ON THE EXISTENCE OF POSITIVE SOLUTIONS FOR A CLASS OF INFINITE SEMIPOSITONE PROBLEMS

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We discuss the existence of a positive solution to the infinite semipositone problem

$$-\Delta u = -au + bu^2 - du^3 - f(u) - \frac{c}{u^\alpha},\ x \in \Omega, \quad u = 0,\ x \in \partial\Omega,$$

where $\alpha \in (0,1)$, a, b, d and c are positive constants, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, Δ is the Laplacian operator, and $f : [0,\infty) \to \mathbb{R}$ is a nondecreasing continuous function such that $f(u) \to \infty$ and $f(u)/u \to 0$ as $u \to \infty$. We obtain our result via the method of sub- and supersolutions. We also extend our result to classes of infinite semipositone system and p-Laplacian problem.

Keywords: Positive solution; Infinite semipositone; Sub- and supersolutions
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1. Introduction

Consider the boundary value problem

$$\begin{cases} -\Delta u = -au + bu^2 - du^3 - f(u) - \dfrac{c}{u^\alpha}, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\alpha \in (0,1)$, a, b, d and c are positive constants, and Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, Δ is the Laplacian operator, and $f : [0,\infty) \to \mathbb{R}$ is a continuous function. We make the following assumptions:

(H1) $f : [0,+\infty) \to \mathbb{R}$ is nondecreasing continuous functions such that $\lim_{s\to+\infty} f(s) = \infty$.

(H2) $\lim_{s\to+\infty} \frac{f(s)}{s} = 0$.

Note that (1.1) is as an infinite semipositone problems ($\lim_{u\to 0} F(u) = -\infty$, where $F(u) := -au + bu^2 - du^3 - f(u) - (c/u^\alpha)$).

In [9], the authors have studied the case when $F(u) := g(u) - (c/u^\alpha)$ where g is nonnegative and nondecreasing and $\lim\limits_{u\to\infty} g(u) = \infty$. The case $g(u) := au - f(u)$

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has been studied in [8], where $f(u) \geq au - M$ and $f(u) \leq Au^p$ on $[0, \infty)$ for some $M, A > 0, p > 1$ and this g may have a falling zero. A simple example of this g is $g(u) = u - u^p$, where $p > 1$. Note that this g has a falling zero at $u = 1$, in fact g is negative for $u > 1$. In this article, we consider the case when $g(u) := -au + bu^2 - du^3 - f(u)$ and we study more challenging infinite semipositone problem. A example of f satisfying our hypotheses is $f(x) = u^p$; $0 < p < 1$. Further, let $0, R_1$ and R_2 denote the zeros of $-au + bu^2 - du^3$ (such that $R_1 < R_2$), then $g(u) = -au + bu^2 - du^3 - u^p$ is negative for $u < R_1$ and $u > R_2$.

In recent years, there has been considerable progress on the study of semipositione problems ($F(0) < 0$ but finite)(see [2],[3],[6]). Many results have been obtained of infinite semipositone problems; see for example [7], [8], [9] and [10].

In [1], the authors establish the existence of a positive solution to $-\Delta u = -au+bu^2-du^3-ch(x)$ with Dirichlet boundary conditions and the method employed in it uses the fact that $-\inf_{s\in[0,R_2]}(-au+bu^2-du^3) < ar$, where r is the first positive zero of $(-au + bu^2 - du^3)'$. We will use in this paper this fact, too. The main tool used in this study is the method of sub- and supersolutions ([4]).

2. The main result

In this section, we shall establish our existence result via the method of sub - supersolution. A function ψ is said to be a subsolution of (1.1) if it is in $C^2(\Omega)\cap C(\overline{\Omega})$ such that $\psi = 0$ on $\partial\Omega$ and

$$-\Delta\psi \leq -a\psi + b\psi^2 - d\psi^3 - f(\psi) - \frac{c}{\psi^\alpha} \quad \text{in } \Omega,$$

and z is said supersolution of (1.1) if it is in $C^2(\Omega) \cap C(\overline{\Omega})$ such that $z = 0$ on $\partial\Omega$ and

$$-\Delta z \geq -az + bz^2 - dz^3 - f(z) - \frac{c}{z^\alpha} \qquad \text{in } \Omega.$$

Then it is well known that if there exist a subsolution ψ and supersolution z such that $\psi \leq z$ in Ω then (1.1) has a solution u such that $\psi \leq u \leq z$, see [4].

Theorem 2.1. *Let* (H1) *and* (H2) *hold, Then there exists positive constants* $b_0 := b_0(a, d, \Omega)$ *and* $c_0 := c_0(a, b, d, \Omega)$ *such that for* $b \geq b_0$ *and* $c \leq c_0$*, problem* (1.1) *has a positive solution.*

Proof. Let $\lambda_1 > 0$ be the first eigenvalue of the operator $-\Delta$ with Dirichlet boundary condition and ϕ_1 be the corresponding eigenfunction satisfying $\phi_1 > 0$ in Ω and $\frac{\partial\phi_1}{\partial\nu} < 0$ on $\partial\Omega$, where ν is outward normal vector on $\partial\Omega$ and $\|\phi_1\|_\infty = 1$, see [5]. Note that λ_1 and ϕ_1 satisfy:

$$\begin{aligned} -\Delta\phi_1 &= \lambda_1\phi_1 \quad \text{in } \Omega \\ \phi_1 &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Let $\delta > 0$, $\mu > 0$, $m > 0$ be such that

$$(\frac{2}{1+\alpha})\{(\frac{1-\alpha}{1+\alpha})|\nabla\phi_1|^2 - \lambda_1{\phi_1}^2\} \geq m \quad \text{in } \overline{\Omega}_\delta, \tag{2.1}$$

and $\phi_1 \in [\mu, 1]$ in $\Omega \setminus \overline{\Omega}_\delta$, where $\overline{\Omega}_\delta := \{x \in \Omega : d(x, \partial\Omega) \leq \delta\}$. This is possible since $|\nabla\phi_1| \neq 0$ on $\partial\Omega$ while $\phi_1 = 0$ on $\partial\Omega$.

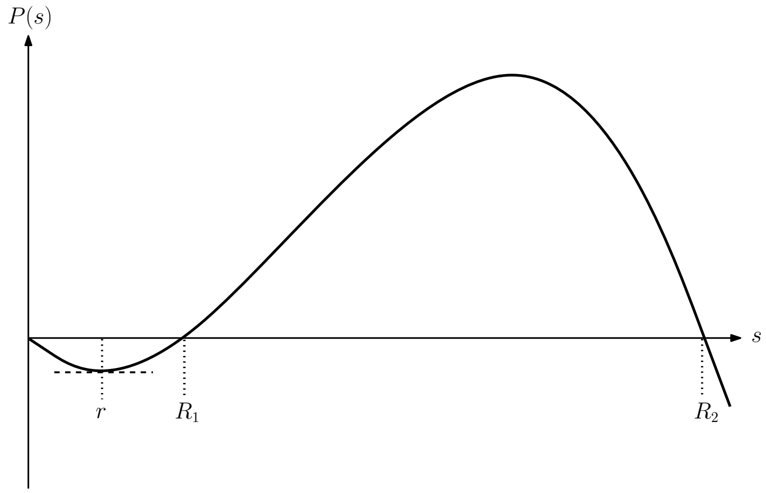

FIGURE 1. Graph of $P(s)$.

Let $b_0 > 2\sqrt{ab}$ and $P(s) = -as + bs^2 - ds^3$. Then the zeros of $P(s)$ are $0, R_1 = \frac{b-\sqrt{b^2-4ad}}{2d}$ and $R_2 = \frac{b+\sqrt{b^2-4ad}}{2d}$. We note that $P(s)$ can be factored as $P(s) = -ds(s-R_1)(s-R_2)$. Let $r = \frac{b-\sqrt{b^2-3ad}}{3d}$ denote the first positive zero of $P'(s)$. since $P(s)$ is convex on $(0, \frac{b}{3d})$ and $r < \frac{b}{3d}$, we have $\rho := -\inf_{s\in[0,R_2]} P(s) < a(b-\sqrt{b^2-3ad}/3d) = ar$ (see Fig 1). We note that

$$\frac{\rho}{R_2} < \frac{a(b-\sqrt{b^2-3ad}/3d)}{b+\sqrt{b^2-4ad}/2d} = \frac{2a^2d}{(b+\sqrt{b^2-4ad})(b+\sqrt{b^2-3ad})} \to 0 \text{ as } \ b \to \infty,$$

$$\frac{R_2}{R_1} = \frac{b+\sqrt{b^2-4ad}}{b-\sqrt{b^2-4ad}} = \frac{(b+\sqrt{b^2-4ad})^2}{4ad} \to \infty \text{ as } \ b \to \infty$$

Hence there exists $b_0^{(1)} := b_0^{(1)}(a, d, \Omega)$ such that for every $b > b_0^{(1)}$ we have

$$\frac{\rho}{R_2} < \frac{m}{6}, \tag{2.2}$$

$[\frac{R_2}{2}\mu^{\frac{2}{1+\alpha}}, \frac{R_2}{2}] \subset (R_1, R_2)$ and $k_\mu := \inf_{s\in[\frac{R_2}{2}\mu^{\frac{2}{1+\alpha}}, \frac{R_2}{2}]} P(s) > 0$. Next we see that

$$\begin{aligned}\frac{k_\mu}{R_2} &= \frac{\min\left\{P(\frac{R_2}{2}\mu^{\frac{2}{1+\alpha}}), P(\frac{R_2}{2})\right\}}{R_2} \\ &= \min\left\{d\frac{R_2}{2}\mu^{\frac{2}{1+\alpha}}(\frac{R_2}{2}\mu^{\frac{2}{1+\alpha}} - R_1)(1 - \frac{\mu^{\frac{2}{1+\alpha}}}{2}), d\frac{R_2}{4}(\frac{R_2}{2} - R_1)\right\} \to \infty \text{ as } \ b \to \infty,\end{aligned}$$

and hence there exists $b_0^{(2)} := b_0^{(2)}(a, d, \Omega)$ such that for every $b > b_0^{(2)}$ we have

$$\frac{k_\mu}{R_2} > \frac{2\lambda_1}{1+\alpha}.$$

Finally from (H1) and (H2), $f(R_2) \to \infty$ and $f(R_2/2)/(R_2/2) \to 0$ as $b \to \infty$. Thus there exists $b_0^{(3)} := b_0^{(3)}(a, d, \Omega)$ such that for every $b > b_0^{(3)}$ we have $f(R_2) \geq 0$ and

$$f(\frac{R_2}{2}\phi_1{}^{\frac{2}{1+\alpha}}) \;\leq\; f(\frac{R_2}{2}) \;\leq\; \min\left\{\lambda_1, \frac{m}{3}\right\}(\frac{R_2}{2}). \tag{2.3}$$

For a given $a, d > 0$, define $b_0 := \max\{b_0^{(1)}, b_0^{(2)}, b_0^{(3)}\}$ and $c_0 := c_0(a, b, d, \Omega) := \min\left\{\frac{m}{3}(\frac{R_2}{2})^{1+\alpha}, (\frac{R_2}{2})^{\alpha}\mu^{2\alpha/1+\alpha}(k_\mu - \frac{2\lambda_1}{1+\alpha}R_2)\right\}$, and let $b \geq b_0$ and $c \leq c_0$. We will show that $\psi := R\phi_1{}^{2/1+\alpha}$ is a subsolution of (1.1), where $R := \dfrac{R_2}{2}$.

We first note that

$$\nabla\psi = R(\frac{2}{1+\alpha})\phi_1{}^{\frac{1-\alpha}{1+\alpha}}\nabla\phi_1$$

and

$$\begin{aligned}
-\Delta\psi &= -R(\frac{2}{1+\alpha})\{\phi_1{}^{\frac{1-\alpha}{1+\alpha}}\Delta\phi_1 + (\frac{1-\alpha}{1+\alpha})\phi_1{}^{-\frac{2\alpha}{1+\alpha}}|\nabla\phi_1|^2\} \\
&= R(\frac{2}{1+\alpha})\frac{1}{(\phi_1{}^{\frac{2}{1+\alpha}})^{\alpha}}\{\lambda_1\phi_1{}^2 - (\frac{1-\alpha}{1+\alpha})|\nabla\phi_1|^2\}.
\end{aligned}$$

Next for $x \in \overline{\Omega}_\delta$ since $\frac{1}{(\phi_1{}^{\frac{2}{1+\alpha}})^{\alpha}} \geq 1$, from (2.1),(2.2),(2.3) and $c \leq c_0$ we have

$$\begin{aligned}
-\Delta\psi &= R(\frac{2}{1+\alpha})\frac{1}{(\phi_1{}^{\frac{2}{1+\alpha}})^{\alpha}}\{\lambda_1\phi_1{}^2 - (\frac{1-\alpha}{1+\alpha})|\nabla\phi_1|^2\} \\
&\leq -mR\frac{1}{(\phi_1{}^{\frac{2}{1+\alpha}})^{\alpha}} \\
&= -\frac{mR}{3(\phi_1{}^{\frac{2}{1+\alpha}})^{\alpha}} - \frac{mR}{3(\phi_1{}^{\frac{2}{1+\alpha}})^{\alpha}} - \frac{mR}{3(\phi_1{}^{\frac{2}{1+\alpha}})^{\alpha}} \\
&\leq -\frac{mR}{3} - \frac{mR}{3} - \frac{mR}{3(\phi_1{}^{\frac{2}{1+\alpha}})^{\alpha}} \\
&\leq -\rho - f(R\phi_1{}^{\frac{2}{1+\alpha}}) - \frac{mR^{1+\alpha}/3}{(R\phi_1{}^{\frac{2}{1+\alpha}})^{\alpha}} \\
&\leq -a\psi + b\psi^2 - d\psi^3 - f(\psi) - \frac{c}{\psi^{\alpha}}.
\end{aligned} \tag{2.4}$$

Also for $x \in \Omega \setminus \overline{\Omega}_\delta$, since $0 < \mu \leq \phi$, from (2.3) and $c \leq c_0$,

$$\begin{aligned}
-\Delta\psi &= R(\frac{2}{1+\alpha})\frac{1}{(\phi_1{}^{\frac{2}{1+\alpha}})^\alpha}\{\lambda_1\phi_1{}^2 - (\frac{1-\alpha}{1+\alpha})|\nabla\phi_1|^2\} \\
&\leq R(\frac{2}{1+\alpha})\lambda_1\phi^{\frac{2}{1+\alpha}} \\
&\leq R(\frac{2}{1+\alpha})\lambda_1 \\
&= 2\Big[R(\frac{2}{1+\alpha})\lambda_1\Big] - R(\frac{2}{1+\alpha})\lambda_1 \\
&\leq \frac{4\lambda_1}{1+\alpha}R - R\lambda_1 \\
&\leq k_\mu - \frac{c}{(R\mu^{\frac{2}{1+\alpha}})^\alpha} - f(R\phi_1{}^{\frac{2}{1+\alpha}}) \\
&\leq -a\psi + b\psi^2 - d\psi^3 - f(\psi) - \frac{c}{\psi^\alpha}. \qquad (2.5)
\end{aligned}$$

According to (2.4) and (2.5), we can conclude that ψ is a subsolution of (1.1). We also show that $z := R_2$ is a supersolution, by noting that

$$-\Delta z = 0 \geq -f(z) - \frac{c}{z^\alpha} = -az + bz^2 - dz^3 - f(z) - \frac{c}{z^\alpha}.$$

Further $z \geq \psi$. Thus, (1.1) has a positive solution. This completes the proof of Theorem 2.1. □

3. Extension of (1.1) to system (3.1)

In this section, we consider the extension of (1.1) to the following system:

$$\begin{cases} -\Delta u = -a_1u + b_1u^2 - d_1u^3 - f_1(u) - \dfrac{c_1}{v^\alpha}, & x \in \Omega, \\ -\Delta v = -a_2v + b_2v^2 - d_2v^3 - f_2(v) - \dfrac{c_2}{u^\alpha}, & x \in \Omega, \\ u = 0 = v, & x \in \partial\Omega, \end{cases} \qquad (3.1)$$

where $\alpha \in (0,1)$, $a_1, a_2, b_1, b_2, d_1, d_2, c_1$ and c_2 are positive constants, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, and $f_i : [0,\infty) \to \mathbb{R}$ is a continuous function for $i = 1, 2$. We make the following assumptions:

(H3) $f_i : [0, +\infty) \to \mathbb{R}$ is nondecreasing continuous functions such that $\lim_{s\to+\infty} f_i(s) = \infty$ for $i = 1, 2$.

(H4) $\lim_{s\to+\infty} \frac{f_i(s)}{s} = 0$ for $i = 1, 2$.

We prove the following result by finding sub-super solutions to infinite semipositone system (3.1).

Theorem 3.1. *Let* (H3) *and* (H4) *hold, Then there exists positive constants* $b_0^* := b_0^*(a_1, a_2, d_1, d_2, \Omega)$ *and* $c_0^* := c_0^*(a_1, a_2, b_1, b_2, d_1, d_2, \Omega)$ *such that for* $\min\{b_1, b_2\} \geq b_0^*$ *and* $\max\{c_1, c_2\} \leq c_0^*$*, problem* (3.1) *has a positive solution.*

Proof. Let $(R_1^{(i)}, R_2^{(i)}, \rho^{(i)}, k_\mu^{(i)})$, $P_i(s) := -a_i s + b_i s^2 - d_i s^3$ for $i = 1, 2$ be given, as in section 2. By the same argument as in section 2, there exists $b_0^* := b_0^*(a_1, a_2, d_1, d_2, \Omega)$ such that for $\min\{b_1, b_2\} > b_0^*$ we have

$$\frac{\rho^{(i)}}{R_2^{(i)}} < \frac{m}{6}, \quad \frac{k_\mu^{(i)}}{R_2^{(i)}} > \frac{2\lambda_1}{1+\alpha},$$

and $f_i(\frac{R_2^{(i)}}{2}\phi_1{}^{\frac{2}{1+\alpha}}) \leq \min\{\lambda_1, \frac{m}{3}\}(\frac{R_2^{(i)}}{2})$ for $i = 1, 2$. Define

$$\begin{aligned} c_0^* &:= c_0^*(a_1, a_2, b_1, b_2, d_1, d_2, \Omega) \\ &:= \min\Big\{\frac{m}{3}(\frac{R_2^{(1)}}{2})(\frac{R_2^{(2)}}{2})^\alpha, \frac{m}{3}(\frac{R_2^{(1)}}{2})^\alpha(\frac{R_2^{(2)}}{2}), (\frac{R_2^{(2)}}{2})^\alpha \mu^{2\alpha/1+\alpha}(k_\mu^{(1)} - \frac{2\lambda_1}{1+\alpha}R_2^{(1)}), \\ &\quad (\frac{R_2^{(1)}}{2})^\alpha \mu^{2\alpha/1+\alpha}(k_\mu^{(2)} - \frac{2\lambda_1}{1+\alpha}R_2^{(2)})\Big\} \end{aligned}$$

and $(\psi_1, \psi_2) := (R^{(1)}\phi_1{}^{2/1+\alpha}, R^{(2)}\phi_1{}^{2/1+\alpha})$, where $R^{(i)} = R_2^{(i)}/2$. Let $\min\{b_1, b_2\} > b_0^*$ and $\max\{c_1, c_1\} \leq c_0^*$, then for $x \in \overline{\Omega}_\delta$ we have

$$\begin{aligned} -\Delta\psi_1 &= R^{(1)}(\frac{2}{1+\alpha})\frac{1}{(\phi_1{}^{\frac{2}{1+\alpha}})^\alpha}\{\lambda_1\phi_1{}^2 - (\frac{1-\alpha}{1+\alpha})|\nabla\phi_1|^2\} \\ &\leq -mR^{(1)}\frac{1}{(\phi_1{}^{\frac{2}{1+\alpha}})^\alpha} \\ &\leq -\frac{mR^{(1)}}{3} - \frac{mR^{(1)}}{3} - \frac{mR^{(1)}}{3(\phi_1{}^{\frac{2}{1+\alpha}})^\alpha} \\ &\leq -\rho^{(1)} - f(R^{(1)}\phi_1{}^{\frac{2}{1+\alpha}}) - \frac{mR^{(1)}[R^{(2)}]^\alpha/3}{(R^{(2)}\phi_1{}^{\frac{2}{1+\alpha}})^\alpha} \\ &\leq -a\psi_1 + b\psi_1^2 - d\psi_1^3 - f(\psi_1) - \frac{c_1}{\psi_2^\alpha}. \end{aligned}$$

And for $x \in \Omega \setminus \overline{\Omega}_\delta$, we have

$$\begin{aligned} -\Delta\psi_1 &= R^{(1)}(\frac{2}{1+\alpha})\frac{1}{(\phi_1{}^{\frac{2}{1+\alpha}})^\alpha}\{\lambda_1\phi_1{}^2 - (\frac{1-\alpha}{1+\alpha})|\nabla\phi_1|^2\} \\ &\leq R^{(1)}(\frac{2}{1+\alpha})\lambda_1 \\ &= 2\Big[R^{(1)}(\frac{2}{1+\alpha})\lambda_1\Big] - R^{(1)}(\frac{2}{1+\alpha})\lambda_1 \\ &\leq \frac{4\lambda_1}{1+\alpha}R^{(1)} - R^{(1)}\lambda_1 \\ &\leq k_\mu^{(1)} - \frac{c_2}{(R^{(2)}\mu^{\frac{2}{1+\alpha}})^\alpha} - f(R^{(1)}\phi_1{}^{\frac{2}{1+\alpha}}) \\ &\leq -a_1\psi_1 + b_1\psi_1^2 - d_1\psi_1^3 - f(\psi_1) - \frac{c_1}{\psi_2^\alpha}. \end{aligned}$$

Similarly

$$-\Delta\psi_2 \le -a_2\psi_2 + b_2\psi_2^2 - d_2\psi_2^3 - f(\psi_2) - \frac{c_2}{\psi_1^\alpha}, \quad x \in \Omega.$$

Thus the (ψ_1, ψ_2) is a subsolution of (3.1). It is obvious that $(z_1, z_2) := (R_2^{(1)}, R_2^{(2)})$ is a supersolution of (3.1), such that $(z_1, z_2) \ge (\psi_1, \psi_2)$. Thus Theorem 3.1 is proven. □

4. **Extension of** (1.1) **to problem** (4.1)

In this section, we consider the extension of (1.1) to the following problem:

$$\begin{cases} -\Delta_p u = -au + bu^2 - du^3 - f(u) - \dfrac{c}{u^\alpha}, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{4.1}$$

where $\Delta_p z = div(|\nabla z|^{p-2}\nabla z)$, $p > 1$, $\alpha \in (0, 1)$, a, b, d and c are positive constants, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, and $f : [0, \infty) \to \mathbb{R}$ is a continuous function. Then we have the following result.

Theorem 4.1. *Let* (H1) *and* (H2) *hold, Then there exists positive constants* $b_0^{**} := b_0^{**}(a, d, \Omega)$ *and* $c_0^{**} := c_0^{**}(a, b, d, \Omega)$ *such that for* $b \ge b_0^{**}$ *and* $c \le c_0^{**}$*, problem* (4.1) *has a positive solution.*

Proof. We shall establish Theorem 4.1 by constructing positive sub-super solutions to equation (4.1). Let λ_1 be the first eigenvalue of the problem

$$-\Delta_p\phi_1 = \lambda_1\phi_1^{p-1}, \ x \in \Omega, \quad \phi_1 = 0, x \in \partial\Omega,$$

where ϕ_1 denote the corresponding eigenfunction, satisfying $\phi_1 > 0$ in Ω and $|\nabla\phi_1| > 0$ on $\partial\Omega$, see [5]. Without loss of generality, we let $\|\phi_1\|_\infty = 1$. Let $\delta > 0$, $\mu > 0$, $m > 0$ be such that

$$(\frac{p}{p-1+\alpha})^{p-1}\left\{\frac{(1-\alpha)(p-1)}{p-1+\alpha}|\nabla\phi_1|^p - \lambda_1{\phi_1}^p\right\} \ge m \quad \text{in } \overline{\Omega}_\delta,$$

and $\phi_1 \in [\mu, 1]$ in $\Omega \setminus \overline{\Omega}_\delta$, where $\overline{\Omega}_\delta := \{x \in \Omega : d(x, \partial\Omega) \le \delta\}$. This is possible since $|\nabla\phi_1| \ne 0$ on $\partial\Omega$ while $\phi_1 = 0$ on $\partial\Omega$. Also let R_1, R_2 be as in section 2 and b_0^{**} be such that for every $b > b_0^{**}$

$$\frac{\rho}{{R_2}^{p-1}} < \frac{m}{6}\ , \quad \frac{k_\mu}{{R_2}^{p-1}} > (\frac{\lambda_1}{2})(\frac{p}{p-1+\alpha})^{p-1},$$

and

$$f([\frac{R_2}{2}]^{p-1}{\phi_1}^{\frac{p}{p-1+\alpha}}) \le \min\left\{\lambda_1, \frac{m}{3}\right\}(\frac{R_2}{2})^{p-1}.$$

Define

$$\begin{aligned} c_0^{**} &:= c_0^{**}(a, b, d, \Omega) \\ &:= \min\left\{\frac{m}{3}(\frac{R_2}{2})^{(p-1)(1+\alpha)}, (\frac{R_2}{2})^{\alpha(p-1)}\mu^{\frac{\alpha p}{p-1+\alpha}}[k_\mu - R_2\lambda_1(\frac{p}{p-1+\alpha})^{p-1}]\right), \end{aligned}$$

and $\psi := R{\phi_1}^{\frac{p}{p-1+\alpha}}$. Then

$$\nabla\psi = R(\frac{p}{p-1+\alpha}){\phi_1}^{\frac{1-\alpha}{p-1+\alpha}}\nabla\phi_1$$

and

$$\begin{aligned}\Delta_p\psi &= div(|\nabla\psi|^{p-2}\nabla\psi)\\ &= R^{p-1}(\frac{p}{p-1+\alpha})^{p-1}div(\phi_1{}^{\frac{(1-\alpha)(p-1)}{p-1+\alpha}}|\nabla\phi_1|^{p-2}\nabla\phi_1)\\ &= R^{p-1}(\frac{p}{p-1+\alpha})^{p-1}\left\{\nabla\left(\phi_1{}^{\frac{(1-\alpha)(p-1)}{p-1+\alpha}}\right)|\nabla\phi_1|^{p-2}\nabla\phi_1+\phi_1{}^{\frac{(1-\alpha)(p-1)}{p-1+\alpha}}\Delta_p\phi_1\right\}\\ &= R^{p-1}(\frac{p}{p-1+\alpha})^{p-1}\left\{\frac{(1-\alpha)(p-1)}{p-1+\alpha}\phi_1{}^{\frac{-\alpha p}{p-1+\alpha}}|\nabla\phi_1|^p-\lambda_1\phi_1{}^{\frac{p(p-1)}{p-1+\alpha}}\right\}\\ &= R^{p-1}(\frac{p}{p-1+\alpha})^{p-1}\frac{1}{\left(\phi_1{}^{\frac{p}{p-1+\alpha}}\right)^{\alpha}}\left\{\frac{(1-\alpha)(p-1)}{p-1+\alpha}|\nabla\phi_1|^p-\lambda_1\phi_1{}^p\right\},\end{aligned}$$

thus

$$-\Delta_p\psi = R^{p-1}(\frac{p}{p-1+\alpha})^{p-1}\frac{1}{\left(\phi_1{}^{\frac{p}{p-1+\alpha}}\right)^{\alpha}}\left\{\lambda_1\phi_1{}^p-\frac{(1-\alpha)(p-1)}{p-1+\alpha}|\nabla\phi_1|^p\right\}.$$

By the same argument as in the proof of theorem 2.1, we can show that ψ is a subsolution of (4.1) for $b \geq b_0^{**}$ and $c \leq c_0^{**}$. Next, it is easy to check that $z := R_2$ is a supersolution of (4.1) with $z \geq \psi$. Hence (4.1) has a positive solution and the proof is complete. □

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