GENERALIZED NOTIONS OF AMENABILITY AND CHARACTER AMENABILITY OF A CERTAIN CLASS OF BANACH ALGEBRAS

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Suppose A is a Banach algebra and $\epsilon \in \overline{B_1^{(0)}}$ (the closed unit ball of A). In this paper we generalize the notions of amenability, ϕ-amenability, ϕ-contractibility, biprojectivity and biflatness of a new Banach algebra A_ϵ. Moreover we investigate ϕ-pseudo amenability, ϕ-Johnson amenability, ϕ-inner amenability, ϕ-biflatness and ϕ-biprojectivity of A_ε.

Keywords: Banach algebra, approximate amenability, approximate ϕ-amenability, approximate ϕ-contractibility, approximate biprojectivity, approximate biflatness.

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Suppose that A is a Banach algebra and X is a Banach A-bimodule. A derivation from A into X is a linear operator $D : A \longrightarrow X$ satisfying

$$D(ab) = D(a) \cdot b + a \cdot D(b) \qquad (a, b \in A).$$

For every $x \in X$ we define ad_x by $ad_x(a) = a \cdot x - x \cdot a$ $(a \in A)$. Note that ad_x is a derivation which is called an inner derivation. A derivation D is said to be inner if there exists $x \in X$ such that $D(a) = ad_x(a)$ $(a \in A)$ and is approximately inner if there exists a net $(x_i) \subseteq X$ such that $D(a) = \lim_\alpha ad_{x_\alpha}(a)$ $(a \in A)$. A Banach algebra A is amenable if for any Banach A-bimodule X, every continuous derivation $D : A \longrightarrow X^*$ is inner and A is called approximately amenable if D is approximately inner.

The concepts of approximate amenability and approximate weak amenability of Banach algebras was introduced and extensively studied by Ghahramani and Loy in [3]. In [2] they also introduced and studied the notions of approximate semi-amenable and approximate semi-ontractible Banach algebras.

Let A be a Banach algebra and $\phi \in \Delta(A)$ (the character space of A). Kaniuth et al. [8] have introduced and studied the interesting notion of ϕ-amenability (see also [9]). A Banach algebra A is called ϕ-amenable if for every Banach A-bimodule X with the left module action $a \cdot x = \phi(a)x$ $(a \in A, x \in X)$, every continuous derivation from A into X^* is inner. Hu et al. [5] introduced and studied the notion of ϕ-contractibility of A. In fact, A is called ϕ-contractible if there exists a (right) ϕ-diagonal for A; that is, an element m in the projective tensor product $A\widehat{\otimes}A$ such that $\phi(\pi_A(m)) = 1$ and $a \cdot m = \phi(a)m$ for all $a \in A$, where π_A denotes the product morphism from $A\widehat{\otimes}A$ into A given by $\pi_A(a \otimes b) = ab$ $(a, b \in A)$. Furthermore, several authors have investigated the concepts of essential ϕ-amenability, essential left ϕ-contractibility, ϕ-pseudo amenability, ϕ-Johnson amenability and ϕ-inner amenability of Banach algebras; see for example [12], [14], [11], [15] and [6].

Moreover, H. Pourmahmood Aghababa et al. [13] was introduced and studied the concepts of approximate character amenability and approximate character contractibility of Banach algebras and investigated the relations between these concepts.

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Suppose A is a Banach algebra and $\varepsilon \in A$ with $\|\varepsilon\| \leq 1$. Recently, authors in [7], defined a new product on A by $a \odot b = a\varepsilon b$ $(a, b \in A)$. A with this product is denoted by A_ε. They studied the algebraic properties, arens regularity and amenability of A_ε. Also A. R. Khoddami in [10], investigated the relation between biflatness, biprojectivity, φ-amenability and φ-contractibility of A and A_ε.

In this paper, we study the relation between approximate amenability, approximate semi-amenability (approximate semi-contractibility), approximate weak amenability, approximate φ-amenability, approximate φ-contractibility, ϕ-pseudo amenability (ϕ-Johnson amenability), ϕ-inner amenability, approximate biflatness (ϕ-biflatness) and approximate biprojectivity (ϕ-biprojectivity) of A and A_ε.

1. Generalized notions of amenability and character amenability of A_ε

We commence this section with the following definition:

Definition 1.1. [7] *Let A be a Banach algebra and $\epsilon \in \overline{B_1^{(0)}}$ (the closed unit ball of A) with $\|\epsilon\| \leq 1$. Then A with the product $a \odot b = a\epsilon b$ $(a, b \in A)$ is an associative Banach algebra which is denoted by A_ϵ.*

Let A be a Banach algebra. The net $\{e_\alpha\}$ in A is called a ϕ-weak approximate identity if, for every $a \in A$, $|\phi(e_\alpha a) - \phi(a)| \longrightarrow 0$.

Note that if $\phi \in \Delta(A)$, then $\psi = \phi(\epsilon)\phi \in \Delta(A_\epsilon)$ (see Proposition 2.4 of [7]).

Proposition 1.1. *Let A be a Banach algebra, $\phi \in \Delta(A)$, $\phi(\epsilon) \neq 0$ and $\psi = \phi(\epsilon)\phi$. Then A has a ϕ-weak approximate identity if and only if A_ϵ has a ψ-weak approximate identity.*

Proof. Let $\{e_\alpha\}$ be a ϕ-weak approximate identity for A. Then $|\phi(e_\alpha a) - \phi(a)| \longrightarrow 0$ for all $a \in A$. Suppose that $e'_\alpha = \frac{e_\alpha}{\phi(\epsilon)}$ for all α. So for every $a \in A$, we have

$$\begin{aligned} |\psi(e'_\alpha \odot a) - \psi(a)| &= |\psi(\frac{e_\alpha}{\phi(\epsilon)} \odot a) - \psi(a)| \\ &= |\phi(\epsilon)\phi(e_\alpha a) - \phi(\epsilon)\phi(a)| \\ &= |\phi(\epsilon)||\phi(e_\alpha a) - \phi(a)| \longrightarrow 0. \end{aligned}$$

It follows that $\{e'_\alpha\}$ is a ψ-weak approximate identity for A_ϵ.

Conversely, let $\{e'_\alpha\}$ be a ψ-weak approximate identity for A_ϵ. Hence $|\psi(e'_\alpha \odot a) - \psi(a)| \longrightarrow 0$ for all $a \in A$. Choose $e_\alpha = e'_\alpha \epsilon$ for all α. For every $a \in A$

$$\begin{aligned} |\phi(e_\alpha a) - \phi(a)| &= |\phi(e'_\alpha \epsilon a) - \phi(a)| \\ &= |\frac{1}{\phi(\epsilon)}||\phi(\epsilon)\phi(e'_\alpha \epsilon a) - \phi(\epsilon)\phi(a)| \\ &= |\frac{1}{\phi(\epsilon)}||\psi(e'_\alpha \odot a) - \psi(a)| \longrightarrow 0. \end{aligned}$$

Therefore $\{e_\alpha\}$ is a ϕ-weak approximate identity for A. □

Theorem 1.1. *Let A be a Banach algebra and ϵ be an idempotent element of the algebraic center of A. Then the following statements are valid:*

(i) *If A_ϵ is approximately amenable, then so is A.*
(ii) *If A_ϵ is approximately semi-amenable, then so is A.*

Proof. Suppose that A_ϵ is approximately amenable. Then $A_\epsilon^\#$ (the unitalization of A_ϵ) is approximately amenable by Proposition 2.4 of [3]. So, Proposition 6.1 of [4] implies that $A_\epsilon^\# \oplus_1 A_\epsilon^\#$ is approximately amenable. Define $h : A_\epsilon^\# \oplus_1 A_\epsilon^\# \longrightarrow A^\#$ by

$$h\big((a, \lambda), (b, \lambda')\big) = (a, \lambda) \qquad ((a, \lambda), (b, \lambda') \in A_\epsilon^\#).$$

Clearly h is a surjection map. Since A_ϵ is approximately amenable, by Lemma 2.2 of [3], A_ϵ has left and right approximate identities. Let (e_α) be a right approximate identities for A_ϵ. So $\{\epsilon e_\alpha\}$ is a right approximate identities for A and thus for every $(a_1,\lambda_1),(a_2,\lambda_2),(b_1,\lambda_1')$ and $(b_2,\lambda_2')\in A_\epsilon^{\#}$, we get

$$\begin{aligned}
h\big((a_1,\lambda_1),&(b_1,\lambda_1')\big)h\big((a_2,\lambda_2),(b_2,\lambda_2')\big)\\
&=(a_1,\lambda_1)(a_2,\lambda_2)\\
&=(a_1a_2+\lambda_1a_2+\lambda_2a_1,\lambda_1\lambda_2)\\
&=\lim_\alpha(a_1a_2\epsilon e_\alpha+\lambda_1a_2+\lambda_2a_1,\lambda_1\lambda_2)\\
&=\lim_\alpha(a_1\epsilon a_2\epsilon e_\alpha+\lambda_1a_2+\lambda_2a_1,\lambda_1\lambda_2)\\
&=(a_1\epsilon a_2+\lambda_1a_2+\lambda_2a_1,\lambda_1\lambda_2)\\
&=h\Big((a_1\odot a_2+\lambda_1a_2+\lambda_2a_1,\lambda_1\lambda_2),(b_1\odot b_2+\lambda_1'b_2+\lambda_2'b_1,\lambda_1'\lambda_2')\Big)\\
&=h\Big(\big((a_1,\lambda_1),(b_1,\lambda_1')\big)\big((a_2,\lambda_2),(b_2,\lambda_2')\big)\Big).
\end{aligned}$$

That is h is a homomorphism. Moreover, for every (a,λ) and $(b,\lambda')\in A_\epsilon^{\#}$,

$$\begin{aligned}
\|h\big((a,\lambda),(b,\lambda')\big)\|&=\|(a\epsilon,\lambda)\|\le\|(a,\lambda)\|\\
&\le\|(a,\lambda)\|+\|(b,\lambda')\|\\
&=\|((a,\lambda),(b,\lambda'))\|.
\end{aligned}$$

Consequently, h is a continuous epimorphism. Therefore, from Proposition 2.2 of [3], it follows that $A^{\#}$ is approximately amenable. Again Proposition 2.4 of [3] yields that A is approximately amenable.

(ii) Assume that A_ϵ is approximately semi-amenable. By Proposition 2.1 of [17], $A_\epsilon^{\#}$ is approximately semi-amenable and so Theorem 5.1 of [2] yields that $A_\epsilon^{\#}\oplus_1 A_\epsilon^{\#}$ is approximately semi-amenable. By Lemma 2.2 of [17], A_ϵ has an approximate identity. Now if we define h as part (i). Then Proposition 3.11 of [2], implies that $A^{\#}$ is approximately semi-amenable. Now from the Proposition 2.1 of [17], we conclude that A is approximately semi-amenable. □

Theorem 1.2. *Let A be an unital Banach algebra and ϵ be an invertible element of A. Then the following statements are valid:*

(i) *If A is approximately amenable, then so is A_ϵ.*
(ii) *If A is approximately semi-amenable, then so is A_ϵ.*
(iii) *If A is approximate semi-contractible, then so is A_ϵ.*
(iv) *Let A be a commutative Banach algebra. If A is approximately weakly amenable, then so is A_ϵ.*

Proof. Suppose that A is approximately amenable. Then by Proposition 6.1 of [4], $A\oplus_1 A$ is approximately amenable. Define $h: A\oplus_1 A\longrightarrow A_\epsilon$ by

$$h(a,b)=a\epsilon^{-1}\qquad (a\in A).$$

For every $a_1,a_2\in A$ and $b_1,b_2\in B$, we have

$$\begin{aligned}
h(a_1,b_1)\odot h(a_2,b_2)&=a_1\epsilon^{-1}\odot a_2\epsilon^{-1}=a_1a_2\epsilon^{-1}\\
&=h(a_1a_2,b_1b_2)=h\big((a_1,b_1)(a_2,b_2)\big),
\end{aligned}$$

and for every $a\in A_\epsilon$, $h(a\epsilon,b)=a\epsilon\epsilon^{-1}=a$. So h is an epimorphism. Since A is unital and ϵ is invertible, from Proposition 2.3 of [7], it follows that ϵ^{-1} is the unit of A_ϵ. Thus

$\|\epsilon^{-1}\| \leq 1$. Hence for every $a, b \in A$,

$$\|h(a,b)\| = \|a\epsilon^{-1}\| \leq \|a\| \leq \|a\| + \|b\| = \|(a,b)\|.$$

Consequently, h is continuous. Thus h is a continuous epimorphism. Therefore, by using Proposition 2.2 of [3], we deduce that A_ϵ is approximately amenable.

(ii) Suppose that A is approximately semi-amenable. By Theorem 5.1 of [2], $A \oplus_1 A$ is approximately semi-amenable. Let h be defined as part (i). Then Proposition 3.11 of [2], implies that A_ϵ is approximately semi-amenable.

(iii) Suppose that A is approximate semi-contractible. By a similar argument as part (ii), if we apply Theorem 2.14 of [17] and Proposition 3.11 of [2], one can prove that A_ϵ is approximate semi-contractible.

(iv) Suppose that A is approximately weakly amenable. Then $A \oplus_1 A$ is approximately weakly amenable by Theorem 2.3 of [18]. Let h be defined as part (i). Therefore approximate weak amenability of A_ϵ follows from Theorem 2.1 of [1].

□

Definition 1.2. [13] *A Banach algebra A is called ϕ-approximately amenable if there exists a net $\{m_\alpha\} \subseteq A^{**}$ such that $m_\alpha(\phi) = 1$ and $\|a \cdot m_\alpha - \phi(a)m_\alpha\| \longrightarrow 0$ for all $a \in A$.*

Also A is called ϕ-approximately contractible if there exists a net $\{m_\alpha\} \subseteq A$ such that $\phi(m_\alpha) = 1$ and $\|am_\alpha - \phi(a)m_\alpha\| \longrightarrow 0$ for all $a \in A$.

Proposition 1.2. *Let A be a Banach algebra and $\phi \in \Delta(A)$. Then the following statements are valid:*

- (i) *If A is approximately ϕ-contractible and $\phi(\epsilon) \neq 0$, then A_ϵ is approximately ψ-contractible, where $\psi = \phi(\epsilon)\phi$.*
- (ii) *If A_ϵ is unital and approximately ψ-contractible, then A is approximately ϕ-contractible, where $\phi(a) = \psi(\epsilon^{-1}a)$ $(a \in A)$.*

Proof. (i) Suppose that A is approximately ϕ-contractible. Then there exists a net $(m_\alpha) \subset A$ such that $\phi(m_\alpha) = 1$ and $\|am_\alpha - \phi(a)m_\alpha\| \longrightarrow 0$ for all $a \in A$. Let $n_\alpha = \frac{m_\alpha}{\phi(\epsilon)}$. Hence for every $a \in A$

$$\begin{aligned}\|a \odot n_\alpha - \psi(a)n_\alpha\| &= \|a\epsilon n_\alpha - \phi(\epsilon)\phi(a)n_\alpha\| \\ &= \|a\epsilon\frac{m_\alpha}{\phi(\epsilon)} - \phi(a)m_\alpha\| \\ &= \|\frac{a\epsilon}{\phi(\epsilon)}m_\alpha - \phi(\frac{a\epsilon}{\phi(\epsilon)})m_\alpha\| \longrightarrow 0.\end{aligned}$$

Also

$$\psi(n_\alpha) = \psi(\frac{m_\alpha}{\phi(\epsilon)}) = \frac{1}{\phi(\epsilon)}\phi(\epsilon)\phi(m_\alpha) = 1,$$

for all α. So A_ϵ is approximately ψ-contractible.

(ii) Suppose that A_ϵ is unital and approximately ψ-contractible. Then there exists a net $(n_\alpha) \subset A_\epsilon$ such that $\psi(n_\alpha) = 1$ and $\|a \odot n_\alpha - \psi(a)n_\alpha\| \longrightarrow 0$ for all $a \in A_\epsilon$. Since $1 = \psi(n_\alpha) = \phi(\epsilon)\phi(n_\alpha)$ it follows that $\phi(n_\alpha) \neq 0$. Also since A_ϵ is unital, by Proposition 2.3 of [7], ϵ^{-1} is the unit of A_ϵ and thus $\psi(\epsilon^{-1}) \neq 0$. Choose $m_\alpha = \frac{n_\alpha}{\phi(n_\alpha)}$. Then $\phi(m_\alpha) = 1$

for every α and for every $a \in A$, we obtain

$$\begin{aligned}\|am_\alpha - \phi(a)m_\alpha\| &= \|a\frac{n_\alpha}{\phi(n_\alpha)} - \phi(a)\frac{n_\alpha}{\phi(n_\alpha)}\| \\ &= \|a\frac{n_\alpha}{\psi(\epsilon^{-1}n_\alpha)} - \phi(a)\frac{n_\alpha}{\psi(\epsilon^{-1}n_\alpha)}\| \\ &= |\frac{1}{\psi(\epsilon^{-1})}|\|an_\alpha - \phi(a)n_\alpha\| \\ &= |\frac{1}{\psi(\epsilon^{-1})}|\|a\epsilon^{-1}\odot n_\alpha - \psi(a\epsilon^{-1})n_\alpha\| \longrightarrow 0.\end{aligned}$$

Therefore A is approximately ϕ-contractible and the proof is now complete. □

The proof of the following proposition is omitted, since it can be proved in the same direction of Proposition 1.2.

Proposition 1.3. *Let A be a Banach algebra and $\phi \in \Delta(A)$. Then the following statements are valid:*

(i) *If A is approximately ϕ-amenable and $\phi(\epsilon) \neq 0$, then A_ϵ is approximately ψ-amenable, where $\psi = \phi(\epsilon)\phi$.*
(ii) *If A_ϵ is unital and approximately ψ-amenable, then A is approximately ϕ-amenable, where $\phi(a) = \psi(\epsilon^{-1}a)$ $(a \in A)$.*

Definition 1.3. [15] *Let A be a Banach algebra and $\phi \in \Delta(A)$. A is called ϕ-Johnson amenable if there exists a bounded net $\{m_\alpha\} \subseteq A\widehat{\otimes}A$ such that $\phi\circ\pi_A(m_\alpha) \longrightarrow 1$ and $\|a\cdot m_\alpha - m_\alpha\cdot a\| \longrightarrow 0$, for every $a \in A$.*

Definition 1.4. [11] *Let A be a Banach algebra and $\phi \in \Delta(A)$. A is called ϕ-pseudo amenable if there exists a net $\{m_\alpha\} \subseteq A\widehat{\otimes}A$ such that $\phi\circ\pi_A(m_\alpha) \longrightarrow 1$ and $\|a\cdot m_\alpha - \phi(a)m_\alpha\| \longrightarrow 0$, for every $a \in A$.*

Before turning the next theorem we note that if A is an unital Banach algebra and $\epsilon \in \overline{B_1^{(0)}}$, then $A = (A_\epsilon)_{\epsilon^{-2}}$ (see Proposition 2.3 of [10]).

Theorem 1.3. *Let A be a Banach algebra, A_ϵ be unital and let $\phi \in \Delta(A)$ be such that $\phi(\epsilon) \neq 0$. Then the following statements are valid:*

(i) *A is ϕ-Johnson amenable if and only if A_ϵ is ψ-Johnson amenable, where $\psi = \phi(\epsilon)\phi$.*
(ii) *A is ϕ-pseudo amenable if and only if A_ϵ is ψ-pseudo amenable, where $\psi = \phi(\epsilon)\phi$.*

Proof. (i) Suppose that A is ϕ-Johnson amenable. Then there exists a bounded net $\{m_\alpha\} \subseteq A\widehat{\otimes}A$ such that $\phi\circ\pi_A(m_\alpha) \longrightarrow 1$ and $\|a\cdot m_\alpha - m_\alpha\cdot a\| \longrightarrow 0$, for every $a \in A$. Let $k : A_\epsilon\widehat{\otimes}A_\epsilon \longrightarrow A_\epsilon\widehat{\otimes}A_\epsilon$ be the bounded linear map such that $k(a\otimes c) = a\epsilon^{-1}\otimes c$ $(a, c \in A_\epsilon)$. k is an A_ϵ-bimodule map (see the proof of Theorem 2.3 of [10]). Consider $n_\alpha = \frac{k(m_\alpha)}{\phi(\epsilon)}$ and let $m_\alpha = \sum_{i=0}^\infty a_i^\alpha \otimes b_i^\alpha$. So,

$$\begin{aligned}\psi\circ\pi_{A_\epsilon}(n_\alpha) = \psi\circ\pi_{A_\epsilon}(\frac{k(m_\alpha)}{\phi(\epsilon)}) &= \psi\circ\pi_{A_\epsilon}(\frac{\sum_{i=0}^\infty a_i^\alpha\epsilon^{-1}\otimes b_i^\alpha}{\phi(\epsilon)}) \\ &= \frac{\psi}{\phi(\epsilon)}\sum_{i=0}^\infty a_i^\alpha\epsilon^{-1}\epsilon b_i^\alpha = \frac{\psi}{\phi(\epsilon)}\sum_{i=0}^\infty a_i^\alpha b_i^\alpha \\ &= \phi\circ\pi_A(m_\alpha) \longrightarrow 1.\end{aligned}$$

Moreover, for every $a \in A$, we have

$$\begin{aligned}\|a \odot n_\alpha - n_\alpha \odot a\| &= \|a \odot \frac{k(m_\alpha)}{\phi(\epsilon)} - \frac{k(m_\alpha)}{\phi(\epsilon)} \odot a\| \\ &= \frac{1}{|\phi(\epsilon)|}\|a \odot k(m_\alpha) - k(m_\alpha) \odot a\| \\ &= \frac{1}{|\phi(\epsilon)|}\|k(a \odot m_\alpha) - k(m_\alpha \odot a)\| \\ &\le \frac{\|k\|}{|\phi(\epsilon)|}\|a\epsilon m_\alpha - m_\alpha \epsilon a\| \\ &= \frac{\|k\|}{|\phi(\epsilon)|}\|a\epsilon \cdot m_\alpha - m_\alpha \cdot \epsilon a\| \longrightarrow 0.\end{aligned}$$

Therefore A_ϵ is ψ-Johnson amenable.

Conversely, suppose that A_ϵ is ψ-Johnson amenable. Since $(A_\epsilon)_{\epsilon^{-2}} = A$ and $\phi = \psi(\epsilon^{-2})\psi$, the proof is an immediate consequence of above argument.

(ii) Suppose that A is ϕ-pseudo amenable. So there exists a net $\{m_\alpha\} \subseteq A\widehat{\otimes}A$ such that $\phi \circ \pi_A(m_\alpha) \longrightarrow 1$ and $\|a \cdot m_\alpha - \phi(a)m_\alpha\| \longrightarrow 0$, for every $a \in A$. Choose $n_\alpha = \frac{k(m_\alpha)}{\phi(\epsilon)}$. Similar arguments to the proof of part (i), show that $\psi \circ \pi_{A_\epsilon}(n_\alpha) \longrightarrow 1$ and for every $a \in A$, we get

$$\begin{aligned}\|a \odot n_\alpha - \psi(a)n_\alpha\| &= \|a \odot \frac{k(m_\alpha)}{\phi(\epsilon)} - \psi(a)\frac{k(m_\alpha)}{\phi(\epsilon)}\| \\ &= \|k\big(a \odot \frac{m_\alpha}{\phi(\epsilon)}\big) - k\big(\psi(a)\frac{m_\alpha}{\phi(\epsilon)}\big)\| \\ &\le \|k\|\|a \odot \frac{m_\alpha}{\phi(\epsilon)} - \psi(a)\frac{m_\alpha}{\phi(\epsilon)}\| \\ &\le \|k\|\|\frac{a\epsilon}{\phi(\epsilon)} \cdot m_\alpha - \phi(\frac{a\epsilon}{\phi(\epsilon)})m_\alpha\| \longrightarrow 0.\end{aligned}$$

Consequently, A_ϵ is ψ-pseudo amenable.

Conversely, suppose that A_ϵ is ψ-pseudo amenable. Since $(A_\epsilon)_{\epsilon^{-2}} = A$ and $\phi = \psi(\epsilon^{-2})\psi$, it follows that A is ϕ-pseudo amenable. □

Definition 1.5. [6] *Let A be a Banach algebra, $\varphi \in \Delta(A)$ and $A_\phi = \{a \in A : \phi(a) = 1\}$. A is called ϕ-inner amenable if there exists a bounded linear functional m on A^* satisfying $m(\phi) = 1$ and $m(f \cdot a) = m(a \cdot f)$ for all $f \in A^*$ and $a \in A_\phi$.*

Note that A is ϕ-inner amenable if and only if there is a bounded net (v_α) in A_ϕ such that $\|v_\alpha a - av_\alpha\| \longrightarrow 0$ for all $a \in A_\phi$ (see Theorem 2.1 of [6]).

Proposition 1.4. *Let A be a Banach algebra and let $\phi \in \Delta(A)$ be such that $\phi(\epsilon) \neq 0$. If A is ϕ-inner amenable, then A_ϵ is $\psi = \phi(\epsilon)\phi$-inner amenable. In the case that ϵ is an element of the algebraie center of A, the converse is also valid.*

Proof. Suppose that A is ϕ-inner amenable. Then there is a bounded net (v_α) in A_ϕ such that $\|v_\alpha a - av_\alpha\| \longrightarrow 0$ for all $a \in A_\phi$. Choose $w_\alpha = \frac{v_\alpha}{\phi(\epsilon)}$. Hence, for every α

$$\psi(w_\alpha) = \psi(\frac{v_\alpha}{\phi(\epsilon)}) = \phi(v_\alpha) = 1.$$

That is $w_\alpha \in (A_\epsilon)_\psi$. Now let $a \in A$ be such that $\psi(a) = 1$. So

$$\phi(\epsilon a) = \phi(\epsilon)\phi(a) = \psi(a) = 1,$$

and it follows that

$$\|w_\alpha \odot a - a \odot w_\alpha\| = \|\frac{v_\alpha}{\phi(\epsilon)} a\epsilon - a\epsilon \frac{v_\alpha}{\phi(\epsilon)}\| = |\frac{1}{\phi(\epsilon)}| \|v_\alpha a\epsilon - a\epsilon v_\alpha\| \longrightarrow 0.$$

This means that $\|w_\alpha \odot a - a \odot w_\alpha\| \longrightarrow 0$ for all $a \in (A_\epsilon)_\psi$. Therefore A_ϵ is ψ-inner amenable.

Conversely, Suppose that ϵ is an element of the algebraie center of A and A_ϵ is ψ-inner amenable. So there is a bounded net (w_α) in $(A_\epsilon)_\psi$ such that $\|w_\alpha \odot a - a \odot w_\alpha\| \longrightarrow 0$ for all $a \in (A_\epsilon)_\psi$. Define $v_\alpha := \epsilon w_\alpha$. Thus $\phi(v_\alpha) = \phi(\epsilon)\phi(w_\alpha) = \psi(w_\alpha) = 1$. Also since $\psi(\frac{a}{\phi(\epsilon)}) = 1$ for every $a \in A_\phi$, it follows that

$$\begin{aligned} \|v_\alpha a - a v_\alpha\| &= \|\epsilon w_\alpha a - a\epsilon w_\alpha\| \\ &= \|w_\alpha \odot a - a \odot w_\alpha\| \\ &= |\phi(\epsilon)| \|w_\alpha \odot \frac{a}{\phi(\epsilon)} - \frac{a}{\phi(\epsilon)} \odot w_\alpha\| \longrightarrow 0. \end{aligned}$$

Therefore A is ϕ-inner amenable. □

2. Generalized biprojectivity and biflatness of A_ϵ

We start this section with the following definitions:

Definition 2.1. [16] *A Banach algebra A is called approximately biprojective if there is a net $\{\rho_\alpha\} \subseteq A$ of continuous A-bimodule maps from A into $A\widehat{\otimes}A$ such that $\pi_A \circ \rho_\alpha(a) \longrightarrow a$.*

A is called approximately biflat if there is a net $\{\theta_\alpha\}$ of continuous A-bimodule maps from $(A\widehat{\otimes}A)^*$ into A^* such that $w^* - \lim_\alpha \theta_\alpha \circ \pi_{A^*} = id_{A^*}$ where w^* is the weak operator topology on $B(A^*)$.*

Let $\phi \in \Delta(A)$. Then ϕ has a unique extension on A^{**} denoted by $\widetilde{\phi}$ and defined by $\widetilde{\phi}(F) = F(\phi)$ for every $F \in A^{**}$. Clearly this extension remains to be a character on A^{**}.

Definition 2.2. [15] *Let A be a Banach algebra and $\phi \in \Delta(A)$. A is called ϕ-biprojective if there exists a continuous A-bimodule map $\rho : A \longrightarrow A\widehat{\otimes}A$ such that $\phi \circ \pi_A \circ \rho = \phi$.*

Also A is called ϕ-biflat if there exists a continuous A-bimodule map $\rho : A \longrightarrow (A\widehat{\otimes}A)^{**}$ such that $\widetilde{\phi} \circ \pi_A^{**} \circ \rho = \phi$.

Note that if A is a Banach algebra and $\epsilon \in \overline{B_1^{(0)}}$, then $f \odot a = f \cdot a\epsilon$ and $a \odot f = \epsilon a \cdot f$, for all $a \in A_\epsilon$ and $f \in A_\epsilon^*$. Also $a \odot u = a\epsilon \cdot u$ and $u \odot a = u \cdot \epsilon a$ for every $a \in A_\epsilon$ and $u \in A_\epsilon \widehat{\otimes} A_\epsilon$ (see Proposition 2.4 of [10]).

The proof idea of the following Theorem is taken from the proof of Theorem 2.3 and Theorem 2.4 of [10].

Theorem 2.1. *Let A be a Banach algebra and A_ϵ be unital. Then the following statements are valid:*

(i) *A is approximately biprojective if and only if A_ϵ is approximately biprojective.*
(ii) *A is approximately biflat if and only if A_ϵ is approximately biflat.*

Proof. (i) Suppose that A is a approximately biprojective Banach algebra. Then there exists a net $\{\rho_\alpha\}$ of continuous A-bimodule maps from A into $A\widehat{\otimes}A$ such that $\pi_A \circ \rho_\alpha(a) \longrightarrow a$ for every $a \in A$. Set $\rho_\alpha^\epsilon = k \circ \rho_\alpha$ such that $k : A_\epsilon \widehat{\otimes} A_\epsilon \longrightarrow A_\epsilon \widehat{\otimes} A_\epsilon$ be the bounded linear map defined by $k(a \otimes c) = a\epsilon^{-1} \otimes c$ $(a, c \in A_\epsilon)$. By the same argument as in the proof of the theorem 2.3 of [10], one can show that ρ_α^ϵ is an A_ϵ-bimodule map for every α. Let

$\rho_\alpha(a) = \sum_{i=1}^{\infty} a_i^\alpha \otimes b_i^\alpha$. So, for every $a \in A$, we have

$$\begin{aligned}
\lim_\alpha \Big(\pi_{A_\epsilon} \circ \rho_\alpha^\epsilon(a)\Big) &= \lim_\alpha \Big(\pi_{A_\epsilon} \circ k \circ \rho_\alpha(a)\Big) \\
&= \lim_\alpha \Big(\pi_{A_\epsilon} \circ k\big(\sum_{i=1}^{\infty} a_i^\alpha \otimes b_i^\alpha\big)\Big) \\
&= \lim_\alpha \Big(\pi_{A_\epsilon}\big(\sum_{i=1}^{\infty} a_i^\alpha \epsilon^{-1} \otimes b_i^\alpha\big)\Big) \\
&= \lim_\alpha \Big(\sum_{i=1}^{\infty} a_i^\alpha \epsilon^{-1} \odot b_i^\alpha\Big) = \lim_\alpha \Big(\sum_{i=1}^{\infty} a_i^\alpha b_i^\alpha\Big) \\
&= \lim_\alpha \Big(\pi_A\big(\sum_{i=1}^{\infty} a_i^\alpha \otimes b_i^\alpha\big)\Big) = \lim_\alpha \Big(\pi_A(\rho_\alpha(a))\Big) \\
&= \lim_\alpha \Big(\pi_A \circ \rho_\alpha(a)\Big) = a.
\end{aligned}$$

Therefore A_ϵ is approximately biprojective.

Conversely, suppose that A_ϵ is approximately biprojective. Since $(A_\epsilon)_{\epsilon^{-2}} = A$, from the above argument we conclude that A is approximately biprojective.

(ii) Suppose that A is approximately biflat. Then there is a net $\{\theta_\alpha\}_\alpha$ of continuous A-bimodule maps from $(A\widehat{\otimes}A)^*$ into A^* such that $w^* - \lim_\alpha \theta_\alpha \circ \pi_{A^*} = id_{A^*}$. Suppose that $l : A_\epsilon\widehat{\otimes}A_\epsilon \longrightarrow A_\epsilon$ and $\sigma : A_\epsilon\widehat{\otimes}A_\epsilon \longrightarrow A_\epsilon\widehat{\otimes}A_\epsilon$ are the bounded linear maps such that $l(a\otimes b) = a\epsilon\otimes b$ $(a, b \in A_\epsilon)$ and $\sigma(a\otimes b) = a\epsilon^{-1}\otimes b$ $(a, b \in A_\epsilon)$. Define $\theta_\alpha^\epsilon : (A_\epsilon\widehat{\otimes}A_\epsilon)^* \longrightarrow A_\epsilon^*$ by $\theta_\alpha^\epsilon(f) = \theta_\alpha \circ \sigma^*(f)$ $(f \in (A_\epsilon\widehat{\otimes}A_\epsilon)^*)$. A similar argument as in the proof of theorem 2.4 of [10], shows that θ_α^ϵ is an A_ϵ-bimodule map for every α. Now from the fact that $l^* \circ \pi_A^*(f) = \pi_{A_\epsilon}^*(f)$ and $l^*(\pi_A^*(f)) \circ \sigma = \pi_A^*(f)$ $(f \in A^*)$, it follows that

$$\begin{aligned}
w^* - \lim_\alpha \Big(\theta_\alpha^\epsilon \circ \pi_{A_\epsilon}^*)(f)\Big) &= w^* - \lim_\alpha \Big(\theta_\alpha^\epsilon \circ l^* \circ \pi_A^*(f)\Big) \\
&= w^* - \lim_\alpha \Big(\theta_\alpha \circ \sigma^* \circ l^* \circ \pi_A^*(f)\Big) \\
&= w^* - \lim_\alpha \Big(\theta_\alpha\big(l^*(\pi_A^*(f)) \circ \sigma\big)\Big) \\
&= w^* - \lim_\alpha \Big(\theta_\alpha\big(\pi_A^*(f)\big)\Big) \\
&= f,
\end{aligned}$$

for every $f \in A^*$. Therefore A_ϵ is approximately biflat.

Conversely, suppose that A_ϵ is approximately biflat. From the facts that $(A_\epsilon)_{\epsilon^{-2}} = A$, we deduce that A is approximately biflat. □

Theorem 2.2. *Let A be a Banach algebra and A_ϵ be unital. Then the following statements are valid:*

(i) *A is ϕ-biprojective if and only if A_ϵ is ψ-biprojective, where $\psi = \phi(\epsilon)\phi$.*
(ii) *A is ϕ-biflat if and only if A_ϵ is ψ-biflat, where $\psi = \phi(\epsilon)\phi$.*

Proof. (i) Suppose that A is ϕ-biprojective. So there exists a continuous A-bimodule map $\rho : A \longrightarrow A\widehat{\otimes}A$ such that $\phi \circ \pi \circ \rho = \phi$. By a similar argument as in the proof of part(i) of Theorem 2.1, if we define $\rho_{A_\epsilon} = k \circ \rho_A$, one can show that $\psi \circ \pi_{A_\epsilon} \circ \rho_{A_\epsilon} = \psi$. So A_ϵ is ψ-biprojective.

Conversely, ϕ-biprojectivity of A follows from the facts that $(A_\epsilon)_{\epsilon^{-2}} = A$ and $\phi = \psi(\epsilon^{-2})\psi$.

(ii) Suppose that A is ϕ-biflat. Hence there exists a continuous A-bimodule map $\rho_A : A \longrightarrow (A\widehat{\otimes}A)^{**}$ such that $\widetilde{\phi} \circ \pi_A^{**} \circ \rho = \phi$. Clearly, ρ_A is an A_ϵ-bimodule map. Indeed

$$\rho_A(a \odot b) = \rho_A(a\epsilon b) = \rho(a)\epsilon b = \rho(a) \odot b,$$

Similarly

$$\rho_A(b \odot a) = \rho_A(b\epsilon a) = b\epsilon\rho_A(a) = b \odot \rho_A(a).$$

Let l and σ be A_ϵ-bimodule maps defined as in the proof of Theorem 2.1. Obviously, $\pi_{A_\epsilon} = \pi_A \circ l$. Consequently, $\pi_{A_\epsilon}^{**} = \pi_A^{**} \circ l^{**}$. Now define $\rho_{A_\epsilon} : A_\epsilon \longrightarrow (A_\epsilon\widehat{\otimes}A_\epsilon)^{**}$, by

$$\rho_{A_\epsilon}(a) = \sigma^{**} \circ \rho_A(a) \ (a \in A).$$

Since ρ_A and σ are two A_ϵ-bimodule maps, it follows that ρ_{A_ϵ} is A_ϵ-bimodule map. Moreover, for every $a \in A$, we have

$$\begin{aligned}\widetilde{\psi} \circ \pi_{A_\epsilon}^{**} \circ \rho_{A_\epsilon}(a) &= \widetilde{\psi} \circ \pi_A^{**} \circ l^{**} \circ \sigma^{**} \circ \rho_A(a) \\ &= (\widetilde{\phi(\epsilon)\phi}) \circ \pi_A^{**} \circ \rho_A(a) \\ &= \phi(\epsilon)\widetilde{\phi} \circ \pi_A^{**} \circ \rho_A(a) \\ &= \phi(\epsilon)\phi(a) \\ &= \psi(a).\end{aligned}$$

So A_ϵ is ψ-biflat.

The converse follows from the facts that $(A_\epsilon)_{\epsilon^{-2}} = A$ and $\phi = \psi(\epsilon^{-2})\psi$. □

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