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ITERATIVE ALGORITHMS FOR GENERALIZED VARIATIONAL INEQUALITIES

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A generalized variational inequality problem is considered. An algorithm for finding the solutions of the generalized variational inequality is formally constructed. Strong convergence analysis of the suggested algorithm is given.

Keywords: generalized variational inequality, iterative algorithm, strong convergence.

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1. **Introduction**

Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. Let $\mathcal{C}\subset\mathcal{H}$ be a nonempty closed convex set. Let $\mathcal{A}\colon\mathcal{C}\to\mathcal{H}$ and $\psi\colon\mathcal{C}\to\mathcal{C}$ be two nonlinear operators. Recall that the generalized variational inequality (in short, GVI) is to find a point $x^\dagger\in\mathcal{C}$ such that

$$\langle\mathcal{A}x^\dagger,\psi(y)-\psi(x^\dagger)\rangle\geq 0,\ \forall y\in\mathcal{C}. \tag{1}$$

The solution set of (1) is denoted by $GVI(\mathcal{A},\psi,\mathcal{C})$.

If $\psi\equiv\mathcal{I}$, then GVI (1) reduces to the variational inequality of finding $x^\dagger\in\mathcal{C}$ such that

$$\langle\mathcal{A}x^\dagger,y-x^\dagger\rangle\geq 0,\ \forall y\in\mathcal{C}. \tag{2}$$

The solution set of (2) is denoted by $VI(\mathcal{A},\mathcal{C})$.

Variational inequalities were introduced by Stampacchia [18] and provide a convenient mathematical tool for researching a large variety of interesting problems arising in physics, finance, economics, network analysis, elasticity, optimization, water resources, medical images and structural analysis ([4, 14, 15, 20, 21, 27, 28, 35, 36, 41]). There are several iterative methods for solving VI (2). See, e.g., [2, 5, 7, 10, 19, 26, 31, 32, 37, 38]. The simplest one is the natural extension of the projected gradient algorithm for solving optimization problems by replacing the operator $\mathcal{A}$ with the gradient, so that we obtain a sequence $\{u_k\}$ generated

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the following manner: for given initial value u_0,

$$u_{k+1} = \mathrm{proj}_{\mathcal{C}}[u_k - \nu \mathcal{A} u_k],\ k \geq 0,$$

where ν is some positive real number and $proj_{\mathcal{C}}$, is the metric projection from $\mathcal{H}$ onto $\mathcal{C}$.

Note that the above algorithm can acquire convergence under quite strict hypotheses. In order to overcome this flaw, Korpelevich suggested in [11] an algorithm of the following form: for given initial value u_0,

$$\begin{cases} v_k = \mathrm{proj}_{\mathcal{C}}[u_k - \nu \mathcal{A} u_k], \\ u_{k+1} = \mathrm{proj}_{\mathcal{C}}[u_k - \nu \mathcal{A} v_k],\ k \geq 0. \end{cases}$$

Consequently, Korpelevich's algorithm and its variant form have been presented and studied in the literature, see for instance, [1, 3, 6, 8, 13, 16, 17, 22, 23, 25, 29, 33, 39, 40]. In this article, we will study the following generalized variational inequalities of finding a point $\tilde{x}$ such that

$$\tilde{x} \in GVI(\mathcal{A}, \psi, \mathcal{C}) \cap GVI(\mathcal{B}, \psi, \mathcal{C}). \tag{3}$$

Motivated by the work of [5, 22, 42], in this paper, we introduce a new iterative algorithm for solving (3). We prove the strong convergence of the presented algorithm under some mild conditions.

2. **Notation and Lemmas**

Let $\mathcal{C}$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. An operator $S\colon \mathcal{C} \to \mathcal{C}$ is said to be L-Lipschitz if $\|Sx^\dagger - Sy^\dagger\| \leq L\|x^\dagger - y^\dagger\|$, $\forall x^\dagger, y^\dagger \in \mathcal{C}$, where $L > 0$ is a constant.

Definition 2.1. *An operator $\mathcal{A} : \mathcal{C} \to \mathcal{H}$ is said to be*

- *Monotone if $\langle \mathcal{A}u - \mathcal{A}v, u - v\rangle \geq 0$, $\forall u, v \in \mathcal{C}$.*
- *Strongly monotone if $\langle \mathcal{A}u - \mathcal{A}v, u - v\rangle \geq \delta\|u - v\|^2$, $\forall u, v \in \mathcal{C}$, where $\delta > 0$ is a constant.*
- *λ-inverse strongly monotone if $\langle \mathcal{A}u - \mathcal{A}v, u - v\rangle \geq \lambda\|\mathcal{A}u - \mathcal{A}v\|^2$, $\forall u, v \in \mathcal{C}$, where $\lambda > 0$ is a constant.*
- *λ-inverse strongly ψ-monotone if $\langle \mathcal{A}u - \mathcal{A}v, \psi(u) - \psi(v)\rangle \geq \lambda\|\mathcal{A}u - \mathcal{A}v\|^2$, $\forall u, v \in \mathcal{C}$, where $\psi : \mathcal{C} \to \mathcal{C}$ is a nonlinear operator and $\lambda > 0$ is a constant.*

An operator $R\colon \mathcal{H} \to 2^{\mathcal{H}}$ is said to be monotone on $\mathcal{H}$ iff $\langle x - y, u - v\rangle \geq 0$ for all $x, y \in \mathcal{H}$, $u \in Rx$, and $v \in Ry$. A monotone operator R on $\mathcal{H}$ is said to be maximal iff its graph is not strictly contained in the graph of any other monotone operator on $\mathcal{H}$.

For $\forall x^\dagger \in \mathcal{H}$, there exists a unique nearest point in $\mathcal{C}$, denoted by $\mathrm{proj}_{\mathcal{C}}[x^\dagger]$ such that $\|x^\dagger - \mathrm{proj}_{\mathcal{C}}[x^\dagger]\| \leq \|y - x^\dagger\|$, for all $y \in \mathcal{C}$. Now it is known that the operator $\mathrm{proj}_{\mathcal{C}} : \mathcal{H} \to \mathcal{C}$ is firmly nonexpansive, that is,

$$\|\mathrm{proj}_{\mathcal{C}}[x^\dagger] - \mathrm{proj}_{\mathcal{C}}[y^\dagger]\|^2 \leq \langle \mathrm{proj}_{\mathcal{C}}[x^\dagger] - \mathrm{proj}_{\mathcal{C}}[y^\dagger], x^\dagger - y^\dagger\rangle,\ \forall x^\dagger, y^\dagger \in \mathcal{H}.$$

Consequently ([34, 35]),

$$\langle x^\dagger - \mathrm{proj}_{\mathcal{C}}[x^\dagger], u^\dagger - \mathrm{proj}_{\mathcal{C}}[x^\dagger]\rangle \leq 0,\ \forall x^\dagger \in \mathcal{H}, u^\dagger \in \mathcal{C}. \tag{4}$$

Recall that an operator S is said to be demiclosed if $w_n \rightharpoonup \tilde{u}$ weakly and $Sw_n \to u$ strongly, imply $S(\tilde{u}) = u$. Next, we collect several conclusions for our main results in the next section.

Lemma 2.1 ([24]). *Suppose* $\{\varpi_n\} \subset [0,\infty)$, $\{\nu_n\} \subset (0,1)$ *and* $\{\varrho_n\}$ *are three real number sequences satisfying*

(i) $\varpi_{n+1} \le (1-\nu_n)\varpi_n + \varrho_n, \forall n \ge 1$;

(ii) $\sum_{n=1}^{\infty} \nu_n = \infty$;

(iii) $\limsup_{n\to\infty} \dfrac{\varrho_n}{\nu_n} \le 0$ *or* $\sum_{n=1}^{\infty} |\varrho_n| < \infty$.

Then $\lim_{n\to\infty} \varpi_n = 0$.

Lemma 2.2 ([12]). *Let* $\{w_n\}$ *be a sequence of real numbers. Assume there exists at least a subsequence* $\{w_{n_k}\}$ *of* $\{w_n\}$ *such that* $w_{n_k} \le w_{n_k+1}$ *for all* $k \ge 0$. *For every* $n \ge N_0$, *define an integer sequence* $\{\tau(n)\}$ *as*

$$\tau(n) = \max\{i \le n : w_{n_i} < w_{n_i+1}\}.$$

Then $\tau(n) \to \infty$ *as* $n \to \infty$ *and for all* $n \ge N_0$, *we have* $\max\{w_{\tau(n)}, w_n\} \le w_{\tau(n)+1}$.

3. Main results

Let $\mathcal{C}$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Let the operators $f, g\colon \mathcal{C} \to \mathcal{H}$ be L_1-Lipschitzian and L_2-Lipschitzian, respectively. Let $\psi\colon \mathcal{C} \to \mathcal{C}$ be a weakly continuous and δ-strongly monotone operator such that its range $R(\psi) = \mathcal{C}$. Let $\nu > 0$ and $\mu > 0$ be two constants satisfying $\max\{L_1\nu, L_2\mu\} < \delta$. Let $\mathcal{A}\colon \mathcal{C} \to \mathcal{H}$ be a λ-inverse strongly ψ-monotone operator with coefficient $\lambda > 0$. Let $\mathcal{B}\colon \mathcal{C} \to \mathcal{H}$ be a β-inverse strongly ψ-monotone operator with coefficient $\beta > 0$. Denote the solution set of (3) by Ω, that is, $\Omega = GVI(\mathcal{A}, \psi, \mathcal{C}) \bigcap GVI(\mathcal{B}, \psi, \mathcal{C})$. In the sequel, we assume $\Omega \ne \emptyset$. Now, we first consider the following variational inequality ($VI(f, \psi, \mathcal{C})$, in short) of finding $\tilde{x}$ such that

$$\langle \nu f(\tilde{x}) - \psi(\tilde{x}), \psi(x^\dagger) - \psi(\tilde{x})\rangle \le 0, \ \forall x^\dagger \in \Omega. \tag{5}$$

The solution set of (5) is denoted by $VI(\nu f, \psi, \mathcal{C})$.

Remark 3.1. $VI(\nu f, \psi, \mathcal{C})$ has a unique solution provided that $\nu L_1 < \delta$, see [30].

In the sequel, we assume that $\Gamma := VI(\nu f, \psi, \mathcal{C}) \cap VI(\mu g, \psi, \mathcal{C}) \ne \emptyset$. Next, we present our algorithm for solving the problem (3).

Algorithm 3.1. *For given initial guess* $x_0 \in \mathcal{C}$ *arbitrarily, let the sequence* $\{x_n\}$ *be generated iteratively by*

$$\begin{cases} \psi(u_n) = \mathrm{proj}_{\mathcal{C}}[\lambda_n \nu f(x_n) + (1-\lambda_n)(\psi(x_n) - \varsigma_n \mathcal{A} x_n)], \\ \psi(x_{n+1}) = (1-\sigma_n)\mathrm{proj}_{\mathcal{C}}[\lambda_n \mu g(x_n) + (1-\lambda_n)(\psi(u_n) - \gamma_n \mathcal{B} u_n)] \\ \qquad\qquad +\sigma_n \psi(x_n), \quad n \ge 0, \end{cases} \tag{6}$$

where $\{\lambda_n\}$ *and* $\{\sigma_n\}$ *are two real number sequences in* $[0,1]$ *and* $\{\varsigma_n\}$ *and* $\{\gamma_n\}$ *are two real number sequences in* $(0,\infty)$.

Theorem 3.1. *If the following assumptions are satisfied:*

(i) $\lim_{n\to\infty} \lambda_n = 0$ *and* $\sum_n \lambda_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \sigma_n \le \limsup_{n\to\infty} \sigma_n < 1$;

(iii) $0 < \liminf_{n\to\infty} \varsigma_n \le \limsup_{n\to\infty} \varsigma_n < 2\lambda$;

(iv) $0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < 2\beta$;

then the sequence $\{x_n\}$ *generated by* (6) *converges strongly to* $\tilde{x} \in \Omega$ *which solves variational inequalities* $VI(f, \psi, \mathcal{C})$ *and* $VI(g, \psi, \mathcal{C})$*, that is,* $\tilde{x} \in \Gamma$.

Proof. By Remark 3.1, we know that Γ is a singleton denoted by $\tilde{x}$. By virtue of (4), we obtain $\psi(\tilde{x}) = \mathrm{proj}_{\mathcal{C}}[\psi(\tilde{x}) - \varsigma_n \mathcal{A}\tilde{x}]$ and $\psi(\tilde{x}) = \mathrm{proj}_{\mathcal{C}}[\psi(\tilde{x}) - \gamma_n \mathcal{B}\tilde{x}]$ for all $n \geq 0$. Since $\mathcal{A}$ is λ-inverse strongly ψ-monotone, by Definition 2.1, we have

$$\begin{aligned} \|(\psi(x) - \varsigma\mathcal{A}x) - (\psi(\tilde{x}) - \varsigma\mathcal{A}\tilde{x})\|^2 & \\ &= \|\psi(x) - \psi(\tilde{x})\|^2 - 2\varsigma\langle \mathcal{A}x - \mathcal{A}\tilde{x}, \psi(x) - \psi(\tilde{x})\rangle \\ &\quad + \varsigma^2\|\mathcal{A}x - \mathcal{A}\tilde{x}\|^2 \\ &\leq \|\psi(x) - \psi(\tilde{x})\|^2 - 2\varsigma\lambda\|\mathcal{A}x - \mathcal{A}\tilde{x}\|^2 + \varsigma^2\|\mathcal{A}x - \mathcal{A}\tilde{x}\|^2 \\ &\leq \|\psi(x) - \psi(\tilde{x})\|^2 + \varsigma(\varsigma - 2\lambda)\|\mathcal{A}x - \mathcal{A}\tilde{x}\|^2. \end{aligned} \tag{7}$$

It follows that

$$\begin{aligned} \|(\psi(x_n) - \varsigma_n\mathcal{A}x_n) - (\psi(\tilde{x}) - \varsigma_n\mathcal{A}\tilde{x})\|^2 &\leq \|\psi(x_n) - \psi(\tilde{x})\|^2 \\ &\quad + \varsigma_n(\varsigma_n - 2\lambda)\|\mathcal{A}x_n - \mathcal{A}\tilde{x}\|^2 \\ &\leq \|\psi(x_n) - \psi(\tilde{x})\|^2. \end{aligned} \tag{8}$$

Similarly, we also obtain

$$\begin{aligned} \|(\psi(u_n) - \gamma_n\mathcal{B}u_n) - (\psi(\tilde{x}) - \gamma_n\mathcal{B}\tilde{x})\|^2 &\leq \|\psi(u_n) - \psi(\tilde{x})\|^2 \\ &\quad + \gamma_n(\gamma_n - 2\beta)\|\mathcal{B}u_n - \mathcal{B}\tilde{x}\|^2 \\ &\leq \|\psi(u_n) - \psi(\tilde{x})\|^2. \end{aligned} \tag{9}$$

According to the δ-strong monotonicity of ψ, we deduce

$$\|\psi(x) - \psi(y)\| \geq \delta\|x - y\|, \ \forall x, y \in \mathcal{C}. \tag{10}$$

Set $v_n = \mathrm{proj}_{\mathcal{C}}[\lambda_n \mu g(x_n) + (1 - \lambda_n)(\psi(u_n) - \gamma_n \mathcal{B}u_n)]$ for all $n \geq 0$. From (6), (8) and (10), we derive

$$\begin{aligned} \|\psi(u_n) - \psi(\tilde{x})\| &= \|\mathrm{proj}_{\mathcal{C}}[\lambda_n \nu f(x_n) + (1 - \lambda_n)(\psi(x_n) - \varsigma_n\mathcal{A}x_n)] \\ &\quad - \mathrm{proj}_{\mathcal{C}}[\psi(\tilde{x}) - \varsigma_n\mathcal{A}\tilde{x}]\| \\ &\leq \|(1 - \lambda_n)((\psi(x_n) - \varsigma_n\mathcal{A}x_n) - (\psi(\tilde{x}) - \varsigma_n\mathcal{A}\tilde{x})) \\ &\quad + \lambda_n(\nu f(x_n) - \psi(\tilde{x}) + \varsigma_n\mathcal{A}\tilde{x})\| \\ &\leq \lambda_n\|\nu f(x_n) - \nu f(\tilde{x})\| + \lambda_n\|\nu f(\tilde{x}) - \psi(\tilde{x}) + \varsigma_n\mathcal{A}\tilde{x}\| \\ &\quad + (1 - \lambda_n)\|(\psi(x_n) - \varsigma_n\mathcal{A}x_n) - (\psi(\tilde{x}) - \varsigma_n\mathcal{A}\tilde{x})\| \\ &\leq \lambda_n\nu L_1\|x_n - \tilde{x}\| + \lambda_n\|\nu f(\tilde{x}) - \psi(\tilde{x}) + \varsigma_n\mathcal{A}\tilde{x}\| \\ &\quad + (1 - \lambda_n)\|\psi(x_n) - \psi(\tilde{x})\| \\ &\leq \lambda_n\nu L_1/\delta\|\psi(x_n) - \psi(\tilde{x})\| + \lambda_n\|\nu f(\tilde{x}) - \psi(\tilde{x}) + \varsigma_n\mathcal{A}\tilde{x}\| \\ &\quad + (1 - \lambda_n)\|\psi(x_n) - \psi(\tilde{x})\| \\ &= [1 - (1 - \nu L_1/\delta)\lambda_n]\|\psi(x_n) - \psi(\tilde{x})\| \\ &\quad + \lambda_n\|\nu f(\tilde{x}) - \psi(\tilde{x}) + \varsigma_n\mathcal{A}\tilde{x}\| \\ &\leq [1 - (1 - \nu L_1/\delta)\lambda_n]\|\psi(x_n) - \psi(\tilde{x})\| \\ &\quad + \lambda_n(\|\nu f(\tilde{x}) - \psi(\tilde{x})\| + 2\lambda\|\mathcal{A}\tilde{x}\|) \end{aligned} \tag{11}$$

and

$$\begin{aligned}\|v_n-\psi(\tilde{x})\| &= \|\mathrm{proj}_{\mathcal{C}}[\lambda_n\mu g(x_n)+(1-\lambda_n)(\psi(u_n)-\gamma_n\mathcal{B}u_n)]\| \\ &\le [1-(2-\lambda_n-\nu L_1/\delta-\mu L_2/\delta+\nu L_1\lambda_n/\delta)\lambda_n]\|\psi(x_n)-\psi(\tilde{x})\| \\ &\quad +\lambda_n(\|\nu f(\tilde{x})-\psi(\tilde{x})\|+2\lambda\|\mathcal{A}\tilde{x}\|+\|\mu g(\tilde{x})-\psi(\tilde{x})\|+2\beta\|\mathcal{B}\tilde{x}\|).\end{aligned} \tag{12}$$

By assumption (i), without loss of generality, we can assume that there exists a constant $\tau>0$ such that $\tau<2-\lambda_n-\nu L_1/\delta-\mu L_2/\delta+\nu L_1\lambda_n/\delta$ for all $n\ge 0$. Hence, we get

$$\begin{aligned}\|v_n-\psi(\tilde{x})\| \le \lambda_n(&\|\nu f(\tilde{x})-\psi(\tilde{x})\|+2\lambda\|\mathcal{A}\tilde{x}\|+\|\mu g(\tilde{x})-\psi(\tilde{x})\| \\ &+2\beta\|\mathcal{B}\tilde{x}\|)+(1-\tau\lambda_n)\|\psi(x_n)-\psi(\tilde{x})\|.\end{aligned} \tag{13}$$

In terms of (8) and (11), we obtain

$$\begin{aligned}\|\psi(u_n)-\psi(\tilde{x})\|^2 &\le \|(1-\lambda_n)((\psi(x_n)-\varsigma_n\mathcal{A}x_n)-(\psi(\tilde{x})-\varsigma_n\mathcal{A}\tilde{x})) \\ &\quad +\lambda_n(\nu f(x_n)-\psi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x})\|^2 \\ &\le (1-\lambda_n)\|(\psi(x_n)-\varsigma_n\mathcal{A}x_n)-(\psi(\tilde{x})-\varsigma_n\mathcal{A}\tilde{x}))\|^2 \\ &\quad +\lambda_n\|\nu f(x_n)-\psi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x}\|^2 \\ &\le (1-\lambda_n)[\|\psi(x_n)-\psi(\tilde{x})\|^2+\varsigma_n(\varsigma_n-2\lambda)\|\mathcal{A}x_n-\mathcal{A}\tilde{x}\|^2] \\ &\quad +\lambda_n\|\nu f(x_n)-\psi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x}\|^2.\end{aligned} \tag{14}$$

Similarly, from (9) and (12), we also have

$$\begin{aligned}\|v_n-\psi(\tilde{x})\|^2 \le \lambda_n\|\mu g(x_n)-\psi(\tilde{x})+\gamma_n\mathcal{B}\tilde{x}\|^2+(1-\lambda_n)[&\|\psi(u_n)-\psi(\tilde{x})\|^2 \\ +\gamma_n(\gamma_n-2\beta)\|\mathcal{B}u_n-\mathcal{B}\tilde{x}\|^2&].\end{aligned} \tag{15}$$

Combining (6) with (15), we obtain

$$\begin{aligned}\|\psi(x_{n+1})-\psi(\tilde{x})\| &\le \sigma_n\|\psi(x_n)-\psi(\tilde{x})\|+(1-\sigma_n)\|v_n-\psi(\tilde{x})\| \\ &\le (1-\sigma_n)(1-\tau\lambda_n)\|\psi(x_n)-\psi(\tilde{x})\| \\ &\quad +\sigma_n\|\psi(x_n)-\psi(\tilde{x})\|+(1-\sigma_n)\lambda_n(\|\nu f(\tilde{x})-\psi(\tilde{x})\| \\ &\quad +2\lambda\|\mathcal{A}\tilde{x}\|+\|\mu g(\tilde{x})-\psi(\tilde{x})\|+2\beta\|\mathcal{B}\tilde{x}\|) \\ &= [1-(1-\sigma_n)\tau\lambda_n]\|\psi(x_n)-\psi(\tilde{x})\| \\ &\quad +(1-\sigma_n)\tau\lambda_n(\|\nu f(\tilde{x})-\psi(\tilde{x})\|+2\lambda\|\mathcal{A}\tilde{x}\| \\ &\quad +\|\mu g(\tilde{x})-\psi(\tilde{x})\|+2\beta\|\mathcal{B}\tilde{x}\|)/\tau.\end{aligned} \tag{16}$$

By mathematical induction,

$$\begin{aligned}\|\psi(x_n)-\psi(\tilde{x})\| \le \max\{&\|\psi(x_0)-\psi(\tilde{x})\|,(\|\nu f(\tilde{x})-\psi(\tilde{x})\|+2\lambda\|\mathcal{A}\tilde{x}\| \\ &+\|\mu g(\tilde{x})-\psi(\tilde{x})\|+2\beta\|\mathcal{B}\tilde{x}\|)/\tau\}.\end{aligned}$$

Consequently,

$$\begin{aligned}\|x_n-\tilde{x}\| \le \frac{1}{\delta}\max\{&\|\psi(x_0)-\psi(\tilde{x})\|,(\|\nu f(\tilde{x})-\psi(\tilde{x})\|+2\lambda\|\mathcal{A}\tilde{x}\| \\ &+\|\mu g(\tilde{x})-\psi(\tilde{x})\|+2\beta\|\mathcal{B}\tilde{x}\|)/\tau\}.\end{aligned}$$

Thus, $\{x_n\}$, $\{\psi(x_n)\}$, $\{u_n\}$, $\{v_n\}$, $\{\mathcal{A}x_n\}$ and $\{\mathcal{B}u_n\}$ are all bounded.

From (5), we get

$$\psi(x_{n+1})-\psi(x_n)=(1-\sigma_n)(v_n-\psi(x_n)),\ n\ge 0. \tag{17}$$

By computation, we deduce

$$\begin{aligned}\|\psi(x_{n+1})-\psi(\tilde{x})\|^2 &= \|\psi(x_n)-\psi(\tilde{x})\|^2+\|\psi(x_{n+1})-\psi(x_n)\|^2\\ &\quad+(1-\sigma_n)[\|v_n-\psi(\tilde{x})\|^2-\|\psi(x_n)-\psi(\tilde{x})\|^2\\ &\quad-\|v_n-\psi(x_n)\|^2].\end{aligned} \tag{18}$$

Consequently,

$$\begin{aligned}&\|\psi(x_{n+1})-\psi(\tilde{x})\|^2-\|\psi(x_n)-\psi(\tilde{x})\|^2\\ &=(1-\sigma_n)[\|v_n-\psi(\tilde{x})\|^2-\|\psi(x_n)-\psi(\tilde{x})\|^2\\ &\quad-\|v_n-\psi(x_n)\|^2]+(1-\sigma_n)^2\|v_n-\psi(x_n)\|^2\\ &=(1-\sigma_n)[\|v_n-\psi(\tilde{x})\|^2-\|\psi(x_n)-\psi(\tilde{x})\|^2]\\ &\quad-\sigma_n(1-\sigma_n)\|v_n-\psi(x_n)\|^2.\end{aligned} \tag{19}$$

In light of (11), we get

$$\begin{aligned}\|\psi(u_n)-\psi(\tilde{x})\|^2 &\le [1-(1-\nu L_1/\delta)\lambda_n]\|\psi(x_n)-\psi(\tilde{x})\|^2\\ &\quad+(1-\nu L_1/\delta)\lambda_n\left(\frac{\|\nu f(\tilde{x})-\psi(\tilde{x})\|+2\lambda\|\mathcal{A}\tilde{x}\|}{(1-\nu L_1/\delta)}\right)^2.\end{aligned} \tag{20}$$

Next, we consider two possible cases. Firstly, we assume there exists some integer $m>0$ such that $\{\|\psi(x_n)-\psi(\tilde{x})\|\}$ is decreasing for all $n\ge m$. In this case, we know that $\lim_{n\to\infty}\|\psi(x_n)-\psi(\tilde{x})\|$ exists. From (12) and (19), we have

$$\begin{aligned}\sigma_n(1-\sigma_n)\|v_n-\psi(x_n)\|^2 &\le \|\psi(x_n)-\psi(\tilde{x})\|^2-\|\psi(x_{n+1})-\psi(\tilde{x})\|^2\\ &\quad+(1-\sigma_n)[\|v_n-\psi(\tilde{x})\|^2-\|\psi(x_n)-\psi(\tilde{x})\|^2]\\ &\le \|\psi(x_n)-\psi(\tilde{x})\|^2-\|\psi(x_{n+1})-\psi(\tilde{x})\|^2\\ &\quad+\frac{\lambda_n}{\tau^2}(\|\nu f(\tilde{x})-\psi(\tilde{x})\|+2\lambda\|\mathcal{A}\tilde{x}\|+\|\mu g(\tilde{x})-\psi(\tilde{x})\|\\ &\quad+2\beta\|\mathcal{B}\tilde{x}\|)^2\\ &\to 0.\end{aligned}$$

This together with assumptions (i) and (ii) implies that

$$\lim_{n\to\infty}\|v_n-\psi(x_n)\|=0. \tag{21}$$

Moreover, from (17), we get

$$\lim_{n\to\infty}\|\psi(x_{n+1})-\psi(x_n)\|=0. \tag{22}$$

By (15), we have

$$\begin{aligned}\|\psi(x_{n+1})-\psi(\tilde{x})\|^2 &\le \sigma_n\|\psi(x_n)-\psi(\tilde{x})\|^2+(1-\sigma_n)\|v_n-\psi(\tilde{x})\|^2\\ &\le \lambda_n(\|\nu f(x_n)-\psi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x}\|^2+\|\mu g(x_n)-\psi(\tilde{x})\\ &\quad+\gamma_n\mathcal{B}\tilde{x}\|^2)+\|\psi(x_n)-\psi(\tilde{x})\|^2\\ &\quad+(1-\sigma_n)(1-\lambda_n)[\varsigma_n(\varsigma_n-2\lambda)\|\mathcal{A}x_n-\mathcal{A}\tilde{x}\|^2\\ &\quad+\gamma_n(\gamma_n-2\beta)\|\mathcal{B}u_n-\mathcal{B}\tilde{x}\|^2].\end{aligned} \tag{23}$$

Hence,

$$
\begin{aligned}
&(1-\sigma_n)(1-\lambda_n)[\varsigma_n(2\lambda-\varsigma_n)\|\mathcal{A}x_n-\mathcal{A}\tilde{x}\|^2+\gamma_n(2\beta-\gamma_n)\|\mathcal{B}u_n-\mathcal{B}\tilde{x}\|^2] \\
&\le \|\psi(x_n)-\psi(\tilde{x})\|^2-\|\psi(x_{n+1})-\psi(\tilde{x})\|^2+\lambda_n(\|\nu f(x_n)-\psi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x}\|^2 \\
&\quad +\|\mu g(x_n)-\psi(\tilde{x})+\gamma_n\mathcal{B}\tilde{x}\|^2) \\
&\le (\|\psi(x_n)-\psi(\tilde{x})\|+\|\psi(x_{n+1})-\psi(\tilde{x})\|)\|\psi(x_{n+1})-\psi(x_n)\| \\
&\quad +\lambda_n(\|\nu f(x_n)-\psi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x}\|^2+\|\mu g(x_n)-\psi(\tilde{x})+\gamma_n\mathcal{B}\tilde{x}\|^2) \\
&\to 0 \text{ (by } (i) \text{ and (22))}.
\end{aligned}
$$

This together with assumptions $(i)-(iv)$ implies that

$$
\lim_{n\to\infty}\|\mathcal{A}x_n-\mathcal{A}\tilde{x}\|=0 \ \text{ and } \ \lim_{n\to\infty}\|\mathcal{B}u_n-\mathcal{B}\tilde{x}\|=0. \tag{24}
$$

Set $y_n=\psi(x_n)-\varsigma_n\mathcal{A}x_n-(\psi(\tilde{x})-\varsigma_n\mathcal{A}\tilde{x})$ for all $n\ge 0$.

Applying (4), we get

$$
\begin{aligned}
\|\psi(u_n)-\psi(\tilde{x})\|^2 &\le \langle\lambda_n(\nu f(x_n)-\psi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x})+(1-\lambda_n)y_n,\psi(u_n) \\
&\quad -\psi(\tilde{x})\rangle \\
&= \frac{1}{2}\{\|\lambda_n(\nu f(x_n)-\psi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x})+(1-\lambda_n)y_n\|^2 \\
&\quad +\|\psi(u_n)-\psi(\tilde{x})\|^2-\|\lambda_n(\nu f(x_n)-\psi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x}) \\
&\quad +(1-\lambda_n)y_n-\psi(u_n)+\psi(\tilde{x})\|^2\} \\
&\le \frac{1}{2}\{\lambda_n\|\nu f(x_n)-\psi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x}\|^2+\|\psi(u_n)-\psi(\tilde{x})\|^2 \\
&\quad +(1-\lambda_n)\|\psi(x_n)-\psi(\tilde{x})\|^2-\|\lambda_n(\nu f(x_n)-\psi(\tilde{x}) \\
&\quad +\varsigma_n\mathcal{A}\tilde{x}-y_n)+\psi(x_n)-\psi(u_n)-\varsigma_n(\mathcal{A}x_n-\mathcal{A}\tilde{x})\|^2\} \\
&= \frac{1}{2}\{\lambda_n\|\nu f(x_n)-\psi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x}\|^2\|\psi(u_n)-\psi(\tilde{x})\|^2+ \\
&\quad +(1-\lambda_n)\|\psi(x_n)-\psi(\tilde{x})\|^2-\|\psi(x_n)-\psi(u_n)\|^2 \\
&\quad -\lambda_n^2\|\nu f(x_n)-\psi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x}-y_n\|^2 \\
&\quad -\varsigma_n^2\|\mathcal{A}x_n-\mathcal{A}\tilde{x}\|+2\varsigma_n\lambda_n\langle\mathcal{A}x_n-\mathcal{A}\tilde{x},\nu f(x_n)-\psi(\tilde{x}) \\
&\quad +\varsigma_n\mathcal{A}\tilde{x}-y_n\rangle+2\varsigma_n\langle\psi(x_n)-\psi(u_n),\mathcal{A}x_n-\mathcal{A}\tilde{x}\rangle \\
&\quad -2\lambda_n\langle\psi(x_n)-\psi(u_n),\nu f(x_n)-\psi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x}-y_n\rangle\}.
\end{aligned} \tag{25}
$$

It follows that

$$
\begin{aligned}
\|\psi(u_n)-\psi(\tilde{x})\|^2 &\le \lambda_n\|\nu f(x_n)-\psi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x}\|^2-\|\psi(x_n)-\psi(u_n)\|^2 \\
&\quad +2\lambda_n\|\psi(x_n)-\psi(u_n)\|\|\nu f(x_n)-\psi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x}-y_n\| \\
&\quad +2\varsigma_n\lambda_n\|\mathcal{A}x_n-\mathcal{A}\tilde{x}\|\|\nu f(x_n)-\psi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x}-y_n\| \\
&\quad +2\varsigma_n\|\psi(x_n)-\psi(u_n)\|\|\mathcal{A}x_n-\mathcal{A}\tilde{x}\| \\
&\quad +(1-\lambda_n)\|\psi(x_n)-\psi(\tilde{x})\|^2.
\end{aligned} \tag{26}
$$

In light of (15) and (26), we have

$$\begin{aligned}\|\psi(x_{n+1})-\psi(\tilde{x})\|^2 &\le \sigma_n\|\psi(x_n)-\psi(\tilde{x})\|^2+(1-\sigma_n)\|v_n-\psi(\tilde{x})\|^2\\ &\le \sigma_n\|\psi(x_n)-\psi(\tilde{x})\|^2+(1-\sigma_n)[\|\psi(u_n)-\psi(\tilde{x})\|^2\\ &\quad+\lambda_n\|\mu g(x_n)-\psi(\tilde{x})+\gamma_n\mathcal{B}\tilde{x}\|^2].\end{aligned}$$

It follows that

$$\begin{aligned}\|\psi(x_{n+1})-\psi(\tilde{x})\|^2 &\le (1-\sigma_n)\lambda_n\|\mu g(x_n)-\psi(\tilde{x})+\gamma_n\mathcal{B}\tilde{x}\|^2\\ &\quad+(1-\sigma_n)\lambda_n\|\nu f(x_n)-\psi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x}\|^2\\ &\quad+(1-\lambda_n)(1-\sigma_n)\|\psi(x_n)-\psi(\tilde{x})\|^2\\ &\quad+\sigma_n\|\psi(x_n)-\psi(\tilde{x})\|^2-(1-\sigma_n)\|\psi(x_n)-\psi(u_n)\|^2\\ &\quad+2\varsigma_n(1-\sigma_n)\lambda_n\|\mathcal{A}x_n-\mathcal{A}\tilde{x}\|\|\nu f(x_n)-\psi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x}-y_n\|\\ &\quad+2\varsigma_n(1-\sigma_n)\|\psi(x_n)-\psi(u_n)\|\|\mathcal{A}x_n-\mathcal{A}\tilde{x}\|\\ &\quad+2(1-\sigma_n)\lambda_n\|\psi(x_n)-\psi(u_n)\|\|\nu f(x_n)-\psi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x}-y_n\|\\ &\le \lambda_n(\|\nu f(x_n)-\psi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x}\|^2+\|\mu g(x_n)-\psi(\tilde{x})+\gamma_n\mathcal{B}\tilde{x}\|^2)\\ &\quad+2\varsigma_n\lambda_n\|\mathcal{A}x_n-\mathcal{A}\tilde{x}\|\|\nu f(x_n)-\psi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x}-y_n\|\\ &\quad+2\varsigma_n\|\psi(x_n)-\psi(u_n)\|\|\mathcal{A}x_n-\mathcal{A}\tilde{x}\|-(1-\sigma_n)\|\psi(x_n)-\psi(u_n)\|^2\\ &\quad+2\lambda_n\|\psi(x_n)-\psi(u_n)\|\|\nu f(x_n)-\psi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x}-y_n\|\\ &\quad+\|\psi(x_n)-\psi(\tilde{x})\|^2.\end{aligned}$$

Then,

$$\begin{aligned}(1-\sigma_n)\|\psi(x_n)-\psi(u_n)\|^2 &\le (\|\psi(x_n)-\psi(\tilde{x})\|+\|\psi(x_{n+1})-\psi(\tilde{x})\|)\\ &\quad\times\|\psi(x_{n+1})-\psi(x_n)\|+\lambda_n(\|\nu f(x_n)-\psi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x}\|^2\\ &\quad+\|\mu g(x_n)-\psi(\tilde{x})+\gamma_n\mathcal{B}\tilde{x}\|^2)\\ &\quad+2\varsigma_n\lambda_n\|\mathcal{A}x_n-\mathcal{A}\tilde{x}\|\|\nu f(x_n)-\psi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x}-y_n\|\\ &\quad+2\varsigma_n\|\psi(x_n)-\psi(u_n)\|\|\mathcal{A}x_n-\mathcal{A}\tilde{x}\|\\ &\quad+2\lambda_n\|\psi(x_n)-\psi(u_n)\|\|\nu f(x_n)-\psi(\tilde{x})+\varsigma_n\mathcal{A}\tilde{x}-y_n\|.\end{aligned}$$

The above inequality together with (i), (iii), (22) and (24) implies that

$$\lim_{n\to\infty}\|\psi(x_n)-\psi(u_n)\|=0. \tag{27}$$

Next, we prove $\liminf_{n\to\infty}\langle\nu f(\tilde{x})-\psi(\tilde{x}),\psi(\tilde{x})-\psi(u_n)\rangle\ge 0$. Let $\{\psi(u_{n_i})\}$ be a subsequence of $\{\psi(u_n)\}$ such that

$$\begin{aligned}&\liminf_{n\to\infty}\langle\nu f(\tilde{x})-\psi(\tilde{x}),\psi(\tilde{x})-\psi(u_n)\rangle\\ &=\lim_{i\to\infty}\langle\nu f(\tilde{x})-\psi(\tilde{x}),\psi(\tilde{x})-\psi(u_{n_i})\rangle.\end{aligned} \tag{28}$$

Since $\{\psi(u_{n_i})\}$ is bounded, there exists a subsequence $\{\psi(u_{n_{i_j}})\}$ of $\{\psi(u_{n_i})\}$ which converges weakly to some point $\psi(z)\in\mathcal{C}$. Without loss of generality, we may assume that $\psi(u_{n_i})\rightharpoonup$

$\psi(z)$. Next, we need to prove $z \in GVI(\mathcal{A}, \psi, \mathcal{C})$. Set

$$Rv = \begin{cases} \mathcal{A}v + N_{\mathcal{C}}(v), & v \in \mathcal{C}, \\ \emptyset, & v \notin \mathcal{C}. \end{cases}$$

By [42], we know that R is maximal ψ-monotone. Let $(v, w) \in G(R)$. Since $w - \mathcal{A}v \in N_{\mathcal{C}}(v)$ and $x_n \in \mathcal{C}$, we have $\langle \psi(v) - \psi(x_n), w - \mathcal{A}v\rangle \geq 0$. Noting that $\psi(u_n) = \text{proj}_{\mathcal{C}}[\lambda_n \nu f(x_n) + (1-\lambda_n)(\psi(x_n) - \varsigma_n \mathcal{A}x_n)]$, we get

$$\langle \psi(v) - \psi(u_n), \psi(u_n) - [\lambda_n \nu f(x_n) + (1-\lambda_n)(\psi(x_n) - \varsigma_n \mathcal{A}x_n)]\rangle \geq 0.$$

It follows that

$$\langle \psi(v) - \psi(u_n), \frac{\psi(u_n) - \psi(x_n)}{\varsigma_n} + \mathcal{A}x_n - \frac{\lambda_n}{\varsigma_n}(\nu f(x_n) - \psi(x_n) + \varsigma_n \mathcal{A}x_n)\rangle \geq 0.$$

Thus,

$$\begin{aligned} \langle \psi(v) - \psi(x_{n_i}), w\rangle &\geq \langle \psi(v) - \psi(x_{n_i}), \mathcal{A}v\rangle \\ &\geq \langle \psi(v) - \psi(x_{n_i}), \mathcal{A}v\rangle - \langle \psi(v) - \psi(u_{n_i}), \mathcal{A}x_{n_i}\rangle \\ &\quad + \frac{\lambda_{n_i}}{\varsigma_{n_i}}\langle \psi(v) - \psi(u_{n_i}), \nu f(x_{n_i}) - \psi(x_{n_i}) + \varsigma_{n_i}\mathcal{A}x_{n_i}\rangle \\ &\quad - \langle \psi(v) - \psi(u_{n_i}), \frac{\psi(u_{n_i}) - \psi(x_{n_i})}{\varsigma_{n_i}}\rangle \\ &= \langle \psi(v) - \psi(x_{n_i}), \mathcal{A}v - \mathcal{A}x_{n_i}\rangle + \langle \psi(v) - \psi(x_{n_i}), \mathcal{A}x_{n_i}\rangle \\ &\quad + \frac{\lambda_{n_i}}{\varsigma_{n_i}}\langle \psi(v) - \psi(u_{n_i}), \nu f(x_{n_i}) - \psi(x_{n_i}) + \varsigma_{n_i}\mathcal{A}x_{n_i}\rangle \\ &\quad - \langle \psi(v) - \psi(u_{n_i}), \frac{\psi(u_{n_i}) - \psi(x_{n_i})}{\varsigma_{n_i}}\rangle \\ &\quad - \langle \psi(v) - \psi(u_{n_i}), \mathcal{A}x_{n_i}\rangle \\ &\geq \frac{\lambda_{n_i}}{\varsigma_{n_i}}\langle \psi(v) - \psi(u_{n_i}), \nu f(x_{n_i}) - \psi(x_{n_i}) + \varsigma_{n_i}\mathcal{A}x_{n_i}\rangle \\ &\quad - \langle \psi(v) - \psi(u_{n_i}), \frac{\psi(u_{n_i}) - \psi(x_{n_i})}{\varsigma_{n_i}}\rangle \\ &\quad - \langle \psi(x_{n_i}) - \psi(u_{n_i}), \mathcal{A}x_{n_i}\rangle. \end{aligned} \tag{29}$$

Since $\|\psi(x_{n_i}) - \psi(u_{n_i})\| \to 0$ and $\psi(x_{n_i}) \rightharpoonup \psi(z)$, we deduce that $\langle \psi(v) - \psi(z), w\rangle \geq 0$ by taking $i \to \infty$ in (29). Thus, $z \in R^{-1}0$ by the maximal ψ-monotonicity of R. Hence, $z \in GVI(\mathcal{A}, \psi, \mathcal{C})$.

Note that $\|v_{n_i} - \psi(u_{n_i})\| \to 0$. By the similar argument, we can deduce that $z \in GVI(\mathcal{B}, \psi, \mathcal{C})$. Therefore, $z \in \Omega$.

From (28), we obtain

$$\begin{aligned} \liminf_{n\to\infty}\langle \nu f(\tilde{x}) - \psi(\tilde{x}), \psi(\tilde{x}) - \psi(u_n)\rangle &= \lim_{i\to\infty}\langle \nu f(\tilde{x}) - \psi(\tilde{x}), \psi(\tilde{x}) - \psi(u_{n_i})\rangle \\ &= \langle \nu f(\tilde{x}) - \psi(\tilde{x}), \psi(\tilde{x}) - \psi(z)\rangle \geq 0. \end{aligned} \tag{30}$$

Consequently,

$$\begin{aligned} \liminf_{n\to\infty}\langle \mu g(\tilde{x}) - \psi(\tilde{x}), \psi(\tilde{x}) - v_n\rangle &\geq \lim_{i\to\infty}\langle \mu g(\tilde{x}) - \psi(\tilde{x}), \psi(\tilde{x}) - v_{n_i}\rangle \\ &= \langle \mu g(\tilde{x}) - \psi(\tilde{x}), \psi(\tilde{x}) - \psi(z)\rangle \geq 0. \end{aligned} \tag{31}$$

Applying (4), we obtain

$$\begin{aligned}
\|\psi(u_n)-\psi(\tilde{x})\|^2 &= \|\mathrm{proj}_{\mathcal{C}}[\lambda_n\nu f(x_n)+(1-\lambda_n)(\psi(x_n)-\varsigma_n\mathcal{A}x_n)] \\
&\quad -\mathrm{proj}_C[\psi(\tilde{x})-(1-\lambda_n)\varsigma_n\mathcal{A}\tilde{x}]\|^2 \\
&\le \langle\lambda_n(\nu f(x_n)-\psi(\tilde{x}))+(1-\lambda_n)y_n,\psi(u_n)-\psi(\tilde{x})\rangle \\
&\le (1-\lambda_n)\|\psi(x_n)-\varsigma_n\mathcal{A}x_n-(\psi(\tilde{x})-\varsigma_n\mathcal{A}\tilde{x})\|\|\psi(u_n)-\psi(\tilde{x})\| \\
&\quad +\lambda_n\langle\nu f(\tilde{x})-\psi(\tilde{x}),\psi(u_n)-\psi(\tilde{x})\rangle \\
&\quad +\lambda_n\nu\langle f(x_n)-f(\tilde{x}),\psi(u_n)-\psi(\tilde{x})\rangle \\
&\le \lambda_n L_1\nu\|x_n-\tilde{x}\|\|\psi(u_n)-\psi(\tilde{x})\| \\
&\quad +\lambda_n\langle\nu f(\tilde{x})-\psi(\tilde{x}),\psi(u_n)-\psi(\tilde{x})\rangle \\
&\quad +(1-\lambda_n)\|\psi(x_n)-\psi(\tilde{x})\|\|\psi(u_n)-\psi(\tilde{x})\| \\
&\le \lambda_n(\nu L_1/\delta)\|\psi(x_n)-\psi(\tilde{x})\|\|\psi(u_n)-\psi(\tilde{x})\| \\
&\quad +\lambda_n\langle\nu f(\tilde{x})-\psi(\tilde{x}),\psi(u_n)-\psi(\tilde{x})\rangle \\
&\quad +(1-\lambda_n)\|\psi(x_n)-\psi(\tilde{x})\|\|\psi(u_n)-\psi(\tilde{x})\| \\
&= [1-(1-L_1\nu/\delta)\lambda_n]\|\psi(x_n)-\psi(\tilde{x})\|\|\psi(u_n)-\psi(\tilde{x})\| \\
&\quad +\lambda_n\langle\nu f(\tilde{x})-\psi(\tilde{x}),\psi(u_n)-\psi(\tilde{x})\rangle \\
&\le \frac{1-(1-L_1\nu/\delta)\lambda_n}{2}\|\psi(x_n)-\psi(\tilde{x})\|^2 \\
&\quad +\lambda_n\langle\nu f(\tilde{x})-\psi(\tilde{x}),\psi(u_n)-\psi(\tilde{x})\rangle \\
&\quad +\frac{1}{2}\|\psi(u_n)-\psi(\tilde{x})\|^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|\psi(u_n)-\psi(\tilde{x})\|^2 &\le [1-(1-L_1\nu/\delta)\lambda_n]\|\psi(x_n)-\psi(\tilde{x})\|^2 \\
&\quad +2\lambda_n\langle\nu f(\tilde{x})-\psi(\tilde{x}),\psi(u_n)-\psi(\tilde{x})\rangle.
\end{aligned}$$

Set $z_n=\psi(u_n)-\gamma_n\mathcal{B}u_n)-(\tilde{x})-\gamma_n\mathcal{B}\tilde{x})$ for all $n\ge 0$. By (4), we obtain

$$\begin{aligned}
\|v_n-\psi(\tilde{x})\|^2 &= \|\mathrm{proj}_{\mathcal{C}}[\lambda_n\mu g(x_n)+(1-\lambda_n)(\psi(u_n)-\gamma_n\mathcal{B}u_n)] \\
&\quad -\mathrm{proj}_C[\psi(\tilde{x})-(1-\lambda_n)\gamma_n\mathcal{B}\tilde{x}]\|^2 \\
&\le \langle\lambda_n(\mu g(x_n)-\psi(\tilde{x}))+(1-\lambda_n)z_n,v_n-\psi(\tilde{x})\rangle \\
&\le (1-\lambda_n)\|\psi(u_n)-\gamma_n\mathcal{B}u_n)-(\psi(\tilde{x})-\gamma_n\mathcal{B}\tilde{x})\|\|v_n-\psi(\tilde{x})\| \\
&\quad +\lambda_n\mu\langle g(x_n)-g(\tilde{x}),v_n-\psi(\tilde{x})\rangle \\
&\quad +\lambda_n\langle\mu g(\tilde{x})-\psi(\tilde{x}),v_n-\psi(\tilde{x})\rangle \\
&\le (1-\lambda_n)\|\psi(u_n)-\psi(\tilde{x})\|\|v_n-\psi(\tilde{x})\| \\
&\quad +\lambda_n\langle\mu g(\tilde{x})-\psi(\tilde{x}),v_n-\psi(\tilde{x})\rangle \\
&\quad +\lambda_n L_2\mu\|x_n-\tilde{x}\|\|v_n-\psi(\tilde{x})\|.
\end{aligned}$$

This together with (10) implies that

$$\begin{aligned}\|v_n-\psi(\tilde{x})\|^2 &\le \lambda_n(\mu L_2/\delta)\|\psi(x_n)-\psi(\tilde{x})\|\|v_n-\psi(\tilde{x})\|\\ &\quad+\lambda_n\langle \mu g(\tilde{x})-\psi(\tilde{x}),v_n-\psi(\tilde{x})\rangle\\ &\quad+(1-\lambda_n)\|\psi(u_n)-\psi(\tilde{x})\|\|v_n-\psi(\tilde{x})\|\\ &\le \frac{\lambda_n(\mu L_2/\delta)}{2}\|\psi(x_n)-\psi(\tilde{x})\|^2+\frac{1-\lambda_n}{2}\|\psi(u_n)-\psi(\tilde{x})\|^2\\ &\quad+\frac{1}{2}\|v_n-\psi(\tilde{x})\|^2+\lambda_n\langle \mu g(\tilde{x})-\psi(\tilde{x}),v_n-\psi(\tilde{x})\rangle.\end{aligned}$$

It follows that

$$\begin{aligned}\|v_n-\psi(\tilde{x})\|^2 &\le \lambda_n(\mu L_2/\delta)\|\psi(x_n)-\psi(\tilde{x})\|^2+(1-\lambda_n)\|\psi(u_n)-\psi(\tilde{x})\|^2\\ &\quad+2\lambda_n\langle \mu g(\tilde{x})-\psi(\tilde{x}),v_n-\psi(\tilde{x})\rangle\\ &\le [1-(1-\mu L_2/\delta+(1-\lambda_n)(1-L_1\nu/\delta))\lambda_n]\|\psi(x_n)-\psi(\tilde{x})\|^2\\ &\quad+2(1-\lambda_n)\lambda_n\langle \nu f(\tilde{x})-\psi(\tilde{x}),\psi(u_n)-\psi(\tilde{x})\rangle\\ &\quad+2\lambda_n\langle \mu g(\tilde{x})-\psi(\tilde{x}),v_n-\psi(\tilde{x})\rangle.\end{aligned}$$

Therefore,

$$\begin{aligned}\|\psi(x_{n+1})-\psi(\tilde{x})\|^2 &\le \sigma_n\|\psi(x_n)-\psi(\tilde{x})\|^2+(1-\sigma_n)\|v_n-\psi(\tilde{x})\|^2\\ &\le [1-(1-\sigma_n)(1-\mu L_2/\delta+(1-\lambda_n)\\ &\quad\times(1-L_1\nu/\delta))\lambda_n]\|\psi(x_n)-\psi(\tilde{x})\|^2\\ &\quad+2(1-\sigma_n)(1-\lambda_n)\lambda_n\langle \nu f(\tilde{x})-\psi(\tilde{x}),\psi(u_n)-\psi(\tilde{x})\rangle\\ &\quad+2(1-\sigma_n)\lambda_n\langle \mu g(\tilde{x})-\psi(\tilde{x}),v_n-\psi(\tilde{x})\rangle.\end{aligned} \tag{32}$$

By (30), (31), (32) and Lemma 2.1, we conclude that $\psi(x_n)\to\psi(\tilde{x})$ and $x_n\to\tilde{x}$.

Secondly, assume there exists an integer n_0 such that $\|\psi(x_{n_0})-\psi(\tilde{x})\|\le\|\psi(x_{n_0+1})-\psi(\tilde{x})\|$. Set $\omega_n=\{\|\psi(x_n)-\psi(\tilde{x})\|\}$. Hence, we get $\omega_{n_0}\le\omega_{n_0+1}$. For $n\ge n_0$, let $\{\tau_n\}$ be a sequence defined by $\tau(n)=\max\{l\in\mathbb{N}|n_0\le l\le n,\omega_l\le\omega_{l+1}\}$. We can check easily that $\tau(n)$ is a non-decreasing sequence satisfying $\lim_{n\to\infty}\tau(n)=\infty$ and $\omega_{\tau(n)}\le\omega_{\tau(n)+1}$ for all $n\ge n_0$.

By the similar argument as that of (30), (31) and (32), we can prove that

$$\begin{aligned}&\liminf_{n\to\infty}\langle \nu f(\tilde{x})-\psi(\tilde{x}),\psi(u_{\tau(n)})-\psi(\tilde{x})\rangle\ge 0\quad \textit{and}\\ &\liminf_{n\to\infty}\langle \mu g(\tilde{x})-\psi(\tilde{x}),\psi(\tilde{x})-v_{\tau(n)}\rangle\ge 0,\end{aligned} \tag{33}$$

and

$$\begin{aligned}\omega^2_{\tau(n)+1} &\le [1-(1-\sigma_{\tau(n)})(1-\mu L_2/\delta+(1-\lambda_{\tau(n)})(1-L_1\nu/\delta))\lambda_{\tau(n)}]\omega^2_{\tau(n)}\\ &\quad+2(1-\sigma_{\tau(n)})(1-\lambda_{\tau(n)})\lambda_{\tau(n)}\langle \nu f(\tilde{x})-\psi(\tilde{x}),\psi(u_{\tau(n)})-\psi(\tilde{x})\rangle\\ &\quad+2(1-\sigma_{\tau(n)})\lambda_{\tau(n)}\langle \mu g(\tilde{x})-\psi(\tilde{x}),v_{\tau(n)}-\psi(\tilde{x})\rangle.\end{aligned} \tag{34}$$

Note that $\omega_{\tau(n)}\le\omega_{\tau(n)+1}$. We deduce from (34) that

$$\begin{aligned}\omega^2_{\tau(n)} &\le (1-\lambda_{\tau(n)})\varrho_n\langle \nu f(\tilde{x})-\psi(\tilde{x}),\psi(u_{\tau(n)})-\psi(\tilde{x})\rangle\\ &\quad+\varrho_n\langle \mu g(\tilde{x})-\psi(\tilde{x}),v_{\tau(n)}-\psi(\tilde{x})\rangle,\end{aligned} \tag{35}$$

where $\varrho_n = \frac{2(1-\lambda_{\tau(n)})}{1-\sigma_{\tau(n)})(1-\mu L_2/\delta+(1-\lambda_{\tau(n)})(1-L_1\nu/\delta)}$. In terms of (33) and (35), we derive $\limsup_{n\to\infty}\omega_{\tau(n)} \le 0$, and so

$$\lim_{n\to\infty}\omega_{\tau(n)} = 0. \tag{36}$$

From (33) and (34), we also obtain $\limsup_{n\to\infty}\omega_{\tau(n)+1} \le \limsup_{n\to\infty}\omega_{\tau(n)}$. This together with (36) implies that $\lim_{n\to\infty}\omega_{\tau(n)+1} = 0$. According to Lemma 2.2 to get $0 \le \omega_n \le \max\{\omega_{\tau(n)}, \omega_{\tau(n)+1}\}$. Therefore, $\omega_n \to 0$. That is, $x_n \to \tilde{x}$. This completes the proof. □

Algorithm 3.2. *For given initial guess $x_0 \in \mathcal{C}$ arbitrarily, let the sequence $\{x_n\}$ be generated iteratively by*

$$\begin{cases} u_n = \mathrm{proj}_{\mathcal{C}}[\lambda_n\nu f(x_n) + (1-\lambda_n)(x_n - \varsigma_n\mathcal{A}x_n)], n \ge 0, \\ x_{n+1} = \sigma_n x_n + (1-\sigma_n)\mathrm{proj}_{\mathcal{C}}[\lambda_n\mu g(x_n) + (1-\lambda_n)(u_n - \gamma_n\mathcal{B}u_n)], \end{cases} \tag{37}$$

where $\mathcal{A}, \mathcal{B} : \mathcal{C} \to \mathcal{H}$ are λ-inverse strongly monotone and β-inverse strongly monotone, respectively, $\{\lambda_n\}$ and $\{\sigma_n\}$ are two real number sequences in $[0,1]$ and $\{\varsigma_n\}$ and $\{\gamma_n\}$ are two real number sequences in $(0,\infty)$.

Corollary 3.1. *If the following assumptions are satisfied:*

(i) $\lim_{n\to\infty}\lambda_n = 0$ *and* $\sum_n \lambda_n = \infty$*;*
(ii) $0 < \liminf_{n\to\infty}\sigma_n \le \limsup_{n\to\infty}\sigma_n < 1$*;*
(iii) $0 < \liminf_{n\to\infty}\varsigma_n \le \limsup_{n\to\infty}\varsigma_n < 2\lambda$*;*
(iv) $0 < \liminf_{n\to\infty}\gamma_n \le \limsup_{n\to\infty}\gamma_n < 2\beta$*;*

then the sequence $\{x_n\}$ generated by (37) converges strongly to $\tilde{x} \in VI(f,\mathcal{C}) \cap VI(g,\mathcal{C})$.

4. Conclusions

In this paper, we investigated a generalized variational inequality problem. We suggest a projected type algorithm for finding the common solutions of two variational inequalities. We prove the strong convergence of the algorithm under the mild conditions. Noting that in our suggested iterative sequence, the involved operators A and B require some form of strong monotonicity. A natural question arises, i.e., how to weaken these assumptions?

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