A NEW APPROACH FOR FINDING EXACT SOLUTIONS OF FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

Qinghua FENG[1]

In this paper, by introducing a new ansatz, a new fractional sub-equation method is proposed for finding exact solutions of fractional partial differential equations (FPDEs) in the sense of modified Riemann-Liouville derivative. For illustrating the validity of this method, we apply it to the space-time fractional Whitham-Broer-Kaup (WBK) equations and the space-time fractional Fokas equation. As a result, some new exact solutions for them are successfully established..

Keywords: Fractional sub-equation method; Fractional partial differential equations; exact solutions; Fractional complex transformation

MSC 2010: 35Q51, 35Q53.

1. Introduction

Recently, Fractional differential equations have been the focus of many studies due to their frequent appearance in various applications in physics, biology, engineering, signal processing, systems identification, control theory, finance and fractional dynamics. Among the investigations for fractional differential equations, research for seeking exact solutions and approximate solutions of fractional differential equations is a hot topic. Many powerful and efficient methods have been proposed so far (for example, see [1-12]). Using these methods, solutions with various forms for some given fractional differential equations have been established.

In this paper, we propose a new fractional sub-equation method to establish exact solutions for fractional partial differential equations (FPDEs). The fractional derivative is defined in the sense of modified Riemann-Liouville derivative by Jumarie [13]. This method is based on the following fractional ODE:

$$D_{\xi}^{2\alpha}G(\xi)+\mu G(\xi)=0,\mu\neq 0, \tag{1}$$

where $D_{\xi}^{2\alpha}G(\xi)$ denotes the modified Riemann-Liouville derivative of orde α for $G(\xi)$ with respect to ξ.

The definition and some important properties for the Jumarie's modified Riemann-Liouville derivative of order α are listed as follows [13]:

[1]Prof., School of Science, Shandong University of Technology, China, e-mail: fqhua@sina.com

$$D_\xi^\alpha f(t)=\begin{cases}\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-\xi)^{-\alpha}(f(\xi)-f(0))d\xi, 0<\alpha<1,\\ (f^{(n)}(t))^{(\alpha-n)}, n\le\alpha<n+1, n\ge 1,\end{cases}$$

$$D_t^\alpha t^r=\frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)}t^{r-\alpha}, \tag{2}$$

$$D_t^\alpha(f(t)g(t))=g(t)D_t^\alpha(f(t))+f(t)D_t^\alpha(g(t)), \tag{3}$$

$$D_t^\alpha f[g(t)]=f_g{}'[g(t)]D_t^\alpha g(t)=D_g^\alpha f[g(t)](g'(t))^\alpha. \tag{4}$$

We organize this paper as follows. In Section 2, we derive the expression for $\frac{D_\xi^\alpha G(\xi)}{G(\xi)}$ related to Eq. (1). In Section 3, we give the description of the fractional sub-equation method for solving FPDEs. Then in Section 4 we apply this method to establish exact solutions for the space-time fractional Whitham-Broer-Kaup (WBK) equations and the space-time fractional Fokas equation. Some conclusions are presented at the end of the paper.

2. The general expression for $\frac{D_\xi^\alpha G(\xi)}{G(\xi)}$

In order to obtain the general solutions for Eq. (1), we suppose $G(\xi)=H(\eta)$, and a nonlinear fractional complex transformation $\eta=\frac{\xi^\alpha}{\Gamma(1+\alpha)}$. Then by Eq. (2) and the first equality in Eq. (4), we have $D_\xi^\alpha G(\xi)=D_\xi^\alpha H(\eta)=H'(\eta)D_\xi^\alpha\eta=H'(\eta)$. So Eq. (1) can be turned into the following second ordinary differential equation

$$H''(\eta)+\mu H(\eta)=0. \tag{5}$$

By the general solutions of Eq. (5) we have

$$\frac{H''(\eta)}{H(\eta)}=\begin{cases}\sqrt{-\mu}\left(\frac{C_1\sinh\sqrt{-\mu}\eta+C_2\cosh\sqrt{-\mu}\eta}{C_1\cosh\sqrt{-\mu}\eta+C_2\sinh\sqrt{-\mu}\eta}\right), \mu<0,\\ \sqrt{\mu}\left(\frac{-C_1\sin\sqrt{\mu}\eta+C_2\cos\sqrt{\mu}\eta}{C_1\cos\sqrt{\mu}\eta+C_2\sin\sqrt{\mu}\eta}\right), \mu>0,\end{cases} \tag{6}$$

where C_1, C_2 are arbitrary constants.

Furthermore, we obtain

$$\frac{D_\xi^\alpha G(\xi)}{G(\xi)} = \begin{cases} \sqrt{-\mu}[\dfrac{C_1 \sinh \dfrac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)} + C_2 \cosh \dfrac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}}{C_1 \cosh \dfrac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)} + C_2 \sinh \dfrac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}}], \mu < 0, \\ \sqrt{\mu}[\dfrac{-C_1 \sin \dfrac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)} + C_2 \cos \dfrac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)}}{C_1 \cos \dfrac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)} + C_2 \sin \dfrac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)}}], \mu > 0, \end{cases} \quad (7)$$

3. Description of the fractional sub-equation method

In this section we describe the main steps of the fractional sub-equation method for finding exact solutions of FPDEs.

Suppose that a fractional partial differential equation, say in the independent variables $t, x_1, x_2, ..., x_n$, is given by

$$P(u_1, ..., u_k, ..., D_t^\alpha u_1, ..., D_t^\alpha u_k, D_{x_1}^\alpha u_1, ..., D_{x_1}^\alpha u_k, D_{x_n}^\alpha u_1, ..., \\ D_{x_n}^\alpha u_k, D_t^{2\alpha} u_1, ..., D_t^{2\alpha} u_k, D_{x_1}^{2\alpha} u_1, ...) = 0, \quad (8)$$

where $u_i = u_i(t, x_1, x_2, ..., x_n), i = 1, ..., k$ are unknown functions, P is a polynomial in u_i and their various partial derivatives including fractional derivatives.

Step 1. Suppose that

$$u_i(t, x_1, x_2, ..., x_n) = U_i(\xi), \xi = ct + k_1 x_1 + k_2 x_2 + ... + k_n x_n + \xi_0. \quad (9)$$

Then by the second equality in Eq. (4), Eq. (8) can be turned into the following fractional ordinary differential equation with respect to the variable ξ:

$$\widetilde{P}(U_1, ..., U_k, ..., c^\alpha D_\xi^\alpha U_1, ..., c^\alpha D_\xi^\alpha U_k, k_1^\alpha D_\xi^\alpha U_1, ..., k_1^\alpha D_\xi^\alpha U_k, k_n^\alpha D_\xi^\alpha U_1, ..., \\ k_n^\alpha D_\xi^\alpha u_k, c^{2\alpha} D_\xi^{2\alpha} U_1, ..., c^{2\alpha} D_\xi^{2\alpha} U_k, k_1^{2\alpha} D_\xi^{2\alpha} U_1, ...) = 0. \quad (10)$$

Step 2. Suppose that the solution of (10) can be expressed by a polynomial in $(\frac{D_\xi^\alpha G}{G})$ as follows:

$$U_j(\xi) = a_{j,0} + \sum_{i=1}^{m_j} [a_{j,i} (\frac{D_\xi^\alpha G}{G})^i + b_{j,i} (\frac{D_\xi^\alpha G}{G})^{i-1} \sqrt{\sigma(1 + \frac{1}{\mu} (\frac{D_\xi^\alpha G}{G})^2)}], j = 1, 2, ..., k, \quad (11)$$

where $G = G(\xi)$ satisfies Eq. (1), σ is an constant, and $a_{j,i}, i = 0, 1, ..., m$, $j = 1, 2, ..., k$ are constants to be determined later. The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in (10).

Step 3. Substituting (11) into (10) and using (1), collecting all terms with the same order of $\sqrt{\sigma(1+\frac{1}{\mu}(\frac{D_\xi^\alpha G}{G})^2)}(\frac{D_\xi^\alpha G}{G})$ together, the left-hand side of (10) is converted into another polynomial in $(\frac{D_\xi^\alpha G}{G})$. Equating each coefficient of this polynomial to zero, yields a set of algebraic equations for $a_{j,0}, a_{j,i}, b_{j,i}, i=1,...,m,$ $j=1,2,...,k$.

Step 4. Solving the equations in Step 3, and using (7), we can construct a variety of exact solutions for Eq. (8).

4. Applications

In this section, we will apply the described method in Section 2 to some fractional partial differential equations.

4.1. Space-time fractional Whitham-Broer-Kaup (WBK) equations

We consider the space-time fractional Whitham-Broer-Kaup (WBK) equations [14]

$$\begin{cases} D_t^\alpha u + uD_x^\alpha u + D_x^\alpha v + \beta D_t^{2\alpha} u = 0, \\ D_t^\alpha v + D_x^\alpha (uv) - \beta D_x^{2\alpha} v + \gamma D_x^{3\alpha} u = 0, \end{cases}, 0 < \alpha \leq 1. \tag{12}$$

In [14], the authors solved Eqs. (12) by a proposed fractional sub-equation method based on the fractional Riccati equation, and established some exact solutions for them. Now we will apply the described method above to Eqs. (12). To begin with, we suppose $u(x,t) = U(\xi), v(x,t) = V(\xi)$, where $\xi = kx + ct + \xi_0$. Then by use of the second equality in (4), Eqs. (12) can be turned into

$$\begin{cases} c^\alpha D_\xi^\alpha U + k^\alpha U D_\xi^\alpha U + k^\alpha D_\xi^\alpha V + \beta k^{2\alpha} D_\xi^{2\alpha} U = 0, \\ c^\alpha D_\xi^\alpha V + k^\alpha D_\xi^\alpha (UV) - \beta k^{2\alpha} D_\xi^{2\alpha} V + \gamma k^{3\alpha} D_\xi^{3\alpha} U = 0, \end{cases}. \tag{13}$$

Suppose that the solutions of Eqs. (13) can be expressed by

$$\begin{cases} U(\xi) = a_0 + \sum_{i=1}^{m} [a_i (\frac{D_\xi^\alpha G}{G})^i + b_i (\frac{D_\xi^\alpha G}{G})^{i-1} \sqrt{\sigma(1+\frac{1}{\mu}(\frac{D_\xi^\alpha G}{G})^2)}], \\ V(\xi) = c_0 + \sum_{i=1}^{n} [c_i (\frac{D_\xi^\alpha G}{G})^i + d_i (\frac{D_\xi^\alpha G}{G})^{i-1} \sqrt{\sigma(1+\frac{1}{\mu}(\frac{D_\xi^\alpha G}{G})^2)}], \end{cases} \tag{14}$$

where $G = G(\xi)$ satisfies Eq. (1).

Balancing the order between the highest order derivative term and nonlinear term in Eqs. (13), we can obtain $m=1, n=2$. So we have

$$\begin{cases} U(\xi)=a_0+a_1(\frac{D_\xi^\alpha G}{G})+b_1\sqrt{\sigma(1+\frac{1}{\mu}(\frac{D_\xi^\alpha G}{G})^2)}, \\ V(\xi)=c_0+c_1(\frac{D_\xi^\alpha G}{G})+c_2(\frac{D_\xi^\alpha G}{G})^2+d_1\sqrt{\sigma(1+\frac{1}{\mu}(\frac{D_\xi^\alpha G}{G})^2)}+d_2(\frac{D_\xi^\alpha G}{G})\sqrt{\sigma(1+\frac{1}{\mu}(\frac{D_\xi^\alpha G}{G})^2)}, \end{cases} \tag{15}$$

Substituting (15) along with (1) into (13) and collecting all the terms with the same power of $\sqrt{\sigma(1+\frac{1}{\mu}(\frac{D_\xi^\alpha G}{G})^2)}(\frac{D_\xi^\alpha G}{G})$ together, equating each coefficient to zero, yields a set of algebraic equations. Solving these equations, yields:

Case 1:

$$a_0=-c^\alpha k^\alpha, a_1=\pm k^\alpha\sqrt{\beta^2+\gamma}, b_1=b_1, c_0=b_1^2\sigma(\pm\frac{\beta}{\sqrt{\beta^2+\gamma}}-1), c_1=0,$$

$$c_2=k^{2\alpha}(\pm\beta\sqrt{\beta^2+\gamma}-\beta^2-\gamma), d_1=0, d_2=b_1k^\alpha(\mp\sqrt{\beta^2+\gamma}+\beta), \mu=\frac{k^{-2\alpha}b_1^2\sigma}{\beta^2+\gamma}.$$

Case 2:

$$a_0=-c^\alpha k^\alpha, a_1=0, b_1=b_1, c_0=-\frac{1}{4}b_1^2\sigma, c_1=0,$$

$$c_2=-2k^{2\alpha}(\beta^2+\gamma), d_1=0, d_2=b_1k^\alpha\beta, \mu=\frac{k^{-2\alpha}b_1^2\sigma}{4(\beta^2+\gamma)}.$$

Case 3:

$$a_0=-c^\alpha k^\alpha, a_1=\pm 2k^\alpha\sqrt{\beta^2+\gamma}, b_1=b_1, c_0=2\mu k^{2\alpha}(\pm\beta\sqrt{\beta^2+\gamma}-\beta^2-\gamma), c_1=0,$$

$$c_2=2k^{2\alpha}(\pm\beta\sqrt{\beta^2+\gamma}-\beta^2-\gamma), d_1=0, d_2=0, \mu=\mu.$$

Substituting the results above into (15), and combining with (7) we can obtain the following exact solutions to the space-time fractional Whitham-Broer-Kaup (WBK) equations.

From Case 1 and (7) we obtain:

When $\mu<0, \beta^2+\gamma>0$,

$$U_1(\xi)=-c^\alpha k^\alpha\pm k^\alpha\sqrt{-\mu(\beta^2+\gamma)}[\frac{C_1\sinh\frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}+C_2\cosh\frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}}{C_1\cosh\frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}+C_2\sinh\frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}}]$$

$$+b_1\sqrt{\sigma\{1-[\frac{C_1\sinh\frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}+C_2\cosh\frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}}{C_1\cosh\frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}+C_2\sinh\frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}}]^2\}}$$

$$V_1(\xi)=b_1^2\sigma(\pm\frac{\beta}{\sqrt{\beta^2+\gamma}}-1)-k^{2\alpha}(\pm\beta\sqrt{\beta^2+\gamma}-\beta^2-\gamma)$$
$$\mu[\frac{C_1\sinh\frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}+C_2\cosh\frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}}{C_1\cosh\frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}+C_2\sinh\frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}}]^2 \qquad (16)$$
$$+b_1k^\alpha(\mp\sqrt{\beta^2+\gamma}+\beta)\sqrt{-\mu}[\frac{C_1\sinh\frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}+C_2\cosh\frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}}{C_1\cosh\frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}+C_2\sinh\frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}}]$$
$$\sqrt{\sigma\{1-[\frac{C_1\sinh\frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}+C_2\cosh\frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}}{C_1\cosh\frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}+C_2\sinh\frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}}]^2\}},$$

where $\xi=kx+ct+\xi_0,\mu=\frac{k^{-2\alpha}b_1^2\sigma}{\beta^2+\gamma}$.

When $\mu>0,\beta^2+\gamma>0$,

$$\begin{cases}U_2(\xi)=-c^\alpha k^\alpha\pm k^\alpha\sqrt{\mu(\beta^2+\gamma)}[\frac{-C_1\sin\frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)}+C_2\cos\frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)}}{C_1\cos\frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)}+C_2\sin\frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)}}] \\ +b_1\sqrt{\sigma\{1+[\frac{-C_1\sin\frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)}+C_2\cos\frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)}}{C_1\cos\frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)}+C_2\sin\frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)}}]^2\}} \\ V_2(\xi)=b_1^2\sigma(\pm\frac{\beta}{\sqrt{\beta^2+\gamma}}-1)+k^{2\alpha}(\pm\beta\sqrt{\beta^2+\gamma}-\beta^2-\gamma) \\ \mu[\frac{-C_1\sin\frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)}+C_2\cos\frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)}}{C_1\cos\frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)}+C_2\sin\frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)}}]^2 \\ +b_1k^\alpha(\mp\sqrt{\beta^2+\gamma}+\beta)\sqrt{\mu}[\frac{-C_1\sin\frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)}+C_2\cos\frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)}}{C_1\cos\frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)}+C_2\sin\frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)}}] \\ \sqrt{\sigma\{1+[\frac{-C_1\sin\frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)}+C_2\cos\frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)}}{C_1\cos\frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)}+C_2\sin\frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)}}]^2\}},\end{cases} \qquad (17)$$

where $\xi = kx + ct + \xi_0, \mu = \dfrac{k^{-2\alpha} b_1^2 \sigma}{\beta^2 + \gamma}$.

From Case 2 and (7) we obtain:

When $\mu < 0$,

$$\begin{cases} U_3(\xi) = -c^\alpha k^\alpha + b_1 \sqrt{\sigma\{1 - [\dfrac{C_1 \sinh \frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)} + C_2 \cosh \frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}}{C_1 \cosh \frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)} + C_2 \sinh \frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}}]^2\}} \\ V_3(\xi) = -\dfrac{1}{4} b_1^2 \sigma + 2k^{2\alpha}(\beta^2 + \gamma)\mu [\dfrac{C_1 \sinh \frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)} + C_2 \cosh \frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}}{C_1 \cosh \frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)} + C_2 \sinh \frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}}]^2 \\ + b_1 k^\alpha \beta \sqrt{-\mu} [\dfrac{C_1 \sinh \frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)} + C_2 \cosh \frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}}{C_1 \cosh \frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)} + C_2 \sinh \frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}}] \\ \sqrt{\sigma\{1 - [\dfrac{C_1 \sinh \frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)} + C_2 \cosh \frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}}{C_1 \cosh \frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)} + C_2 \sinh \frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}}]^2\}}, \end{cases} \tag{18}$$

where $\xi = kx + ct + \xi_0, \mu = \dfrac{k^{-2\alpha} b_1^2 \sigma}{4(\beta^2 + \gamma)}$.

When $\mu > 0$,

$$\begin{cases} U_4(\xi) = -c^\alpha k^\alpha + b_1 \sqrt{\sigma\{1 + [\dfrac{-C_1 \sin \frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)} + C_2 \cos \frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)}}{C_1 \cos \frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)} + C_2 \sin \frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)}}]^2\}} \\ V_4(\xi) = -\dfrac{1}{4} b_1^2 \sigma - 2k^{2\alpha}(\beta^2 + \gamma)\mu [\dfrac{-C_1 \sin \frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)} + C_2 \cos \frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)}}{C_1 \cos \frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)} + C_2 \sin \frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)}}]^2 \\ + b_1 k^\alpha \beta \sqrt{-\mu} [\dfrac{-C_1 \sin \frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)} + C_2 \cos \frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)}}{C_1 \cos \frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)} + C_2 \sin \frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)}}] \\ \sqrt{\sigma\{1 + [\dfrac{-C_1 \sin \frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)} + C_2 \cos \frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)}}{C_1 \cos \frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)} + C_2 \sin \frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)}}]^2\}}, \end{cases} \tag{19}$$

where $\xi = kx + ct + \xi_0, \mu = \frac{k^{-2\alpha} b_1^2 \sigma}{4(\beta^2 + \gamma)}$.

From Case 3 and (7) we obtain:

When $\mu < 0, \beta^2 + \gamma > 0$,

$$\begin{cases} U_5(\xi) = -c^\alpha k^\alpha \pm 2k^\alpha \sqrt{-\mu(\beta^2+\gamma)}[\frac{C_1 \sinh \frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)} + C_2 \cosh \frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}}{C_1 \cosh \frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)} + C_2 \sinh \frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}}], \\ V_5(\xi) = 2\mu k^{2\alpha}(\pm\beta\sqrt{\beta^2+\gamma} - \beta^2 - \gamma) - 2k^{2\alpha}(\pm\beta\sqrt{\beta^2+\gamma} - \beta^2 - \gamma) \\ \mu[\frac{C_1 \sinh \frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)} + C_2 \cosh \frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}}{C_1 \cosh \frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)} + C_2 \sinh \frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}}]^2 \end{cases} \quad (20)$$

where $\xi = kx + ct + \xi_0$.

When $\mu > 0, \beta^2 + \gamma > 0$.

$$\begin{cases} U_6(\xi) = -c^\alpha k^\alpha \pm 2k^\alpha \sqrt{\mu(\beta^2+\gamma)}[\frac{-C_1 \sin \frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)} + C_2 \cos \frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)}}{C_1 \cos \frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)} + C_2 \sin \frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)}}], \\ V_6(\xi) = 2\mu k^{2\alpha}(\pm\beta\sqrt{\beta^2+\gamma} - \beta^2 - \gamma) + 2k^{2\alpha}(\pm\beta\sqrt{\beta^2+\gamma} - \beta^2 - \gamma) \\ \mu[\frac{-C_1 \sin \frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)} + C_2 \cos \frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)}}{C_1 \cos \frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)} + C_2 \sin \frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)}}]^2 \end{cases} \quad (21)$$

where $\xi = kx + ct + \xi_0$.

Remark 1. Compared with the results in [14], the established solutions in Eqs. (16)-(21) are new exact solutions for the space-time fractional Whitham-Broer-Kaup (WBK) equations, and have not been reported by other authors in the literature.

4.2. Space-time fractional Fokas equation

We consider the space-time fractional Fokas equation [15]

$$4\frac{\partial^{2\alpha} q}{\partial t^\alpha \partial x_1^\alpha} - \frac{\partial^{4\alpha} q}{\partial x_1^{3\alpha} \partial x_2^\alpha} + \frac{\partial^{4\alpha} q}{\partial x_2^{3\alpha} \partial x_1^\alpha} + 12\frac{\partial^\alpha q}{\partial x_1^\alpha}\frac{\partial^\alpha q}{\partial x_2^\alpha} + 12q\frac{\partial^{2\alpha} q}{\partial x_1^\alpha \partial x_2^\alpha} - 6\frac{\partial^{2\alpha} q}{\partial y_1^\alpha \partial y_2^\alpha} = 0, 0 < \alpha \le 1. \quad (22)$$

In [15], the authors solved Eq. (22) by a fractional Riccati sub-equation method, and obtained some exact solutions for it. Now we will apply the described method in Section 3 to Eq. (22).

Suppose $q(x,t)=U(\xi)$, where $\xi = ct + k_1x_1 + k_2x_2 + l_1y_1 + l_2y_2 + \xi_0, k_1, k_2, l_1, l_2, \xi_0$ are all constants with $k_1, k_2, l_1, l_2, c \neq 0$. Then by use of the second equality in Eq. (4), Eq. (22) can be turned into

$$4c^\alpha k_1^\alpha D_\xi^{2\alpha}U - k_1^{3\alpha}k_2^\alpha D_\xi^{4\alpha}U + k_2^{3\alpha}k_1^\alpha D_\xi^{4\alpha}U + 12k_1^\alpha k_2^\alpha (D_\xi^\alpha U)^2 + 12k_1^\alpha k_2^\alpha U D_\xi^{2\alpha}U - 6l_1^\alpha l_2^\alpha D_\xi^{2\alpha}U = 0. \quad (23)$$

Suppose that the solution of Eq. (23) can be expressed by

$$U(\xi) = a_0 + \sum_{i=1}^{m}[a_i(\frac{D_\xi^\alpha G}{G})^i + b_i(\frac{D_\xi^\alpha G}{G})^{i-1}\sqrt{\sigma(1+\frac{1}{\mu}(\frac{D_\xi^\alpha G}{G})^2)}], \quad (24)$$

where $G = G(\xi)$ satisfies Eq. (1). By Balancing the order between the highest order derivative term and nonlinear term in Eq. (23), we can obtain $m=2$. So we have

$$U(\xi) = a_0 + a_1(\frac{D_\xi^\alpha G}{G}) + a_2(\frac{D_\xi^\alpha G}{G})^2 + b_1\sqrt{\sigma(1+\frac{1}{\mu}(\frac{D_\xi^\alpha G}{G})^2)} + b_2(\frac{D_\xi^\alpha G}{G})\sqrt{\sigma(1+\frac{1}{\mu}(\frac{D_\xi^\alpha G}{G})^2)}. \quad (25)$$

Substituting (25) along with (1) into (23) and collecting all the terms with the same power of $\sqrt{\sigma(1+\frac{1}{\mu}(\frac{D_\xi^\alpha G}{G})^2)}(\frac{D_\xi^\alpha G}{G})$ together, equating each coefficient to zero, yields a set of algebraic equations. Solving these equations, yields:

Case 1:

$$a_0 = \frac{10k_2^\alpha b_2^2\sigma + 2c^\alpha(k_2^{2\alpha} - k_1^{2\alpha}) + 3k_1^\alpha l_1^\alpha l_2^\alpha - 3k_2^{2\alpha}k_1^{-\alpha}l_1^\alpha l_2^\alpha}{6(k_2^\alpha k_1^{2\alpha} - k_2^{3\alpha})}, a_1 = 0,$$

$$a_2 = \frac{k_1^{2\alpha} - k_2^{2\alpha}}{2}, b_1 = 0, b_2 = b_2, \mu = \frac{4b_2^2\sigma}{(k_1^{2\alpha} - k_2^{2\alpha})^2}.$$

Case 2:

$$a_0 = \frac{4k_1^{3\alpha}k_2^\alpha\mu - 4k_1^\alpha k_2^{3\alpha}\mu - 2c^\alpha k_1^\alpha + 3l_1^\alpha l_2^\alpha}{6k_1^\alpha k_2^\alpha}, a_2 = k_1^{2\alpha} - k_2^{2\alpha}, \mu = \mu.$$

Substituting the result above into Eq. (25), and combining with (7) we can obtain the following exact solutions to Eq. (22).

From Case 1 and (7) we obtain:

When $\mu < 0$,

$$U_1(\xi) = \frac{10k_2^\alpha b_2^2\sigma + 2c^\alpha(k_2^{2\alpha} - k_1^{2\alpha}) + 3k_1^\alpha l_1^\alpha l_2^\alpha - 3k_2^{2\alpha}k_1^{-\alpha}l_1^\alpha l_2^\alpha}{6(k_2^\alpha k_1^{2\alpha} - k_2^{3\alpha})}$$

$$-(\frac{k_1^{2\alpha}-k_2^{2\alpha}}{2})\mu[\frac{C_1\sinh\frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}+C_2\cosh\frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}}{C_1\cosh\frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}+C_2\sinh\frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}}]^2$$

$$+b_2\sqrt{-\mu}[\frac{C_1\sinh\frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}+C_2\cosh\frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}}{C_1\cosh\frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}+C_2\sinh\frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}}]\sqrt{\sigma\{1+[\frac{C_1\sinh\frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}+C_2\cosh\frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}}{C_1\cosh\frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}+C_2\sinh\frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}}]^2\}},\tag{26}$$

where $\xi = ct + k_1x_1 + k_2x_2 + l_1y_1 + l_2y_2 + \xi_0,\ \mu = \frac{4b_2^2\sigma}{(k_1^{2\alpha}-k_2^{2\alpha})^2}$.

When $\mu > 0$,

$$U_2(\xi) = \frac{10k_2^\alpha b_2^2\sigma + 2c^\alpha(k_2^{2\alpha}-k_1^{2\alpha}) + 3k_1^\alpha l_1^\alpha l_2^\alpha - 3k_2^{2\alpha}k_1^{-\alpha}l_1^\alpha l_2^\alpha}{6(k_2^\alpha k_1^{2\alpha}-k_2^{3\alpha})}$$

$$+(\frac{k_1^{2\alpha}-k_2^{2\alpha}}{2})\mu[\frac{-C_1\sin\frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)}+C_2\cos\frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)}}{C_1\cos\frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)}+C_2\sin\frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)}}]^2$$

$$+b_2\sqrt{\mu}[\frac{-C_1\sin\frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)}+C_2\cos\frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)}}{C_1\cos\frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)}+C_2\sin\frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)}}]\sqrt{\sigma\{1+[\frac{-C_1\sin\frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)}+C_2\cos\frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)}}{C_1\cos\frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)}+C_2\sin\frac{\sqrt{\mu}\xi^\alpha}{\Gamma(1+\alpha)}}]^2\}},\tag{27}$$

where $\xi = ct + k_1x_1 + k_2x_2 + l_1y_1 + l_2y_2 + \xi_0,\ \mu = \frac{4b_2^2\sigma}{(k_1^{2\alpha}-k_2^{2\alpha})^2}$.

From Case 2 and (7) we obtain:

When $\mu < 0$,

$$U_3(\xi) = \frac{4k_1^{3\alpha}k_2^\alpha\mu - 4k_1^\alpha k_2^{3\alpha}\mu - 2c^\alpha k_1^\alpha + 3l_1^\alpha l_2^\alpha}{6k_1^\alpha k_2^\alpha}$$

$$-(k_1^{2\alpha}-k_2^{2\alpha})\mu[\frac{C_1\sinh\frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}+C_2\cosh\frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}}{C_1\cosh\frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}+C_2\sinh\frac{\sqrt{-\mu}\xi^\alpha}{\Gamma(1+\alpha)}}]^2\tag{28}$$

where $\xi = ct + k_1 x_1 + k_2 x_2 + l_1 y_1 + l_2 y_2 + \xi_0$.

When $\mu > 0$,

$$U_4(\xi) = \frac{4k_1^{3\alpha} k_2^{\alpha} \mu - 4k_1^{\alpha} k_2^{3\alpha} \mu - 2c^{\alpha} k_1^{\alpha} + 3l_1^{\alpha} l_2^{\alpha}}{6k_1^{\alpha} k_2^{\alpha}} + (k_1^{2\alpha} - k_2^{2\alpha})\mu\left[\frac{-C_1 \sin\frac{\sqrt{\mu}\xi^{\alpha}}{\Gamma(1+\alpha)} + C_2 \cos\frac{\sqrt{\mu}\xi^{\alpha}}{\Gamma(1+\alpha)}}{C_1 \cos\frac{\sqrt{\mu}\xi^{\alpha}}{\Gamma(1+\alpha)} + C_2 \sin\frac{\sqrt{\mu}\xi^{\alpha}}{\Gamma(1+\alpha)}}\right]^2 \quad (29)$$

where $\xi = ct + k_1 x_1 + k_2 x_2 + l_1 y_1 + l_2 y_2 + \xi_0$.

Remark 2.
As one can see, the established solutions for the space-time fractional Fokas equation above are different from the results in [15], and are new exact solutions so far to our best knowledge.

5. Conclusions

Based on a new ansatz, we have proposed a new fractional sub-equation method for solving FPDEs. As applications, the space-time fractional Whitham-Broer-Kaup (WBK) equations and the space-time fractional Fokas equation are solved successfully, and new exact solutions for them are established. Being concise and powerful, the proposed method can be applied to solve other fractional partial differential equations.

6. Acknowledgements

The author would like to thank the anonymous referees for their useful and valuable suggestions.

REFERENCES

[1] *C.C. Wu*, "A fractional variational iteration method for solving fractional nonlinear differential equations", in Comput. Math. Appl., **vol.** 61, 2011, pp. 2186-2190
[2] *J. Ji, J.B. Zhang and Y.J. Dong*, "The fractional variational iteration method improved with the Adomian series", in Appl. Math. Lett., **vol.** 25, 2012, pp. 2223-2226
[3] *G. Wu and, E.W.M. Lee*, "Fractional Variational Iteration Method And Its Application", in Phys. Lett. A, **vol.** 374, 2010, pp. 2506-2509
[4] *S. Guo and L. Mei*, "The fractional variational iteration method using He's polynomials", in Phys. Lett. A, **vol.** 375, 2011, pp. 309-313

[5] *A.M.A. El-Sayed, S.H. Behiry and W.E. Raslan*, “Adomian's decomposition method for solving intermediate fractional advection-dispersion equation”, in Comput. Math. Appl., **vol.** 59, 2010, pp. 1759-1765

[6] *A.M.A. El-Sayed and M. Gaber*, “The Adomian decomposition method for solving partial differential equations of fractal order in finite domains”, in Phys. Lett. A, **vol.** 359, 2006, pp. 175-182

[7] *S.S. Ray*, “A new approach for the application of Adomian decomposition method for the solution of fractional space diffusion equation with insulated ends”, in Appl. Math. Comput., **vol.** 202, 2008, pp. 544-549

[8] *M. Merdan*, “A numeric-analytic method for time-fractional Swift-Hohenberg (S-H) equation with modified Riemann-Liouville derivative”, in Appl. Math. Model., **vol.** 37, 2013, pp. 4224-4231

[9] *M. Cui*, “Compact finite difference method for the fractional diffusion equation”, in J. Comput. Phys., **vol.** 228, 2009, pp. 7792-7804

[10] *K.A. Gepreel*, “The homotopy perturbation method applied to the nonlinear fractional Kolmogorov-Petrovskii-Piskunov equations”, in Appl. Math. Lett., **vol.** 24, 2011, pp. 1428-1434

[11] *Q. Huang, G. Huanm and, H. Zhan*, “A finite element solution for the fractional advection-dispersion equation”, in Adv. Water Resour., **vol.** 31, 2008, pp. 1578-1589

[12] *B. Lu*, “B\"{a}cklund transformation of fractional Riccati equation and its applications to nonlinear fractional partial differential equations”, in Phys. Lett. A, **vol.** 376, 2012, pp. 2045-2048

[13] *G. Jumarie*, “Modified Riemann-Liouville derivative and fractional Taylor series of nondifferentiable functions further results”, in Comput. Math. Appl., **vol.** 51, 2006, pp. 1367-1376

[14] *S.M. Guo, L.Q. Mei, Y. L and, Y.F. Sun*, “The improved fractional sub-equation method and its applications to the space–time fractional differential equations in fluid mechanics”, in Phys. Lett. A, **vol.** 376, 2012, pp. 407-411.

[15] *S. Zhang and H.Q. Zhang*, “Fractional sub-equation method and its applications to nonlinear fractional PDEs”, in Phys. Lett. A, **vol.** 375, 2011, pp. 1069-1073