ON LOCALLY CONFORMALLY FLAT WEAKLY-EINSTEIN FOUR-MANIFOLDS

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Weakly-Einstein conditions over four-dimensional conformally flat pseudo-Riemannian algebraic curvature models are considered. In particular, we present the four-dimensional locally conformally flat examples of Walker metrics satisfying weakly-Einstein conditions.

Keywords: Weakly-Einstein space, conformally flat, algebraic pseudo-Riemannian curvature model, Walker metric.
MSC2020: 53C25, 53C18, 58E11.

1. Introduction

For a compact surface (M, g), the Gauss-Bonnet theorem asserts that $\int_M K d\text{vol}_g = 2\pi\chi(M)$, where K is the Gaussian curvature of M, $d\text{vol}_g$ is the volume element of M and $\chi(M)$ is the Euler characteristic. Since the Euler characteristic is a topological invariant, any metric is critical for Hilbert-Einstein functional $\mathcal{E} : g \mapsto \int_M K d\text{vol}_g$. The gradient of this functional is given by $\nabla\mathcal{E} = \varrho - \frac{\tau}{2} g = 0$, where ϱ and τ are the Ricci tensor and the scalar curvature, respectively. Therefore, the two-dimensional curvature identity $\varrho = \frac{\tau}{2} g$ holds for any compact surface, which is the same as the Einstein condition in this dimension. The Gauss-Bonnet theorem was generalized by Chern in [6] to even higher dimensions, which in dimension four is given by

$$\int_M (\|\mathcal{R}\|^2 - 4\|\varrho\|^2 + \tau^2) d\text{vol}_g = 32\pi^2\chi(M), \tag{1}$$

where $\|\mathcal{R}\|^2$ is square norm of the curvature tensor, taken with the sign convention $\mathcal{R}(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$, for all $X, Y \in \mathfrak{X}(M)$, and $\|\varrho\|^2$ is square norm of the Ricci tensor. Using the Gauss-Bonnet-Chern theorem, Berger in [1] derived the curvature identity on a four-dimensional Riemannian compact oriented manifold as follows

$$\left(\check{\mathcal{R}} - \frac{\|\mathcal{R}\|^2}{4} g\right) + \tau\left(\varrho - \frac{\tau}{4} g\right) - 2\left(\check{\varrho} - \frac{\|\varrho\|^2}{4} g\right) - 2\left(\mathcal{R}[\varrho] - \frac{\|\varrho\|^2}{4} g\right) = 0, \tag{2}$$

where $\check{\mathcal{R}}$, $\check{\varrho}$ and $\mathcal{R}[\varrho]$ are the symmetric $(0, 2)$-tensor fields given by

$$\check{\mathcal{R}}_{ij} = \mathcal{R}_{iabc}\mathcal{R}_j^{abc}, \ \check{\varrho}_{ij} = \varrho_{ia}\varrho_j^a, \ \mathcal{R}[\varrho]_{ij} = \mathcal{R}_{iajb}\varrho^{ab}.$$

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Euh et al., in [8], expanded (2) to a non-compact case. Also, Labbi in [12] could expand it to higher dimensions by using an elegant method, but he considered only the compact case. Recently, E. García Río et al., in [9], have pursued the curvature identity on homogeneous Riemannian four-manifolds; especially, locally conformally flat weakly-Einstein Riemannian manifolds were studied in [10]. In [11], authors focused on three-dimensional Lorentzian weakly-Einstein manifolds.

Clearly, if (M, g) is an Einstein manifold, then the equation (2) vanishes identically. Thus, non-trivial cases, i.e., where the manifold is non-Einstein were considered in literature. The tensor fields $\check{\mathcal{R}}$, $\check{\varrho}$ and $\mathcal{R}[\varrho]$ were considered separately. In particular, it was investigated whether they reduce the metric tensor up to scaling or not. Following the definition of weakly-Einstein Riemannian manifolds which is introduced in [9], we extend it to the pseudo-Riemannian four-dimensional manifolds as following.

Definition 1.1. *Let (M, g) be a four-dimensional non-Einstein pseudo-Riemannian manifold. Then, (M, g)*

(i) *is called $\check{\mathcal{R}}$-Einstein if*

$$\check{\mathcal{R}} = \frac{\|\mathcal{R}\|^2}{4} g. \tag{3}$$

(ii) *is called $\check{\varrho}$-Einstein if*

$$\check{\varrho} = \frac{\|\varrho\|^2}{4} g. \tag{4}$$

(iii) *is called $\mathcal{R}[\varrho]$-Einstein if*

$$\mathcal{R}[\varrho] = \frac{\|\varrho\|^2}{4} g. \tag{5}$$

We will use the *weakly-Einstein* condition as a general setting for the $\check{\mathcal{R}}$-Einstein, $\check{\varrho}$-Einstein and $\mathcal{R}[\varrho]$-Einstein conditions.

Four-dimensional pseudo-Riemannian manifolds have been investigated from different aspects (see for example [3, 4, 5]). Up to our knowledge, there is no comprehensive study over weakly-Einstein conditions in four-dimensional pseudo-Riemannian manifolds. The aim of the present work is to exhibit a classification result for weakly-Einstein pseudo-Riemannian locally conformally flat four manifolds. This paper is organized as follows. In section 2, we have considered weakly-Einstein conditions on conformally flat algebraic curvature models of dimension four. We expose classification theorems on Lorentzian and neutral signatures separately. Section 3 is devoted to some geometric examples of weakly-Einstein pseudo-Riemannian manifolds without Riemannian counterpart, i.e., Walker metrics. Finally, locally symmetric examples were studied in the last section.

2. Weakly-Einstein conformally flat algebraic curvature models

Let V be a real vector space of dimension four with a pseudo-Riemannian metric $\langle \cdot, \cdot \rangle$. Then, $\mathfrak{M} = (V, \langle \cdot, \cdot \rangle, \mathcal{A})$ is called an algebraic curvature model, where $\mathcal{A} \in \otimes^4(V^*)$ is an algebraic curvature tensor on V, i.e.,

$$\mathcal{A}(v_1, v_2, v_3, v_4) = -\mathcal{A}(v_2, v_1, v_3, v_4) = \mathcal{A}(v_3, v_4, v_1, v_2),$$

$$\mathcal{A}(v_1, v_2, v_3, v_4) + \mathcal{A}(v_2, v_3, v_1, v_4) + \mathcal{A}(v_3, v_1, v_2, v_4) = 0,$$

for all $v_1, v_2, v_3, v_4 \in V$. The Ricci operator $\widehat{\mathrm{Ric}}$ associated with $\mathcal{A}$ is characterized by $\langle \widehat{\mathrm{Ric}}(v_i), v_j \rangle = \mathcal{A}(v_i, v_k, v_k, v_j)$ for all $v_i, v_j, v_k \in V$ where $i, j, k = 1, 2, 3$. In the Riemannian case, there always exists an orthonormal basis which the Ricci operator $\widehat{\mathrm{Ric}}$ is diagonalizable. While, in the pseudo-Riemannian setting, the Ricci operator require not to be diagonalizable, even though it is self-adjoint [13].

Let $\mathfrak{M} = (V, \langle\cdot,\cdot\rangle, \mathcal{A})$ be an algebraic curvature model and the scalar product $\langle\cdot,\cdot\rangle$ be with Lorentzian signature. Regarding to the Segre type of the Ricci operator, there exists a pseudo-orthonormal basis $\{e_1, \cdots, e_4\}$ with e_4 time-like, such that for real values a, b, c, the Ricci operator $\widehat{\mathrm{Ric}}$ takes one of the following forms [4, Theorem 2.3]:

I) The minimal polynomial of Ricci operator doesn't admit any repeated roots:

Ia) $[\widehat{\mathrm{Ric}}] = \mathrm{diag}\{a, b, c, d\}$.

Ib) $[\widehat{\mathrm{Ric}}] = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & -d \\ 0 & 0 & d & c \end{pmatrix}, \quad d \neq 0.$

II) The minimal polynomial of Ricci operator has a root with multiplicity two:

$$[\widehat{\mathrm{Ric}}] = \begin{pmatrix} b & 0 & 0 & 0 \\ 0 & 1+a & 0 & -1 \\ 0 & 0 & c & 0 \\ 0 & 1 & 0 & a-1 \end{pmatrix}.$$

III) The minimal polynomial of Ricci operator has a root with multiplicity three:

$$[\widehat{\mathrm{Ric}}] = \begin{pmatrix} b & 0 & 0 & 0 \\ 0 & a & 1 & -1 \\ 0 & 1 & a & 0 \\ 0 & 1 & 0 & a \end{pmatrix}.$$

In order to investigate weakly-Einstein conditions, we shall focus generally on all possibilities of the Ricci operator $\widehat{\mathrm{Ric}}$ of both Lorentzian and neutral signatures.

Theorem 2.1. *Let $\mathfrak{M} = (V, \langle\cdot,\cdot\rangle, \mathcal{A})$ be a conformally flat pseudo-Riemannian Lorentzian algebraic curvature model of dimension four, then $\mathfrak{M}$*

- *is $\check{\mathcal{R}}$-Einstein if and only if its scalar curvature vanishes identically.*
- *is $\check{\varrho}$-Einstein if and only if the Ricci operator is either diagonalizable with eigenvalues $\{\kappa, \kappa, -\kappa, -\kappa\}$, $\{\kappa, \kappa, \kappa, -\kappa\}$ or is two-step nilpotent.*
- *is $\mathcal{R}[\varrho]$-Einstein if and only if the Ricci operator is either digonalizable with eigenvalues $\{\kappa, \kappa, -\kappa, -\kappa\}$, $\{\kappa, \kappa, \kappa, 3\kappa\}$or is two-step nilpotent.*

Proof. Since the algebraic curvature model $\mathfrak{M}$ is conformally flat, so the Weyl conformal tensor will vanish identically, and thus, the curvature tensor $\mathcal{A}$ will be calculated due to the Ricci tensor by the following Weyl tensor relation

$$\begin{aligned} W(x,y,z,w) = \;& \mathcal{A}(x,y,z,w) \\ & + \tfrac{\tau}{(n-1)(n-2)}\{\langle x,w\rangle\langle y,z\rangle - \langle x,z\rangle\langle y,w\rangle\} \\ & - \tfrac{1}{n-2}\{\langle x,w\rangle\varrho(y,z) - \varrho(x,z)\langle y,w\rangle \\ & + \varrho(x,w)\langle y,z\rangle - \langle x,z\rangle\varrho(y,w)\}, \end{aligned} \tag{6}$$

where τ and ϱ are the scalar curvature and the Ricci tensor respectively, and n is the dimension of V. To study the $\check{\mathcal{A}}$-Einstein condition $\check{\mathcal{A}} = \frac{\|\mathcal{A}\|^2}{n}\langle\cdot,\cdot\rangle$, we need to calculate the curvature tensor $\mathcal{A}$ through the different cases. For instance, we bring the details of the case (II), the other cases could be treated by similar arguments.

Let the Ricci operator be of the form (II), by using the equation (6), the non-zero components of the $(0,4)$-curvature tensor $\mathcal{A}$ are

$$\begin{array}{lll} \mathcal{A}_{1212}=\frac{1}{6}(a+2b-c+3), & \mathcal{A}_{1214}=-\frac{1}{2}, & \mathcal{A}_{1313}=\frac{1}{3}(b-a+c), \\ \mathcal{A}_{1414}=\frac{1}{6}(c-a-2b+3), & \mathcal{A}_{2323}=\frac{1}{6}(a-b+2c+3), & \mathcal{A}_{2334}=\frac{1}{2}, \\ \mathcal{A}_{2424}=\frac{1}{6}(b-4a+c), & \mathcal{A}_{3434}=\frac{1}{6}(b-a-2c+3). & \end{array}$$

Now, direct calculations gives the symmetric $(0,2)$-tensor $\check{\mathcal{A}}$ by the following non-zero components

$$\begin{aligned} \check{\mathcal{A}}_{11} &= \tfrac{1}{3}\left(a^2+2b^2+c^2-2ac\right), \\ \check{\mathcal{A}}_{22} &= \tfrac{1}{3}\left(3a^2+b^2+c^2+2a+b+c-ba-bc-ac\right), \\ \check{\mathcal{A}}_{24} &= -\tfrac{1}{3}\left(2a+c+b\right), \\ \check{\mathcal{A}}_{33} &= \tfrac{1}{3}\left(a^2+b^2+2c^2-2ba\right), \\ \check{\mathcal{A}}_{44} &= -\tfrac{1}{3}\left(3a^2+b^2+c^2-2a-b-c-ba-bc-ac\right). \end{aligned}$$

We immediately have

$$\|\mathcal{A}\|^2=\frac{1}{3}\left(8a^2+5b^2+5c^2-4ba-2bc-4ac\right),$$

and thus, the $\check{\mathcal{A}}$-Einstein condition satisfies if and only if $\tau=2a+b+c=0$.

Now, direct calculations yield that

$$\check{\varrho}=\begin{pmatrix} b^2 & 0 & 0 & 0 \\ 0 & 2a+a^2 & 0 & -2a \\ 0 & 0 & c^2 & 0 \\ 0 & -2a & 0 & 2a-a^2 \end{pmatrix},$$

and thus, $\|\varrho\|^2=2a^2+b^2+c^2$. By applying the equation (4), this model is $\check{\varrho}$-Einstein if and only if $a=b=c=0$ which concludes that the Ricci operator is two-step nilpotent. From the last statement, we calculate the symmetric tensor $\mathcal{A}[\varrho]$ as follows

$$\begin{aligned} \mathcal{A}[\varrho]_{11} &= \tfrac{1}{3}(a^2+c^2+2ab-2ac+bc), \\ \mathcal{A}[\varrho]_{22} &= \tfrac{1}{3}(2a^2+b^2+c^2+2b-2a+2c-bc), \\ \mathcal{A}[\varrho]_{24} &= \tfrac{2}{3}(a-b-c), \\ \mathcal{A}[\varrho]_{33} &= \tfrac{1}{3}(a^2+b^2-2ab+2ac+bc), \\ \mathcal{A}[\varrho]_{44} &= -\tfrac{1}{3}(2a^2+b^2+c^2+2a-2b-2c-bc). \end{aligned}$$

Now, according to the equation (5), this case is $\mathcal{A}[\varrho]$-Einstein if and only if $a=b=c=0$. □

Regarding to the scalar products with neutral signature, there exists a pseudo-orthonormal basis $\{e_1,\cdots,e_4\}$ with e_3,e_4 time-like such that for real values a,b,c,d, the Ricci operator have one of the following forms [5, Theorem 2.2]:

I) The minimal polynomial of Ricci operator does not have repeated roots:

Ia) $[\widehat{\mathrm{Ric}}]=\mathrm{diag}\{a,b,c,d\}$,

Ib) $[\widehat{\mathrm{Ric}}]=\begin{pmatrix} a & 0 & 0 & b \\ 0 & d & 0 & 0 \\ 0 & 0 & c & 0 \\ -b & 0 & 0 & a \end{pmatrix}, (b\neq 0),$

Ic) $$[\widehat{\mathrm{Ric}}] = \begin{pmatrix} a & 0 & 0 & b \\ 0 & c & d & 0 \\ 0 & -d & c & 0 \\ -b & 0 & 0 & a \end{pmatrix}, (b \neq 0, d \neq 0).$$

II) The minimal polynomial of Ricci operator has a root with multiplicity two:

IIa) $$[\widehat{\mathrm{Ric}}] = \begin{pmatrix} 1+a & 0 & 0 & 1 \\ 0 & c & 0 & 0 \\ 0 & 0 & b & 0 \\ -1 & 0 & 0 & a-1 \end{pmatrix},$$

IIb) $$[\widehat{\mathrm{Ric}}] = \begin{pmatrix} 1+a & 0 & 0 & 1 \\ 0 & b+1 & 1 & 0 \\ 0 & -1 & b-1 & 0 \\ -1 & 0 & 0 & a-1 \end{pmatrix},$$

IIc) $$[\widehat{\mathrm{Ric}}] = \begin{pmatrix} 1+a & 0 & 0 & 1 \\ 0 & b & c & 0 \\ 0 & -c & b & 0 \\ -1 & 0 & 0 & a-1 \end{pmatrix}, (c \neq 0),$$

IId) $$[\widehat{\mathrm{Ric}}] = \begin{pmatrix} a & 1 & b-1 & 0 \\ 1 & a & 0 & -b-1 \\ 1-b & 0 & a & -1 \\ 0 & b+1 & -1 & a \end{pmatrix}, (b \neq 0).$$

III) The minimal polynomial of Ricci operator has a root with multiplicity three:

IIIa) $$[\widehat{\mathrm{Ric}}] = \begin{pmatrix} a & 1 & 0 & 1 \\ 1 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ -1 & 0 & 0 & a \end{pmatrix}.$$

IIIb) $$[\widehat{\mathrm{Ric}}] = \begin{pmatrix} a & 0 & 0 & 1 \\ 0 & b & 0 & 0 \\ 0 & 0 & a & 1 \\ -1 & 0 & 1 & a \end{pmatrix}.$$

IV) The minimal polynomial of Ricci operator has a root with multiplicity four:

$$[\widehat{\mathrm{Ric}}] = \begin{pmatrix} a-1 & 0 & 1 & 0 \\ 0 & a & -1 & 0 \\ -1 & 1 & 1+a & 1 \\ 0 & 0 & 1 & a \end{pmatrix}.$$

Now, study of weakly-Einstein conditions on the conformally flat algebraic curvature models of neutral signature gives the following result.

Theorem 2.2. *Let $\mathfrak{M} = (V, \langle\cdot,\cdot\rangle, \mathcal{A})$ be a conformally flat pseudo-Riemannian neutral signature algebraic curvature model of dimension four, then $\mathfrak{M}$*

- *is $\check{\mathcal{R}}$-Einstein if and only if its scalar curvature vanishes identically.*
- *is $\check{\varrho}$-Einstein if and only if the Ricci operator is either diagonalizable with eigenvalues $\{\kappa,\kappa,-\kappa,-\kappa\}$, $\{\kappa,\kappa,\kappa,-\kappa\}$, $\{ib, ib, -ib, -ib\}$ or is two-step nilpotent.*
- *is $\mathcal{R}[\varrho]$-Einstein if and only if the Ricci operator is either diagonalizable with eigenvalues $\{\kappa,\kappa,-\kappa,-\kappa\}$, $\{\kappa,\kappa,\kappa,3\kappa\}$, $\{ib, ib, -ib, -ib\}$ or is two-step nilpotent.*

Proof. By similar arguments to the Lorentzian signature, the proof is based on case by case study. We bring here the details of the case (IIIa) and other cases could be handled by similar arguments.

Let the Ricci operator be of the form (IIIa). By using the equation (6), the components of the curvature tensor $\mathcal{A}$ are calculated as

$$\begin{aligned}
&\mathcal{A}_{1414} = \mathcal{A}_{2424} = -\mathcal{A}_{1212} = \tfrac{1}{6}(-3a + b),\\
&\mathcal{A}_{1224} = \mathcal{A}_{1323} = \mathcal{A}_{1424} = -\mathcal{A}_{1334} = -\tfrac{1}{2},\\
&\mathcal{A}_{1313} = \mathcal{A}_{2323} = \mathcal{A}_{3434} = -\tfrac{1}{3}b,
\end{aligned}$$

and the Ricci tensor is

$$\varrho = \begin{pmatrix} a & 1 & 0 & 1 \\ 1 & a & 0 & 0 \\ 0 & 0 & -b & 0 \\ 1 & 0 & 0 & -a \end{pmatrix}.$$

The tensor field $\check{\mathcal{A}}$ is calculated as

$$\check{\mathcal{A}} = \begin{pmatrix} \frac{1}{3}(3a^2 - 2ab + b^2) & a + \frac{1}{3}b & 0 & a + \frac{1}{3}b \\ a + \frac{1}{3}b & \frac{1}{3}(3a^2 - 2ab + b^2) & 0 & 0 \\ 0 & 0 & -\frac{2}{3}b^2 & 0 \\ a + \frac{1}{3}b & 0 & 0 & -\frac{1}{3}(3a^2 - 2ab + b^2) \end{pmatrix},$$

and have $\|\mathcal{A}\|^2 = 3a^2 - 2ab + \frac{5}{3}b^2$. According to the equation (3), by direct calculations the curvature model $\mathfrak{M}$ is $\check{\mathcal{A}}$-Einstein if and only if $b = -3a$ which concludes that $\tau = 3a + b = 0$.

Now, we calculate the tensor filed $\check{\varrho}$ as follows

$$\check{\varrho} = \begin{pmatrix} a^2 & 2a & 0 & 2a \\ 2a & a^2 + 1 & 0 & 1 \\ 0 & 0 & -b^2 & 0 \\ 2a & 1 & 0 & 1 - a^2 \end{pmatrix}.$$

Since $\|\varrho\|^2 = 3a^2 + b^2$, by using the equation (4), the curvature model $\mathfrak{M}$ is never $\check{\varrho}$-Einstein. Direct calculation yields that the non-zero components of the tensor $\mathcal{A}[\varrho]$ are

$$\begin{aligned}
&\mathcal{A}[\varrho]_{11} = \tfrac{1}{3}(3a^2 - ab + b^2),\\
&\mathcal{A}[\varrho]_{12} = \mathcal{A}[\varrho]_{14} = \tfrac{2}{3}b,\\
&\mathcal{A}[\varrho]_{22} = \tfrac{1}{3}(3a^2 + b^2 - ab - 3),\\
&\mathcal{A}[\varrho]_{24} = -1,\\
&\mathcal{A}[\varrho]_{33} = -ab,\\
&\mathcal{A}[\varrho]_{44} = -\tfrac{1}{3}(3a^2 + b^2 - ab + 3).
\end{aligned}$$

By using the equation (5), the curvature model $\mathfrak{M}$ is never $\mathcal{A}[\varrho]$-Einstein in this case. □

Remark 2.1. *In the theorems 2.1 and 2.2, we studied conditions for the conformally flat algebraic curvature model $\mathfrak{M}$ to be weakly Einstein. By a more accurate view, since the condition $\check{\varrho}$-Einstein is just related to the Ricci tensor and its components, so in fact, we can discard the assumption of conformally flatness in this case.*

3. Weakly-Einstein Walker manifolds of dimension four

A Walker manifold is a pseudo-Riemannian manifold admitting a null parallel distribution $\mathcal{D}$. Walker in [14], proved the existence of local coordinates $(x_1, \cdots, x_n)$ on a Walker manifold (M, g) which the metric tensor is given by

$$g = \begin{pmatrix} 0 & 0 & \mathrm{Id}_r \\ 0 & A & H \\ \mathrm{Id}_r & {}^tH & B \end{pmatrix},$$

where A and B are symmetric matrices, A and H are independent of the coordinates $(x_1, \cdots, x_r)$ and the null parallel r-plane $\mathcal{D}$ is locally generated by the coordinate vector fields $\{\partial_{x_1}, \cdots, \partial_{x_r}\}$. The Walker manifold (M, g) is called strictly Walker whenever the distribution $\mathcal{D}$ is parallel by any of its generators. With respect to the above canonical coordinates, this property means that the matrix B is also independent of the coordinates $(x_1, \cdots, x_r)$ (see [2] and the references therein).

In dimension four, we choose local coordinates (x, y, z, t) on the Walker manifold (M, g). When the null parallel distribution $\mathcal{D}$ admits the maximum dimension, the metric g is of neutral signature and has the following form

$$g = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & c \\ 0 & 1 & c & b \end{pmatrix}, \tag{7}$$

where a, b, c are smooth functions of coordinates x, y, z, t. Moreover, the two-dimensional null parallel distribution $\mathcal{D}$ is strictly parallel if and only if the defining functions are just depended to the coordinates z, t.

In the case where the distribution $\mathcal{D}$ is of dimension one, the Walker metric g with respect to the local coordinates (x, y, z, t) is given by

$$g = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & p & q & k \\ 0 & q & s & h \\ 1 & k & h & f \end{pmatrix}, \tag{8}$$

where the defining functions p, q, s, k, h are arbitrary functions of coordinates y, z, t and f is depended to x, y, z, t. This metric is Lorentzian or of neutral signature.

To see some geometric realizations of results of the previous section, we study weakly-Einstein conditions on four-dimensional locally conformally flat Walker manifolds in this section.

Let (M, g) be a Walker manifold, where g is described by the equation (7). In [7], Davidow and Muškarov considered the four-dimensional conformally flat Walker manifolds of this kind. They specified functions a, b, c as

$$\begin{aligned} a &= x^2C + 2xyD + xE + yF + G, \\ b &= -y^2C + 2xyL + xM + yN + P, \\ c &= x^2L + y^2D + xQ + yR + S, \end{aligned}$$

where $C, D, \cdots$, are arbitrary smooth functions of z, t, satisfying several extra conditions. For simplicity, we set the functions $E, F, G, M, N, P, Q, R, S$ equal to zero and rewrite the results of [7] as following.

Theorem 3.1. [7] *A four-dimensional Walker metric of the equation* (7) *is conformally flat if the functions* a, b, c *have the form*

$$\begin{aligned} a &= x^2C + 2xyD, \\ b &= -y^2C + 2xyL, \\ c &= x^2L + y^2D. \end{aligned} \tag{9}$$

where C, D, L *are smooth functions of* z *and* t *obeying the following equations*

$$\begin{aligned} C_t - 2L_z &= 0, \\ C_z + 2D_t &= 0. \end{aligned} \tag{10}$$

Now, we take our attention to the Walker metrics of the equation (7) which satisfy in the equations (9) and (10). Summarizing, by using the local coordinates (x, y, z, t), the locally conformally flat Walker metric is

$$g = 2dxdz + 2dydt + (x^2C + 2xyD)dz^2 + 2(x^2L + y^2D)dzdt + (2xyL - y^2C)dt^2, \tag{11}$$

where the equation (10) is valid.

In this case, the non-zero components of the Levi-Civita connection are as

$$\begin{aligned}
\nabla_{\partial_x}\partial_z &= (xC + yD)\partial_x + xL\partial_y, \quad \nabla_{\partial_x}\partial_t = xL\partial_x + yL\partial_y, \\
\nabla_{\partial_y}\partial_z &= xD\partial_x + yD\partial_y, \quad \nabla_{\partial_y}\partial_z = yD\partial_x - (yC - xL)\partial_y, \\
\nabla_{\partial_z}\partial_z &= x(\frac{1}{2}xC_z + yD_z + x^2DL + 2x^2C^2 + 3y^2D^2 + 3xyCD)\partial_x \\
&\quad + (-\frac{1}{2}x^2C_t + x^2L_z + y^2D_z - xyD_t + 3x^2yDL + x^3CL + y^3D^2)\partial_y \\
&\quad - (xC + yD)\partial_z - xD\partial_t, \\
\nabla_{\partial_z}\partial_t &= (\frac{1}{2}x^2C + xyD_t + 3x^2yDL + x^3CL + y^3D^2)\partial_x \\
&\quad + (-\frac{1}{2}y^2C_z + xyL_z - y^3CD + 3xy^2DL + x^3L^2)\partial_y \\
&\quad - xL\partial_z - yD\partial_t, \\
\nabla_{\partial_t}\partial_t &= (\frac{1}{2}y^2C_z - xyL_z + x^2L_t + y^2D_t + x^3L^2 - y^3CD + 3xy^2DL)\partial_x \\
&\quad + y(-\frac{1}{2}yC_t + xL_t + y^2C^2 - 3xyCL + 3x^2L^2 + 2y^2DL)\partial_y \\
&\quad - yL\partial_z - (xL - yC)\partial_t.
\end{aligned}$$

Also, the non-zero components of the curvature tensor are

$$\begin{aligned}
&\mathcal{R}_{\partial_x\partial_z}\partial_x = C\partial_x + L\partial_y, \quad \mathcal{R}_{\partial_x\partial_z}\partial_y = \mathcal{R}_{\partial_y\partial_z}\partial_x = D\partial_x, \quad \mathcal{R}_{\partial_x\partial_t}\partial_x = L\partial_x, \\
&\mathcal{R}_{\partial_y\partial_t}\partial_y = D\partial_x - C\partial_y, \quad \mathcal{R}_{\partial_x\partial_t}\partial_y = \mathcal{R}_{\partial_y\partial_t}\partial_x = L\partial_y, \quad \mathcal{R}_{\partial_y\partial_z}\partial_y = D\partial_y, \\
&\mathcal{R}_{\partial_z\partial_t}\partial_x = (xL_z - xC_t - yD_t)\partial_x + (yL_z - xL_t)\partial_y, \\
&\mathcal{R}_{\partial_z\partial_t}\partial_y = (yD_z - xD_t)\partial_x - (yC_z - xL_z + yD_t)\partial_y, \\
&\mathcal{R}_{\partial_x\partial_z}\partial_t = (2xyDL + x^2CL + xC_t + yD_t - xL_z)\partial_x + (x^2L^2 + y^2DL)\partial_y - L\partial_z, \\
&\mathcal{R}_{\partial_x\partial_t}\partial_z = (2xyDL + x^2CL)\partial_x + (x^2L^2 + y^2DL - xL_t + yL_z)\partial_y - L\partial_z, \\
&\mathcal{R}_{\partial_x\partial_t}\partial_t = (x^2L^2 + y^2DL - yL_z + xL_t)\partial_x + (-y^2CL + 2xyL^2)\partial_y - L\partial_t, \\
&\mathcal{R}_{\partial_y\partial_z}\partial_z = (2xyD^2 + x^2CD)\partial_x + (y^2D^2 + x^2DL + yD_z - xD_t)\partial_y - D\partial_z,
\end{aligned}$$

$$
\begin{aligned}
\mathcal{R}_{\partial_y\partial_z}\partial_t &= (x^2DL + y^2D^2 + xD_t - yD_z)\partial_x + (2xyDL - y^2CD)\partial_y - D\partial_t,\\
\mathcal{R}_{\partial_y\partial_t}\partial_z &= (y^2D^2 + x^2DL)\partial_x + (-y^2CD + 2xyDL + xL_z - yC_z - yD_t)\partial_y - D\partial_t,\\
\mathcal{R}_{\partial_x\partial_z}\partial_z &= (x^2DL + x^2C^2 + 2xyCD + y^2D^2)\partial_x + (2xyDL + x^2CL + xL_z - xC_t\\
&\quad - yD_t)\partial_y - C\partial_z - D\partial_t,\\
\mathcal{R}_{\partial_y\partial_t}\partial_t &= (-y^2CD + 2xyDL - xL_z + yC_z + yD_t)\partial_x + (y^2C^2 - 2xyCL + x^2L^2\\
&\quad + y^2DL)\partial_y - L\partial_z + C\partial_t,\\
\mathcal{R}_{\partial_z\partial_t}\partial_z &= (2x^2yDL_z - 2x^2yC_tD + x^2yD_zL - x^3CC_t - x^2yCD_t - 3xy^2DD_t + y^3DD_z\\
&\quad + x^3CL_z - x^3LD_t)\partial_x - (2x^2yDL_t - 4xy^2DL_z - xyL_{zz} + x^2L_{zt} + y^2D_{zt}\\
&\quad - \frac{1}{2}x^2C_{tt} - xyD_{tt} + 2y^3DC_t + x^3CL_t - \frac{1}{2}x^3LC_t + \frac{1}{2}x^2yCC_t + \frac{1}{2}xy^2C_tD\\
&\quad - 2x^2yCL_z + xy^2CD_t)\partial_y - (xL_z - xC_t - yD_t)\partial_z - (yD_z + xD_t)\partial_t,\\
\mathcal{R}_{\partial_z\partial_t}\partial_t &= (-4x^2yD_tL - 2x^2yD_zL - xyL_{zz} - y^3CD_z + 2x^3LL_z + x^2L_{zt} + y^2D_{zt}\\
&\quad - \frac{1}{2}x^2C_{tt} - xyD_{tt} - \frac{3}{2}x^3LC_t + \frac{1}{2}x^2yCC_t - \frac{1}{2}xy^2C_tD - x^2yCL_z + 2xy^2CD_t)\partial_x\\
&\quad - (xy^2CL_z - 3x^2yLL_z + xy^2DL_t + 2xy^2D_tL + x^3LL_t - y^3DL_z - y^3CD_t)\partial_y\\
&\quad + (xL_t - yL_z)\partial_z - (xL_z + yD_t)\partial_t.
\end{aligned}
$$

Then, the Ricci operator $\widehat{\mathrm{Ric}}$, with respect to the basis $\{\partial_x, \partial_y, \partial_z, \partial_t\}$ is described as follows

$$
\widehat{\mathrm{Ric}} = \begin{pmatrix} C & 2D & 2yD_z - 2xD_t & xC_t - yC_z \\ 2L & -C & xC_t - yC_z & -2yL_z + 2xL_t \\ 0 & 0 & C & 2L \\ 0 & 0 & 2D & -C \end{pmatrix}. \tag{12}
$$

Theorem 3.2. *Let (M, g) be a four-dimensional locally conformally flat Walker metric of the equation* (11)*. Then*

- *(M, g) is $\check{\mathcal{R}}$-Einstein.*
- *(M, g) is $\check{\varrho}$-Einstein if and only if one of the following cases occurs:*

  (1) $C = \kappa_1, \quad D = \kappa_2, \quad L = \kappa_3,$
  (2) $C = L = 0, \quad D = f_1(z),$
  (3) $C = D = 0, \quad L = f_2(t),$

  where $\kappa_i, i = 1, \cdots, 3$ are arbitrary real constants and $f_1(z), f_2(t)$ are arbitrary smooth functions on M.
- *(M, g) is $\mathcal{R}[\varrho]$-Einstein if and only if be $\check{\varrho}$-Einstein.*

Proof. According to the algebraic results of the Theorem 2.2, since the scalar curvature is equal to zero, the four-dimensional conformally flat Walker metric (11) is always $\check{\mathcal{R}}$-Einstein. For the $\check{\varrho}$-Einstein condition, the only possibility for the Ricci operator to be digonalizable is the case with eigenvalues $\kappa, \kappa, -\kappa, -\kappa$ which happens if and only if the functions C, D, L be real constants. In this case, the conformally flat equations (10) will be valid trivially and so case (1) of the statement is deduced. On the other hand, according to the equation (12), the Ricci operator of confornally flat manifold (M, g) is two-step nilpotent if and only if

$$\begin{cases} C^2 + 4DL = 0, \\ C(yD_z - xD_t) + D(xC_t - yC_z) = 0, \\ D(-yL_z + xL_t) + L(yD_z - xD_t) = 0, \\ L(xC_t - yC_z) - C(-yL_z + xL_t) = 0. \end{cases}$$

Therefore, a straightforward computation shows that one of the following cases will occur

(1) $C = L = 0, \quad D = f_1(z),$
(2) $C = D = 0, \quad L = f_2(t),$

which results in the cases (2)-(3) of the statement.

Finally, the $\mathcal{R}[\varrho]$-Einstein condition is valid whenever one of the forms of diagonalizable Ricci operator establish or be two-step nilpotent. These conditions are exactly the same for $\check{\varrho}$-Einstein condition, so the proof is complete. □

Next, we examine the weakly-Einstein conditions on four-dimensional Walker metrics admitting a one-dimensional null parallel distribution. In order to make the results more elegant and readable we set in the metric g of the equation (8), $p = s = 1, q, k, h = 0$. In this case, we study the following Walker metric in the local coordinates (x, y, z, t) which is Lorentzian.

$$g = 2dxdt + dy^2 + dz^2 + f(x, y, z, t)dt^2. \tag{13}$$

By using the well known Koszul formula, the non-zero components of the Levi-Civita connection of the Walker metric (13) are given by

$$\begin{aligned} &\nabla_{\partial_x}\partial_t = \frac{1}{2}f_x\partial_x, \quad \nabla_{\partial_y}\partial_t = \frac{1}{2}f_y\partial_x, \quad \nabla_{\partial_z}\partial_t = \frac{1}{2}f_z\partial_x, \\ &\nabla_{\partial_t}\partial_t = \frac{1}{2}((ff_x + f_t)\partial_x - f_y\partial_y - f_z\partial_z - f_x\partial_t). \end{aligned}$$

Then, non-zero components of the curvature tensor are calculated immediately as

$$\begin{aligned} &\mathcal{R}_{\partial_x\partial_t}\partial_x = \frac{1}{2}f_{xx}\partial_x, \quad \mathcal{R}_{\partial_x\partial_t}\partial_y = \frac{1}{2}f_{yx}\partial_x, \quad \mathcal{R}_{\partial_x\partial_t}\partial_z = \frac{1}{2}f_{zx}\partial_x \\ &\mathcal{R}_{\partial_y\partial_t}\partial_x = \frac{1}{2}f_{yx}\partial_x, \quad \mathcal{R}_{\partial_y\partial_t}\partial_y = \frac{1}{2}f_{yy}\partial_x, \quad \mathcal{R}_{\partial_y\partial_t}\partial_z = \frac{1}{2}f_{zy}\partial_x, \\ &\mathcal{R}_{\partial_z\partial_t}\partial_x = \frac{1}{2}f_{zx}\partial_x, \quad \mathcal{R}_{\partial_z\partial_t}\partial_y = \frac{1}{2}f_{zy}\partial_x, \quad \mathcal{R}_{\partial_z\partial_t}\partial_z = \frac{1}{2}f_{zz}\partial_x, \\ &\mathcal{R}_{\partial_x\partial_t}\partial_t = \frac{1}{2}(ff_{xx}\partial_x - f_{yx}\partial_y - f_{zx}\partial_z - f_{xx}\partial_t), \\ &\mathcal{R}_{\partial_y\partial_t}\partial_t = \frac{1}{2}(ff_{yx}\partial_x - f_{yy}\partial_y - f_{zy}\partial_z - f_{yx}\partial_t), \\ &\mathcal{R}_{\partial_z\partial_t}\partial_t = \frac{1}{2}(ff_{zx}\partial_x - f_{zy}\partial_y - f_{zz}\partial_z - f_{zx}\partial_t). \end{aligned}$$

Also, the Ricci tensor of the Walker metric (13) is given by

$$\varrho = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & f_{xx} \\ 0 & 0 & 0 & -f_{yx} \\ 0 & 0 & 0 & f_{zx} \\ f_{xx} & f_{yx} & f_{zx} & \Delta \end{pmatrix},$$

where $\Delta = ff_{xx} - f_{yy} - f_{zz}$. The non-zero components of the Weyl tensor of the Walker metric (13) are

$$\begin{aligned}
&W(\partial_x, \partial_y, \partial_y, \partial_t) = \frac{1}{12} f_{xx}, \quad W(\partial_x, \partial_z, \partial_z, \partial_t) = \frac{1}{12} f_{xx}, \quad W(\partial_x, \partial_t, \partial_y, \partial_t) = -\frac{1}{4} f_{yx}, \\
&W(\partial_x, \partial_t, \partial_z, \partial_t) = -\frac{1}{4} f_{zx}, \quad W(\partial_x, \partial_t, \partial_x, \partial_t) = -\frac{1}{6} f_{xx}, \quad W(\partial_y, \partial_z, \partial_y, \partial_z) = \frac{1}{6} f_{xx}, \\
&W(\partial_y, \partial_z, \partial_y, \partial_t) = -\frac{1}{4} f_{zx}, \quad W(\partial_y, \partial_z, \partial_z, \partial_t) = \frac{1}{4} f_{yx}, \quad W(\partial_y, \partial_t, \partial_z, \partial_t) = -\frac{1}{2} f_{zy}, \quad (14) \\
&W(\partial_y, \partial_t, \partial_y, \partial_t) = -\frac{1}{12}(ff_{xx} + 3f_{zz} - 3f_{yy}), \\
&W(\partial_z, \partial_t, \partial_z, \partial_t) = -\frac{1}{12}(ff_{xx} - 3f_{zz} + 3f_{yy}).
\end{aligned}$$

Now, we can calculate the four-dimensional locally conformally flat Lorentzian Walker metric.

Theorem 3.3. *Let (M, g) be a Lorentzian Walker metric, where g is described by the equation* (13)*. The following statements hold*

(i) (M, g) *is flat if and only if*

$$f(x, y, z, t) = f_1(t)x + f_2(t)y + f_3(t)z + f_4(t).$$

(ii) (M, g) *is Einstein if and only if be Ricci-flat if and only if*

$$f(x, y, z, t) = f_1(t)x + f_2(y, z, t),$$

where $(\partial^2_{zz} + \partial^2_{yy})f_2(y, z, t) = 0$.

(iii) (M, g) *is locally conformally flat if and only if*

$$f(x, y, z, t) = f_1(t)x + f_2(t)y + f_3(t)z + f_4(t) + f_5(t)(y^2 + z^2), \quad (15)$$

where $f_i(t), i = 1, \cdots, 5$ *and* $f_2(y, z, t)$ *are arbitrary smooth functions.*

Proof. The Walker metric (13) is flat if and only if its curvature tensor vanishes, which is equivalent to

$$f_{xx} = f_{yx} = f_{zx} = f_{yy} = f_{zy} = f_{zz} = 0.$$

Thus, the Walker metric (13) is flat if and only if the function f satisfies

$$f(x, y, z, t) = f_1(t)x + f_2(t)y + f_3(t)z + f_4(t).$$

Moreover, the Walker metric (13) is Einstein if $\varrho = \frac{\tau}{4} g$. On the other hand, it follows from the Ricci tensor that the Ricci operator is given by

$$\widehat{\mathrm{Ric}} = \frac{1}{2} \begin{pmatrix} f_{xx} & f_{yx} & f_{zx} & -f_{yy} - f_{zz} \\ 0 & 0 & 0 & f_{yx} \\ 0 & 0 & 0 & f_{zx} \\ 0 & 0 & 0 & f_{xx} \end{pmatrix}.$$

Therefore, the scalar curvature of a Walker metric (13) is $\tau = f_{xx}$, and the Einstein equations are as follows

$$f_{xx} = f_{yx} = f_{zx} = 0, \quad f_{zz} - f_{yy} = 0,$$

which are the same equation as for the Ricci-flat Walker metric (13). Thus, the Walker metric (13) is Einstein if and only if be Ricci-flat, so if and only if the function f takes the special form

$$f(x,y,z,t)=f_1(t)x+f_2(y,z,t),$$

where $(\partial^2_{zz}+\partial^2_{yy})f_2(y,z,t)=0$. A Walker metric (13) is conformally flat if and only if the Weyl tensor is zero. So, regarding to the components of the Weyl tensor (14), the Walker metric (13) is conformally flat if and only if

$$\begin{cases} f_{xx}=f_{yx}=f_{zx}=0,\\ ff_{xx}-3f_{yy}+3f_{xz}=0,\\ ff_{xx}-3f_{yy}-3f_{zz}=0.\end{cases}$$

By solving the above equations, f is as claimed in (15) and the proof is complete. □

Now, let (M,g) be a four-dimensional conformally flat Lorentzian Walker manifold, where g is the metric tensor of the equations (13) and (15). Then, the Ricci operator $\widehat{\mathrm{Ric}}$, with respect to the basis $\{\partial_x,\partial_y,\partial_z,\partial_t\}$ is given by

$$\widehat{\mathrm{Ric}}=\begin{pmatrix} 0&0&0&-2f_5(t)\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\end{pmatrix}. \tag{16}$$

Clearly, if $f_5(t)=0$, then the four-dimensional conformally flat Lorentzian Walker manifold is Ricci flat.

Theorem 3.4. *The four-dimensional conformally flat Lorentzian Walker metric is weakly-Einstein (non-Einstein) if and only if* $f_5(t)\neq 0$.

Proof. According to the Theorem 2.1, as the scalar curvature is zero, the four-dimensional conformally flat Lorentzian walker metric is $\check{\mathcal{R}}$-Einstein. On the other hand, since the Ricci operator is two-step nilpotent, Thus the metric is $\check{\varrho}$-Einstein and $\mathcal{R}[\varrho]$-Einstein. □

4. **Locally symmetric examples**

Following the results in [10], locally symmetric spaces constitute a large class of weakly Einstein locally conformally flat manifolds. So, this is good idea to identify locally symmetric examples of the spaces which were considered through the previous section. We undertake this study in the following theorems.

Theorem 4.1. *Let* (M,g) *be a locally symmetric four-dimensional conformally flat Walker metric of the equation* (11). *Then*

- (M,g) *is* $\check{\mathcal{R}}$*-Einstein.*
- (M,g) *is* $\check{\varrho}$*-Einstein.*
- (M,g) *is* $\mathcal{R}[\varrho]$*-Einstein.*

Proof. The Walker metric (11) is locally symmetric if and only if $\nabla\mathcal{R}=0$. So, by calculating the covariant derivative of the curvature tensor, it follows that the Walker metric (11) is locally symmetric if and only if the functions C,D,L are real constants. Thus, the locally symmetric Walker metric (11) is given by

$$g=2dxdz+2dydt+(\kappa_1x^2+2\kappa_2xy)dz^2+2(\kappa_3x^2+\kappa_2y^2)dzdt+(2\kappa_3xy-\kappa_1y^2)dt^2. \tag{17}$$

In this case, the Ricci operator is given by

$$\widehat{\mathrm{Ric}} = \begin{pmatrix} \kappa_1 & 2\kappa_2 & 0 & 0 \\ 2\kappa_3 & -\kappa_1 & 0 & 0 \\ 0 & 0 & \kappa_1 & 2\kappa_3 \\ 0 & 0 & 2\kappa_2 & -\kappa_1 \end{pmatrix}. \tag{18}$$

The Ricci operator is digonalizable with eigenvalues $\kappa, \kappa, -\kappa, -\kappa$. Then, according to the Theorem 3.2, all conditions establish simultaneously. □

About locally symmetric Lorentzian Walker metrics (13) we have the following result.

Theorem 4.2. *A four-dimensional weakly-Einstein locally conformally flat Lorentzian Walker metric* (13) *is locally symmetric if and only if* $f_1(t) = -\frac{f_5'(t)}{f_5(t)}$.

Proof. The Walker metric (13) is locally symmetric if and only if $\nabla\mathcal{R} = 0$. Long but routine calculations denote that non-zero components of the covariant derivative of $\mathcal{R}$ are given by

$$\begin{aligned}
&(\nabla_{\partial_t}\mathcal{R})(\partial_y, \partial_t, \partial_y) = (\nabla_{\partial_t}\mathcal{R})(\partial_z, \partial_t, \partial_z) = (f_1(t)f_5(t) + f_5'(t))\partial_x, \\
&(\nabla_{\partial_t}\mathcal{R})(\partial_t, \partial_y, \partial_t) = (f_1(t)f_5(t) + f_5'(t))\partial_y, \\
&(\nabla_{\partial_t}\mathcal{R})(\partial_t, \partial_z, \partial_t) = (f_1(t)f_5(t) + f_5'(t))\partial_z.
\end{aligned}$$

Thus, the Walker metric (13) is locally symmetric if and only if $f_1(t) = -\frac{f_5'(t)}{f_5(t)}$. □

Funding: This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors.

Data Availability: Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest: We wish to confirm that there are no known conflicts of interest associated with this publication and there has been no financial support for this work that could have influenced its outcome.

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