INFLUENCE OF THE DISCRETIZATION STEP ON THE POSITIVITY OF CONFORMABLE FRACTIONAL LINEAR SYSTEMS

Kamel Benuettou[1], Mohammed Amine Ghezzar[2], Djillali Bougada[3]

The effectiveness of this work lies in the influence of the discretization step on the positivity of the one-dimensional conformable fractional linear systems. A piecewise constant approximation method which converts a continuous fractional order derivative into a discrete time equation is presented as an alternative approach. It aims at investigating whether, how and when this step affects the positive continuous times models. To accomplish this research, a new test was developed and implemented in which the positivity conditions of the considered system was examined both before and after being exposed to the sampling step. i.e., under which conditions the one-dimensional discrete-time linear system obtained by discretization from the one-dimensional continuous time linear systems will be also positive if the one-dimensional continuous time linear system is positive. The new method presented is suggested to test the influence of the discretization step on the positivity of this class of systems. Finally, we discuss some numerical examples to illustrate the validity of the provided results.

Keywords: Fractional Linear systems, Conformable derivative, Positivity, Discretization, Discrete systems.

MSC2020: 37M05, 37M1, 44AXX, 15B4.

1. Introduction

In recent years, there has been a growing interest in fractional systems that are subjected to positivity constraints on their dynamical variables. Positive fractional systems appear naturally in a variety of control theory applications, including for examples: circuits systems and signal processing, image analysis, heat transfer equation, digital filter, automatic, Boolean networks, neural computing and applications and population dynamics [5, 7, 10, 12, 14, 16]. The most known fractional derivatives do not satisfy the fundamental properties of the classical derivative. To overcome certain of these problems, Khalil et al. in [1, 11, 15, 17] have introduced the concept of a new class of fractional derivatives known as conformable fractional derivatives which offers two improvements over the classical derivatives in fractional calculus, such as Caputo or Riemann-Liouville. First, the new concept of conformable fractional derivative serves the majority of the properties of the mathematical essential derivative, including Rolle's theorem ,linearity , the fractional derivative of a constant function, product rule, quotient rule, chain rule, and power rule [11, 15]. Additionally, the conformable derivative is useful for modelling many physical problems. Note that

[1]Abdelhamid Ibn Badis University Mostaganem. ACSY Team-Laboratory of Pure and Applied Mathematics, Department of Mathematics and Computer Science, Faculty SEI-BP 227/118 Mostaganem 27000, Algeria, e-mail: kamel.benyattou.etu@univ-mosta.dz

[2]National Higher School of Mathematics. ACSY Team-Laboratory of Pure and Applied Mathematics, e-mail: amine.ghezzar@nhsm.edu.dz

[3]Abdelhamid Ibn Badis University Mostaganem. ACSY Team-Laboratory of Pure and Applied Mathematics, Department of Mathematics and Computer Science, Faculty SEI-BP 227/118 Mostaganem 27000, Algeria, e-mail: djillali.bouagada@univ-mosta.dz

the differential equations with conformable fractional derivatives are simpler to solve computationally compared to those using Caputo or Riemann-Liouville fractional derivatives [1, 3, 11, 17, 19]. Various applications of the conformable fractional derivative have been investigated in many areas, such as: artificial neural networks, thermal and environmental processes, Laplace's Equation, diversity of wave structures, digital and image processing, and chemical reaction problems [8, 9, 11, 18, 20, 21].

The positivity conditions in dynamical systems and control theory have been studied and illustrated by several practical examples and some applications in [5, 10, 14]. Indeed, positive systems have attracted several researchers, Kaczorek in [11] proposed an important result concerning the positivity and stability conditions for the conformable fractional systems, Lam et al. in [16] have presented an innovative contributions on positive switched systems and stochastic positive systems. Meanwhile, The influence of discretization steps on positivity for a certain class of fractional continuous-discrete time systems has been presented in [4, 7, 11] using the finite difference method to discretize the Caputo fractional derivative. Many of the fractional models having positive fractional linear systems behaviour can be found in control theory, engineering, mechanics, medicine, chemical reaction problems, economics, management, and electrical circuits [11, 12, 13, 14].

The paper is organized as follows. In subsection 3.**1** we studies the discretization and the solution of the conformable systems using the piecewise constant approximation. Then, the main purpose of this work is to propose the necessary and sufficient conditions for preserving the positivity of the model obtained by the discretization. Eventually, some numerical examples are provided to illustrate the effectiveness of our results.

2. Preliminaries

There are several types of fractional derivatives in mathematics, including Caputo, Riemann-Liouville, and Hadamard, as well as others. Recently, a new definition of the fractional derivative known as the conformable derivative has been presented by Khalil and Kaczorek [11, 15]. In the following we recall some needed definitions and properties of the considered fractional derivative based on [1, 11, 15].

Definition 2.1. [1] *Let x be a function $x : [0, +\infty[\to \mathbb{R}$. The conformable derivative of the function x of order α where $0 < \alpha \leq 1$ is defined by the following relation*

$$T^\alpha x(t) = \lim_{\varepsilon \to 0} \frac{x(t + \varepsilon t^{1-\alpha}) - x(t)}{\varepsilon} \tag{1}$$

If (1) exists, then the function x is called $\alpha-$differentiable.

Theorem 2.1. [15] *Let $x(t)$ and $y(t)$ be defined on $[0, +\infty[$ $\alpha-$differentiable, where $0 < \alpha \leq 1$. Then for all a, b real numbers, we have the following relations*

$$T^\alpha[ax(t) + by(t)] = aT^\alpha x(t) + bT^\alpha y(t) \qquad \forall t \in [0, +\infty[\tag{2}$$

$$T^\alpha[x(t)y(t)] = x(t)T^\alpha y(t) + y(t)T^\alpha x(t) \tag{3}$$

$$T^\alpha \left[\frac{x(t)}{y(t)}\right] = \frac{T^\alpha[x(t)]y(t) - T^\alpha[y(t)]x(t)}{[y(t)]^2} \tag{4}$$

$$T^\alpha[t^q] = qt^{q-\alpha} \tag{5}$$

$$T^\alpha[e^{qt}] = qt^{1-\alpha}e^{qt} \tag{6}$$

3. Discretization and positivity of the conformable fractional systems

In this section, the positivity problem for the discrete-time systems is developed. Necessary and sufficient conditions are given which guarantee whether the discrete-time system obtained by discretization retains positivity when the continuous system is assumed to be positive. New results on the influence of discretization step on the positivity of conformable fractional linear systems are then established.

We consider in the following the conformable fractional continuous-time linear systems described by

$$T^{\alpha}x(t) = Ax(t) + Bu(t) \tag{7}$$

$$y(t) = Cx(t) + Du(t) \tag{8}$$

where $T^{\alpha}x(t)$ stands for the conformable fractional derivative of the function $x(t)$, $0 < \alpha \leq 1$, $t \in \mathbb{R}^{+}$, $x(t) \in \mathbb{R}^{n}$ is the state vector, $u(t) \in \mathbb{R}^{m}$ is the input vector, $y(t)$ the vector of the output. $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$,$C \in \mathbb{R}^{q\times n}$, $D \in \mathbb{R}^{q\times m}$, the boundary condition $x(0)$ is given.

3.1. Discretization and solution of conformable system

This section is designed to present some needed definitions and results that concern the problem of discretization of the conformable fractional continuous time linear systems. In this work, a piecewise constant approach is applied to discretize the considered model, contrary to what is mentioned in other research, which uses the finite difference approach to discretize the fractional derivative [7, 12].

Definition 3.1. [11] *The systems* (7) *and* (8) *are said to be positive if and only if all the states and all the outputs are positive:* $x(t) \in \mathbb{R}^{n}_{+}, y(t) \in \mathbb{R}^{n}_{+}$ *for all* $x_0 \in \mathbb{R}^{n}_{+}$ *et* $u_0 \in \mathbb{R}^{n}_{+}$.

Definition 3.2. [11] $A = [a_{ij}]$ *is a Metzler matrix if* $a_{ij} \geq 0$ *for* $i \neq j$ *and* $i, j = \overline{1 \cdots n}$, *the set of Metzler matrices will be denoted by* M_n.

Theorem 3.1. [11] *The systems* (7) *and* (8) *are said to be positive if and only if:*

$$A \in M_n,\ B \in \mathbb{R}^{n\times m}_{+},\ C \in \mathbb{R}^{q\times n}_{+}\ ,D \in \mathbb{R}^{q\times m}_{+} \tag{9}$$

Based on [19], the following discretization of the systems (7), (8) is given.

Theorem 3.2. [19] *Consider the following conformable derivative system*

$$T^{\alpha}x(t) = f(x(t)), \quad 0 < t \leq T,\, x(0) = x_0 \tag{10}$$

using the conformable discretization by piecewise constant approximation, we obtain the following discretization of (10)

$$\begin{aligned} x_{n+1} &= x_n + \frac{h^{\alpha}}{\alpha} f(x_n) \\ y_n &= Cx_n + Dx_n \end{aligned} \tag{11}$$

where h *is the steps of discretization and* $t = nh$, *with* $n \in \mathbb{N}^{*}$

By applying the theorem (3.2) to the systems (7) and (8) we deduce the next result.

Theorem 3.3. *Let's consider* $h > 0$. *The systems* (7) *and* (8) *are discretized to the corresponding discrete-time systems defined by*

$$x_{n+1} = \widetilde{A}x_n + \widetilde{B}u_n \tag{12}$$

$$y_n = Cx_n + Du_n \tag{13}$$

with

$$\widetilde{A} = \left[\frac{h^{\alpha}}{\alpha}A + I\right] \text{ and } \widetilde{B} = \frac{h^{\alpha}}{\alpha}B \tag{14}$$

Proof. We take $f(x(t)) = Ax(t) + Bu(t)$ in the equation (10), and by the formula (11) we deduce that

$$\begin{aligned} x_{n+1} =& x_n + \frac{h^\alpha}{\alpha}\left[Ax_n + Bu_n\right] \\ x_{n+1} =& \left[\frac{h^\alpha}{\alpha}A + I\right]x_n + \frac{h^\alpha}{\alpha}Bu_n \end{aligned} \tag{15}$$

If we put

$$\widetilde{A} = \left[\frac{h^\alpha}{\alpha}A + I\right] \; and \; \widetilde{B} = \frac{h^\alpha}{\alpha}B \tag{16}$$

Hence

$$x_{n+1} = \widetilde{A}x_n + \widetilde{B}u_n \tag{17}$$

$$y_n = Cx_n + Du_n \tag{18}$$

which completes the proof of the proposed theorem. □

Remark 3.1. *We note that this discretization approach transforms a continuous fractional order system to a classical discrete model.*

The solution of the discrete system obtained by discretization is given by the following theorem.

Theorem 3.4. [2] *The solution of system* (12) *is given by*

$$x_n = \widetilde{A}^n x_0 + \sum_{i=0}^{n-1} \widetilde{A}^{n-1-i}\widetilde{B}u_i \tag{19}$$

and its output

$$y_n = Cx_n + Du_n \tag{20}$$

3.2. Influence of discretization step on the positivity

Our study is based on the following question: if the systems (7)-(8) are positive, does that mean that the systems (12)-(13) remain positive? If so, what are the proposed conditions? The following theorem is given to help us with the concerned condition.

Theorem 3.5. *The systems described by the equations* (12) *and* (13) *are positive if and only if*

$$\widetilde{A} \in \mathbb{R}_+^{n\times n},\; \widetilde{B} \in \mathbb{R}_+^{n\times m},\; C \in \mathbb{R}+^{q\times n}\;, D \in \mathbb{R}_+^{q\times m} \tag{21}$$

Proof. (1) Sufficient condition:

If $A \in R_+^{n\times n}$, $B \in R_+^{n\times m}$, $C \in R_+^{q\times n}$, $D \in R_+^{q\times m}$ and $u_i \in R_+^m$ with all the initial conditions are positive, then $\widetilde{A}^n x_0$ is positive. Similarly, for $\sum_{i=0}^{n-1} \widetilde{A}^{n-1-i}\widetilde{B}u_i$, we deduce that x_n is positive.

(2) Necessary condition:
We assume that $x_0 = e_i$ (the i^{th}column of the matrix identity and $u_n = 0$ of the equation (12) we find,

$$x_1 = \widetilde{A}x_0 = \widetilde{A}e_i = \widetilde{a}_i \tag{22}$$

Knowing that $\widetilde{a}_i$ is the i^{th} column of the matrix $\widetilde{A}$, so $\widetilde{a}_i = x_1 \in R_+^n$. Continuing our reasoning for the other columns, we deduce that $\widetilde{A}$ is positive.

In the same way, for $x_0 = 0$ we find $\widetilde{B}u_0 \in R_+^n$ which implies that $\widetilde{B} \in R_+^n$ since $u_0 \in R_+^n$ is arbitrary.

For the output y_n, if $u_n = 0$ then : $y_0 = CX_0 \in R_+^q$ and $C \in R_+^{q\times n}$ since x_0 is arbitrary. and if $x_0 = 0$ then : $y_0 = Du_0 \in R_+^q$ and $D \in R_+^{q\times m}$ because u_0 is arbitrary. □

By the following theorem, we will show the conditions that guarantee the positivity of the system (12) and (13).

Theorem 3.6. *Let's consider $h > 0$ be the discretization step. If the conditions of theorem 3.1 are satisfied, then one of the following two cases is satisfied*

1): *If A is a Metzler positive matrix, the systems described by the equations* (12) *and* (13) *remains positive.*

2): *If A is Metzler non positive matrix the systems described by the equations* (12) *and* (13) *remains positive if and only if*

$$0 < h \leq \left(\frac{\alpha}{\max|a_{ii}|}\right)^{\frac{1}{\alpha}} \tag{23}$$

where a_{ii} are the diagonal entries of the matrix A.

Proof. Let us consider $h > 0$ and suppose that the system described by the equations (7) and (8) is positive, this if and only if,

$$A \in \mathbb{R}_+^{n\times n}, B \in \mathbb{R}_+^{n\times m},\ C \in \mathbb{R}_+^{q\times n}\ , D \in \mathbb{R}_+^{q\times m}$$

with A is a Metzler matrix. Since $0 < \alpha \leq 1$ and $h > 0$ and using the discretization of our systems, then using the theorem 3.5 the discrete-time systems (12) and (13) remains positive if and only if the matrix $\widetilde{A} \in \mathbb{R}_+^{n\times n}$.

We have $\widetilde{A} = \frac{h^\alpha}{\alpha}A + I$ with A is a Metzler matrix. Then two cases need to be considered,

(1) If A is Metzler positive, $\widetilde{A} = \frac{h^\alpha}{\alpha}A + I$ is positive, then the first relation of the theorem is deduced.

(2) If A is Metzler non-positive, there is at least one diagonal entry of the matrix A which is strictly negative, hence the matrix A is not necessarily positive.

(a) Necessary condition: Since all the non diagonal entries of the matrix A are positive and those of the matrix $\frac{-\alpha}{h^\alpha}I$ are all equal zero. So we need to compare the diagonal elements of A and the value $\frac{-\alpha}{h^\alpha}$. They are two cases,

* For a first case: $a_{ii} \geq 0 \Rightarrow a_{ii} \geq 0 \geq \frac{-\alpha}{h^\alpha}$.

** For the second case: $0 \geq a_{ii} \geq \frac{-\alpha}{h^\alpha}$ the diagonal elements of $\widetilde{A}$ are positive thus

$$\frac{h^\alpha}{\alpha}a_{ii} + 1 \geq 0 \tag{24}$$

$$\frac{h^\alpha}{\alpha}a_{ii} \geq -1 \tag{25}$$

$$\frac{h^\alpha}{\alpha}(-a_{ii}) \leq 1 \tag{26}$$

$$\frac{h^\alpha}{\alpha}|-a_{ii}| \leq 1 \tag{27}$$

this inequality holds true for any value of i, it also holds true for $max|a_{ii}|$.

$$\frac{h^\alpha}{\alpha} max|a_{ii}| \leq 1 \tag{28}$$

$$h^\alpha \leq \frac{\alpha}{max|a_{ii}|} \tag{29}$$

$$h \leq \left(\frac{\alpha}{max|a_{ii}|}\right)^{\frac{1}{\alpha}} \tag{30}$$

(b) Sufficient condition: Suppose that A is a Metzler matrix with at least one diagonal strictly negative coefficient and prove that $\widetilde{A}$ is positive. As a result, the diagonal coefficients of the matrix A are positive if $a_{ii} \geq 0$. For the strictly negative diagonal coefficients of the matrix A verifying

$$h \leq \left(\frac{\alpha}{max|a_{ii}|}\right)^{\frac{1}{\alpha}} \tag{31}$$

in this case we have

$$h^\alpha \leq \frac{\alpha}{max|a_{ii}|} \tag{32}$$

therefore

$$max|a_{ii}| \ \frac{h^\alpha}{\alpha} \leq 1 \tag{33}$$

which gives us,

$$|a_{ii}| \ \frac{h^\alpha}{\alpha} \leq max|a_{ii}| \ \frac{h^\alpha}{\alpha} \leq 1 \tag{34}$$

Hence

$$\frac{h^\alpha}{\alpha} \ |a_{ii}| \leq 1 \tag{35}$$

and since $a_{ii} < 0$

$$-\frac{h^\alpha}{\alpha} \ a_{ii} \leq 1 \tag{36}$$

We get

$$\frac{h^\alpha}{\alpha} \ a_{ii} \geq -1 \tag{37}$$

thus

$$\frac{h^\alpha}{\alpha} \ a_{ii} + 1 \geq 0 \tag{38}$$

We deduce that all the diagonal coefficients of the matrix $\widetilde{A}$ is positive.

□

Example 3.1. *Consider the systems* (7) *and* (8) *for* $\alpha = 0.5$ *and system matrices*

$$A = \begin{bmatrix} -8 & 2 \\ 1 & -7 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ u(t) = 1$$

Using a positivity theorem from [11], *we find that our system is positive, and since A is a non-positive Metzler matrix, we must ensure that the value step verifies the positivity of the system obtained by discretization*

$$h \leq \left(\frac{0.5}{max|a_{ii}|}\right)^{\frac{1}{0.5}} \tag{39}$$

then

$$0 < h \leq 0.0039 \tag{40}$$

(1) *Let* $\epsilon_1 = 0.03$, $h_1 = 0.0039 + \epsilon_1 = 0.0339$. *The systems obtained by discretization with this value step* h_1 *is not positive because*

$$\widetilde{A} = \begin{bmatrix} -1.9462 & 0.7365 \\ 0.3683 & -1.5779 \end{bmatrix}$$

is not positive. Hence, the following figure shows us that the solution is not positive. (see Figure 1)

(2) *Let* $\epsilon_2 = 0.001$, $h_2 = 0.0039 - \epsilon_2 = 0.0029$. *the system obtained by discretization with this value step* h_2 *is positive because*

$$\widetilde{A} = \begin{bmatrix} 0.1374 & 0.2156 \\ 0.1078 & 0.2453 \end{bmatrix}$$

is positive. The figure below illustrates that the solution is positive. (see Figure 2)

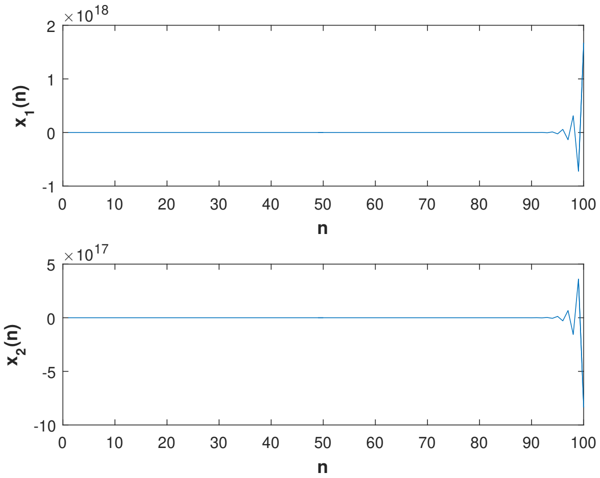

FIGURE 1. State vector of the system from Example 3.1 with $h = 0.0339$.

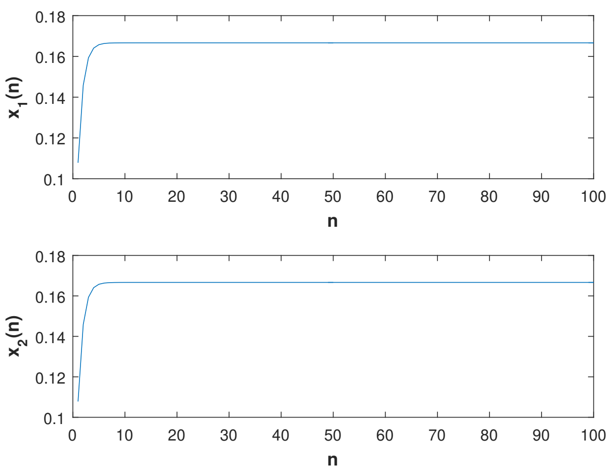

FIGURE 2. State vector of the system from Example 3.1 with $h = 0.0029$.

Example 3.2. *Consider the systems* (7) *and* (8) *for* $\alpha = 0.5$ *and system matrices*

$$A = \begin{bmatrix} -10 & 0 & 0 \\ 1 & -13 & 0 \\ 2 & 0 & -12 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad u(t) = 1$$

Applying a positivity theorem in [11], *we conclude that our systems are positive and since* A *is non-positive Metzler matrix , we must take the step* h *that verifies the positivity of the system obtained by discretization*

$$h \leq \left(\frac{0.5}{max|a_{ii}|} \right)^{\frac{1}{0.5}} \tag{41}$$

then

$$0 < h \leq 0.0015 \tag{42}$$

(1) *Let* $\epsilon_1 = 0.01$, $h_1 = 0.0015 + \epsilon_1 = 0.0115$. *the system obtained by discretization with this value step* h_1 *is not positive because*

$$\widetilde{A} = \begin{bmatrix} -1.1428 & 0 & 0 \\ 0.2143 & -1.7857 & 0 \\ 0.4286 & 0 & -1.5714 \end{bmatrix}$$

is not positive. The figure below proves that the solution is not positive (see Figure 3).

(2) *Let* $\epsilon_2 = 0.001$, $h_2 = 0.0015 - \epsilon_2 = 0.0005$. *the system obtained by discretization with this value step is positive because*

$$\widetilde{A} = \begin{bmatrix} 0.5621 & 0 & 0 \\ 0.0438 & 0.4308 & 0 \\ 0.0876 & 0 & 0.4746 \end{bmatrix}$$

is positive. The figure below confirms that the solution is not positive (see Figure 4).

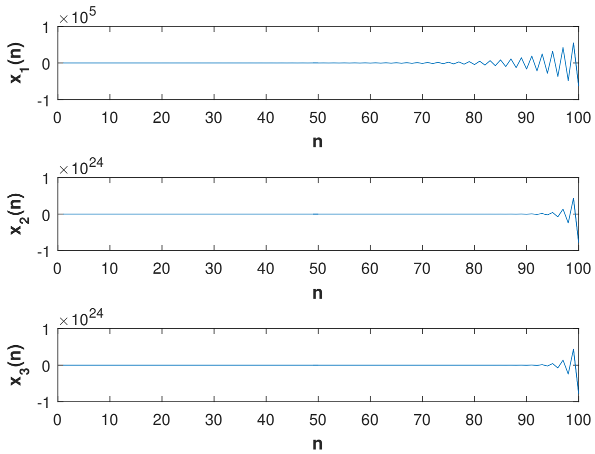

FIGURE 3. State vector of the system from Example 3.2 with $h = 0.0115$.

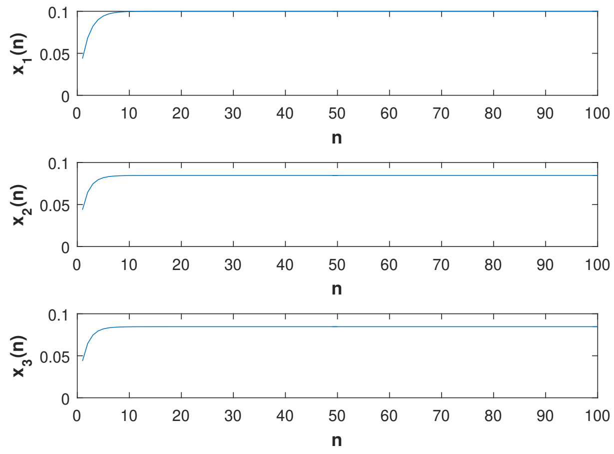

FIGURE 4. State vector of the system from Example 3.2 with $h = 0.0005$.

4. Concluding remarks

In this paper, the influence of discretization step on positivity of conformable fractional linear systems have been investigated. The novelty of our contributions is to use the new approach of discretization by piecewise constants to discretize the one-dimensional conformable fractional continuous systems. Then we use the obtained models to study the influence of the sampling step of discretization on the positivity of the discrete time models. Necessary and sufficient positivity conditions are developed, and two numerical examples that illustrate the applicability and effectiveness of our results are provided.

5. Acknowledgements

This paper presents research results of the ACSY-Team (Analysis & Control systems team) and of the doctorial training on the Operational Research from the Pure and Applied mathematics Laboratory, UMAB. and Decision Support funded by the General Directorate for Scientific Research and Technological Development of Algeria (DGRSDT) and supported by University of Mostaganem Abdelhamid Ibn Badis (UMAB) and initiated by the concerted research project on Control and Systems theory (PRFU Project Code C00L03UN270120200003).

REFERENCES

[1] *T. Abdeljawad,* On conformable fractional calculus, Journal of Computational and Applied Mathematics, **279**(2015), 57-66.

[2] *B. Bamieh,* Lecture 4: Discrete-Time Linear State-Space Models. University of California Department of Mechanical Engineering, 1999. `http://www1.unipa.it/~laura.giarre/corso1/lec4.pdf`

[3] *K. Benyettou, D. Bouagada and M. A. Ghezzar,* Solution of 2D State Space Continuous-Time Conformable Fractional Linear System Using Laplace and Sumudu Transform, Computational Mathematics and Modeling, **32**(2021), 94-106.

[4] *D. Bouagada*, Influence of the value of discretization step on positivity of 2D linear continuous-discrete systems. Far East J. Math. Sci, **15**(2004), 107-112.

[5] *F. Cacace, L. Farina, R. Setola, A. Germani*, Positive Systems _ Theory and Applications (POSTA 2016) Rome Italy, 1st edition. Springer International Publishing, Switzerland 2017.

[6] *A. Christopher, H. B. Jacobs, B. M. Anna,* Discretization of Fractional Differential Equations by a Piecewise Constant Approximation. Mathematical Modelling of Natural Phenomena. **12(6)**, (2017), 10.1051/mmnp/2017063.

[7] *M.A Ghezzar, Djillali Bouagada and Chadli,* Influence of discretization step on positivity of a certain class of two-dimensional continuous-discrete fractional linear systems, IMA Journal of Mathematical Control and Information, **35(3)**, (2018), 845-86.

[8] *G. F Gonzalez, V. I Martinez, I. M Kaabar, P. Silvestre,* On Conformable Laplace's Equation. Mathematical Problems in Engineering, **5514535**(2021), 1-10.

[9] *F. Imbert, A. Napoles, J. Rodriguez, Jo. Sigarreta, José,* Note on the generalized conformable derivative, Revista de la Unión Matemática Argentina. **62**(2021), 443-457.

[10] *T. Kaczorek*, Comparison of approximation methods of positive stable continuous-time linear systems by positive stable discrete-time systems, Archives of Electrical Engineering, **62**(2013), 345-355,

[11] *T. Kaczorek*, Analysis of positive linear continuous–time systems using the conformable derivative, Int. J. Appl. Math. Comput. Sci., 2018, **28(2)**, (2018), 335-340.

[12] *T. Kaczorek*, Selected Problems of Fractional Systems Theory, 1st edition, Springer-Verlag ,Berlin Heidelberg, 2011.

[13] *T. Kaczorek*, *Positive 1 D and 2D Systems*, Springer-Verlag ,London 2002.

[14] *T. Kaczorek, K. Rogowski,* Fractional Linear Systems and Electrical Circuits, Springer International Publishing, Switzerland, 2015.

[15] *R. Khalil , M. Al Horani , A. Yousef a, M. Sababhehb, A new definition of fractional derivative* ,Journal of Computational and Applied Mathematics, **264**(2014), 65-70.

[16] *J. Lam, Y. Chen, X. Liu, X. Zhao, J. Zhang,* Positive Systems, Theory and Applications (POSTA 2018), Lecture Notes in Control and Information Sciences, Springer Nature Switzerland AG, 2019.

[17] *J. L. Matheus and D. F. M. Torres,* Variational Calculus with Conformable Fractional Derivatives, IEEE/CAA Journal of Automatica Sinica, **4(2)**, (2017), 340-352.

[18] *J.E. Solís-Pérez, J.A. Hernández, A. Parrales, J.F. Gómez-Aguilar, A. Huicochea,* Artificial neural networks with conformable transfer function for improving the performance in thermal and environmental processes, Neural Networks, **152**(2022), 44-56.

[19] *B. Xin , W. Peng, Y. Kwon and Y. Liu*, Modeling, discretization, and hyperchaos detection of conformable derivative approach to a financial system with market confidence and ethics risk, Adv. Difference Equ. **138**, (2019).

[20] *U. Younas, J. Ren,* Diversity of wave structures to the conformable fractional dynamical model, Journal of Ocean Engineering and Science, Article In Press (2022), `https://doi.org/10.1016/j.joes.2022.04.014`.

[21] *A. Yosry, A. B Emad, E. Abdel-Salam, M. Nouh,* Artificial Neural Network Modeling of the Conformable Fractional Isothermal Gas Spheres. Revista Mexicana de Fisica, **57**(2021), 189-198.