, Palm Harber, USA, (1994).