A NOTE ON PLANE PARTITION DIAMONDS

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We prove new formulas for $\mathcal{D}_k(n)$, the number of plane partition diamonds of length k of n, and, also, for its polynomial part.

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1. Introduction

In his famous book "Combinatory Analysis" [10, Vol.II, Sect. VIII, pp. 91-170] MacMahon introduced Partition Analysis as a computational method for solving combinatorial problems in connection with systems of linear Diophantine inequalities and equations. He considered partitions of the form (a_1, a_2, a_3, a_4) with

$$a_1 \geq a_2,\ a_1 \geq a_3,\ a_2 \geq a_4 \text{ and } a_3 \geq a_4. \tag{1.1}$$

By using Partition Analysis he derived that

$$\sum x_1^{a_1} x_2^{a_2} x_3^{a_3} x_4^{a_4} = \frac{1 - x_1^2 x_2 x_3}{(1-x_1)(1-x_1x_2)(1-x_1x_2x_3)(1-x_1x_2x_3x_4)}, \tag{1.2}$$

where the sum is taken over all non-negative integers a_i satisfying (1.1). Let

$$\mathcal{D}_1(n) := \#\{(a_1, a_2, a_3, a_4) \ :\ n = a_1 + a_2 + a_3 + a_4 \text{ where } a_i \text{ satisfy (1.1)}\}.$$

Taking $x_1 = x_2 = x_3 = x_4 = q$ in (1.2), MacMahon observed that

$$\sum_{n=0}^{\infty} \mathcal{D}_1(n) q^n = \frac{1}{(1-q)(1-q^2)^2(1-q^3)}. \tag{1.3}$$

In [1], Andrews, Paule, and Riese introduce the family of plane partition diamonds, as a generalization of the above example. A *plane partition diamond* of length k is a sequence of length $3k + 1$ of nonnegative integers

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$\mathbf{a} = (a_1, a_2, \ldots, a_{3k+1})$ satisfying, for $0 \leq i \leq k-1$,

$$a_{3i+1} \geq a_{3i+2},\ a_{3i+1} \geq a_{3i+3},\ a_{3i+2} \geq a_{3i+4},\ a_{3i+3} \geq a_{3i+4}. \tag{1.4}$$

Let $\mathcal{D}_k(n)$ be the number of plane partitions diamonds of length k of n. We mention that several generalizations of plane partition diamonds were studied in [9] and [2] but are beyond the scope of this note.

The paper is organized as follows. In Section 2, we recall the definition and some basic properties of the restricted partition function $p_{\mathbf{a}}(n)$, where $\mathbf{a} = (a_1, \ldots, a_r)$ is a sequence of positive integers. Also, we recall several results which would be used later on.

In Section 3, we study basic properties of the function $\mathcal{D}_k(n)$. For $k \leq 1$ we consider the sequence $\mathbf{a}[k] = (a[k]_1, a[k]_2, \ldots, a[k]_{3k+1})$, where

$$a[k]_j = \begin{cases} j, & j \not\equiv 4 (\text{mod } 6) \\ \frac{j}{2}, & j \equiv 4 (\text{mod } 6) \end{cases}.$$

In Proposition 3.**1** we show that $\mathcal{D}_k(n)$ can be written as

$$\mathcal{D}_k(n) = \sum_{J \subset \{\alpha_k+1, \alpha_k+2, \cdots, k\}} p_{\mathbf{a}[k]}(n - m_J),$$

where $m_J = \sum_{i \in J}(3i-1)$ and $\alpha_k = \lfloor \frac{k+1}{2} \rfloor$. In Proposition 3.**2** we show that

$$\mathcal{D}_k(n) = f_{k,3k}(n)n^{3k} + \cdots + f_{k,1}(n)n + f_{k,0}(n) \text{ for } n \geq n_0(k),$$

where $n_0(k)$ is a constant which depends on k, is a quasi-polynomial of degree $3k$. In Corollary 3.**3** we obtain new formulas for the periodic functions $f_{k,j}$'s and, consequently, for $\mathcal{D}_k(n)$.

In Theorem 4.**2** we prove a concise formula of $\mathcal{D}_k(n)$. In Theorem 5.**1** we prove formulas for the 'Sylvester waves' associated to $\mathcal{D}_k(n)$. Also, in Theorem 5.**2** we prove a concise formula of $\mathcal{P}_k(n)$, the polynomial part of $\mathcal{D}_k(n)$.

2. Restricted partition function

Let $\mathbf{a} := (a_1, a_2, \ldots, a_r)$ be a sequence of positive integers, $r \geq 1$. The *restricted partition function* associated to $\mathbf{a}$ is $p_{\mathbf{a}} : \mathbb{N} \to \mathbb{N}$, $p_{\mathbf{a}}(n) :=$ the number of integer solutions $(x_1, \ldots, x_r)$ of $\sum_{i=1}^r a_i x_i = n$ with $x_i \geq 0$. Note that the generating function of $p_{\mathbf{a}}(n)$ is

$$\sum_{n=0}^{\infty} p_{\mathbf{a}}(n) q^n = \frac{1}{(1-q^{a_1}) \cdots (1-q^{a_r})}. \tag{2.1}$$

Let D be a common multiple of $a_1,\ a_2, \ldots, a_r$. We recall the following well known result:

Proposition 2.1. *(Bell* [4]*)*

$p_{\mathbf{a}}(n)$ is a quasi-polynomial of degree $r-1$, with the period D, i.e.

$$p_{\mathbf{a}}(n) = d_{\mathbf{a},k-1}(n)n^{k-1} + \cdots + d_{\mathbf{a},1}(n)n + d_{\mathbf{a},0}(n),$$

where $d_{\mathbf{a},m}(n+D) = d_{\mathbf{a},m}(n)$ *for* $0 \leq m \leq k-1$ *and* $n \geq 0$, *and* $d_{\mathbf{a},k-1}(n)$ *is not identically zero.*

Sylvester [12, 13, 14] decomposed the restricted partition in a sum of "waves":

$$p_{\mathbf{a}}(n) = \sum_{j\geq 1} W_j(n,\mathbf{a}), \tag{2.2}$$

where the sum is taken over all distinct divisors j of the components of $\mathbf{a}$ and showed that for each such j, $W_j(n,\mathbf{a})$ is the coefficient of t^{-1} in

$$\sum_{0\leq \nu < j,\ \gcd(\nu,j)=1} \frac{\rho_j^{-\nu n} e^{nt}}{(1-\rho_j^{\nu a_1} e^{-a_1 t})\cdots(1-\rho_j^{\nu a_k} e^{-a_k t})},$$

where $\rho_j = e^{\frac{2\pi i}{j}}$ and $\gcd(0,0)=1$ by convention. Note that $W_j(n,\mathbf{a})$'s are quasi-polynomials of period j. Also, $W_1(n,\mathbf{a})$ is called the *polynomial part* of $p_{\mathbf{a}}(n)$ and it is denoted by $P_{\mathbf{a}}(n)$; see also [11, Section 4.4].

The *unsigned Stirling numbers* are defined by

$$\binom{n+r-1}{r-1} = \frac{1}{n(r-1)!} n^{(r)} = \frac{1}{(r-1)!}\left(\begin{bmatrix} r \\ r \end{bmatrix} n^{r-1} + \cdots \begin{bmatrix} r \\ 2 \end{bmatrix} n + \begin{bmatrix} r \\ 1 \end{bmatrix} \right). \tag{2.3}$$

We recall several results which would be used later on:

Theorem 2.2. *(*[6, Theorem 2.8] *and* [7]*)*

(1) *For* $0 \leq m \leq r-1$ *and* $n \geq 0$ *we have*

$$d_{\mathbf{a},m}(n) = \frac{1}{(r-1)!} \sum_{\substack{0\leq j_1 \leq \frac{D}{a_1}-1,\ldots,0\leq j_r\leq \frac{D}{a_r}-1 \\ a_1 j_1+\cdots+a_r j_r \equiv n (\bmod D)}} \sum_{k=m}^{r-1} \begin{bmatrix} r \\ k+1 \end{bmatrix} (-1)^{k-m} \binom{k}{m} \times$$
$$\times D^{-k}(a_1 j_1 + \cdots + a_r j_r)^{k-m}.$$

(2) *We have*

$$p_{\mathbf{a}}(n) = \frac{1}{(r-1)!} \sum_{m=0}^{r-1} \sum_{\substack{0\leq j_1 \leq \frac{D}{a_1}-1,\ldots,0\leq j_r\leq \frac{D}{a_r}-1 \\ a_1 j_1+\cdots+a_r j_r \equiv n (\bmod D)}} \sum_{k=m}^{r-1} \begin{bmatrix} r \\ k+1 \end{bmatrix} (-1)^{k-m} \binom{k}{m} \times$$
$$\times D^{-k}(a_1 j_1 + \cdots + a_r j_r)^{k-m} n^m.$$

Theorem 2.3. *(*[6, Corollary 2.10]*) We have*

$$p_{\mathbf{a}}(n) = \frac{1}{(r-1)!} \sum_{\substack{0\leq j_1 \leq \frac{D}{a_1}-1,\ldots,0\leq j_r\leq \frac{D}{a_r}-1 \\ a_1 j_1+\cdots+a_r j_r \equiv n (\bmod D)}} \prod_{\ell=1}^{r-1} \left(\frac{n - a_1 j_1 - \cdots - a_r j_r}{D} + \ell \right).$$

Proposition 2.4. *(*[8, Proposition 4.2]*) For any positive integer j with $j|a_i$ for some $1 \leq i \leq r$, we have that*

$$W_j(n,\mathbf{a}) = \frac{1}{D(r-1)!}\sum_{m=1}^{r}\sum_{\ell=1}^{j}\rho_j^{\ell}\sum_{k=m-1}^{r-1}\begin{bmatrix} r \\ k+1 \end{bmatrix}(-1)^{k-m+1}\binom{k}{m-1}\cdot$$

$$\cdot \sum_{\substack{0\leq j_1\leq \frac{D}{a_1}-1,\ldots,0\leq j_r\leq \frac{D}{a_r}-1 \\ a_1j_1+\cdots+a_rj_r\equiv \ell (\bmod j)}} D^{-k}(a_1j_1+\cdots+a_rj_r)^{k-m+1}n^{m-1}.$$

Theorem 2.5. *(*[6, Corollary 3.6]*)*
For the polynomial part $P_{\mathbf{a}}(n)$ of the quasi-polynomial $p_{\mathbf{a}}(n)$ we have

$$P_{\mathbf{a}}(n) = \frac{1}{D(r-1)!}\sum_{0\leq j_1\leq \frac{D}{a_1}-1,\ldots,0\leq j_r\leq \frac{D}{a_r}-1}\prod_{\ell=1}^{r-1}\left(\frac{n-a_1j_1-\cdots-a_rj_r}{D}+\ell\right).$$

The *Bernoulli numbers B_ℓ*'s are defined by the identity

$$\frac{t}{e^t-1} = \sum_{\ell=0}^{\infty}\frac{t^\ell}{\ell!}B_\ell.$$

$B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$ and $B_n = 0$ is n is odd and $n \geq 1$.

Theorem 2.6. *(*[6, Corollary 3.11] *or* [3, page 2]*)*
The polynomial part of $p_{\mathbf{a}}(n)$ is

$$P_{\mathbf{a}}(n) := \frac{1}{a_1\cdots a_r}\sum_{u=0}^{r-1}\frac{(-1)^u}{(r-1-u)!}\sum_{i_1+\cdots+i_r=u}\frac{B_{i_1}\cdots B_{i_r}}{i_1!\cdots i_r!}a_1^{i_1}\cdots a_r^{i_r}n^{r-1-u}.$$

3. **Preliminaries**

The number of plane partitions diamonds of length k of n is

$$\mathcal{D}_k(n) := \#\{(a_1,a_2,\ldots,a_{3k+1}) \ : \ n = a_1+a_2+\cdots+a_{3k+1} \text{ where } a_i \text{ satisfy (1.4)}\}.$$

Using partition analysis, the authors in [1] find the generalization of (1.3), namely

$$\sum_{n=0}^{\infty}\mathcal{D}_k(n)q^n = \frac{\prod_{i=1}^{k}(1+q^{3i-1})}{\prod_{i=1}^{3k+1}(1-q^i)}. \tag{3.1}$$

Note that, if $i \leq \frac{k+1}{2}$ then $6i-2 \leq 3k+1$. Since $(1+q^{3i-1})(1-q^{3i-1}) = 1-q^{6i-2}$, from (3.1) it follows that

$$\sum_{n=0}^{\infty} \mathcal{D}_k(n)q^n = \frac{\prod\limits_{i=\alpha_k+1}^{k}(1+q^{3i-1})}{\prod\limits_{i=1}^{\lfloor\frac{k+1}{2}\rfloor}(1-q^{3i-1})\prod\limits_{\substack{1\leq i\leq 3k+1 \text{ and} \\ i\not\equiv 4(\bmod\ 6)}}(1-q^i)}, \tag{3.2}$$

where $\alpha_k := \lfloor\frac{k+1}{2}\rfloor$. Note that, in the case $k=1$, $\alpha_1 = 1$ and (3.2) reduces to (1.3).

For $k \leq 1$ we consider the sequence $\mathbf{a}[k] = (a[k]_1, a[k]_2, \ldots, a[k]_{3k+1})$, where

$$a[k]_j = \begin{cases} j, & j \not\equiv 4(\text{mod } 6) \\ \frac{j}{2}, & j \equiv 4(\text{mod } 6) \end{cases}. \tag{3.3}$$

Proposition 3.1. *Using the notations above, we have that:*

$$\mathcal{D}_k(n) = \sum_{J\subset\{\alpha_k+1,\alpha_k+2,\cdots,k\}} p_{\mathbf{a}[k]}(n-m_J),$$

where $m_J = \sum_{i\in J}(3i-1)$ *and* $\alpha_k = \lfloor\frac{k+1}{2}\rfloor$.

Proof. From (3.2) it follows that

$$\sum_{n=0}^{\infty} \mathcal{D}_k(n)q^n = \sum_{J\subset\{\alpha_k,\alpha_{k+1},\cdots,k\}} \frac{q^{m_J}}{\prod\limits_{i=1}^{\lfloor\frac{k+1}{2}\rfloor}(1-q^{3i-1})\prod\limits_{\substack{1\leq i\leq 3k+1 \text{ and} \\ i\not\equiv 4(\bmod\ 6)}}(1-q^i)}. \tag{3.4}$$

On the other hand, from (2.1) we deduce that

$$\sum_{n=0}^{\infty} p_{\mathbf{a}[k]}(n-m_J)q^n = \frac{q^{m_J}}{\prod\limits_{i=1}^{\lfloor\frac{k+1}{2}\rfloor}(1-q^{3i-1})\prod\limits_{\substack{1\leq i\leq 3k+1 \text{ and} \\ i\not\equiv 4(\bmod\ 6)}}(1-q^i)}. \tag{3.5}$$

The conclusion follows from (3.4) and (3.5). $\square$

Let $D[k] = \operatorname{lcm}(\mathcal{A}_k)$, where $\mathcal{A}_k = \{1 \leq j \leq 3k+1 \ :\ j \not\equiv 4(\text{mod } 6)\}$. For instance, $D[1] = \operatorname{lcm}\{1,2,3\} = 6$, $D[2] = \operatorname{lcm}\{1,2,3,5,7\} = 210$ etc. Note that

$$\mathcal{A}_k = \{a[k]_1, a[k]_2, \ldots, a[k]_{3k+1}\} \tag{3.6}$$

and thus $D[k] = \operatorname{lcm}\{a[k]_1, a[k]_2, \ldots, a[k]_{3k+1}\}$.

Proposition 3.2. $\mathcal{D}_k(n)$ *is a quasi-polynomial of degree* $3k$*, with the period* $D[k]$*, i.e.*

$$\mathcal{D}_k(n) = f_{k,3k}(n)n^{3k} + \cdots + f_{k,1}(n)n + f_{k,0}(n), \text{ for all } n \geq n_0(k),$$

where $n_0(k)=\frac{(k-\alpha_k)(3k+3\alpha_k+1)}{2}$ *and* $f_{k,j}(n+D[k])=f_{k,j}(n)$ *for all* $n\geq n_0(k)$.

Moreover, we have that:

$$f_{k,j}(n)=\sum_{J\subset\{\alpha_k+1,\alpha_k+2,\cdots,k\}} d_{\mathbf{a}[k],j}(n-m_J).$$

Proof. Note that

$$n_0(k)=m_{\{\alpha_k+1,\alpha_k+2,\ldots,k\}}=\sum_{i=\alpha_k+1}^{k}(3i-1)=3\left(\sum_{i=\alpha_k+1}^{k} i\right)-(k-\alpha_k)=$$

$$=\frac{(k-\alpha_k)(3k+3\alpha_k+1)}{2}. \tag{3.7}$$

The expression of $f_{k,j}(n)$ follows from Proposition 3.**1** and (2.**1**).

Now, the conclusion follows from (3.7) and the fact that $d_{\mathbf{a}[k],j}(n+D[k])=d_{\mathbf{a}[k],m}(n)$ for all $0\leq j\leq 3k$ and $n\geq 0$. □

Corollary 3.3. *With the above notations, for* $n\geq n_0(k)$ *we have that*

$$f_{k,j}(n)=\frac{1}{(3k)!}\sum_{J\subset\{\alpha_k+1,\alpha_k+2,\cdots,k\}}\ \sum_{\substack{0\leq j_i\leq \frac{D[k]_i}{a[k]_i}-1,\ 1\leq i\leq 3k+1\\ a[k]_1j_1+\cdots+a[k]_{3k+1}j_{3k+1}\equiv n-m_J(\bmod D[k])}}\ \sum_{\ell=j}^{3k}\begin{bmatrix}3k+1\\ \ell+1\end{bmatrix}\times$$

$$\times(-1)^{\ell-j}\binom{\ell}{j}D[k]^{-\ell}(a[k]_1j_1+\cdots+a[k]_{3k+1}j_{3k+1})^{\ell-j}.$$

In particular, it follows that

$$\mathcal{D}_k(n)=\frac{1}{(3k)!}\sum_{\ell=0}^{3k}\ \sum_{J\subset\{\alpha_k+1,\alpha_k+2,\cdots,k\}}\ \sum_{\substack{0\leq j_i\leq \frac{D[k]_i}{a[k]_i}-1,\ 1\leq i\leq 3k+1\\ a[k]_1j_1+\cdots+a[k]_{3k+1}j_{3k+1}\equiv n-m_J(\bmod D[k])}}\ \sum_{\ell=j}^{3k}\begin{bmatrix}3k+1\\ \ell+1\end{bmatrix}\times$$

$$\times(-1)^{\ell-j}\binom{\ell}{j}D[k]^{-\ell}(a[k]_1j_1+\cdots+a[k]_{3k+1}j_{3k+1})^{\ell-j}(n-m_J)^j.$$

Proof. The conclusion follows from Proposition 3.**2** and Theorem 2.**2**. □

4. **New formulas for the number of plane partition diamonds of length k of n.**

We recall that $\alpha_k=\lfloor\frac{k+1}{2}\rfloor$. We define

$$\beta_k:=\begin{cases}5\alpha_k-2, & k \text{ is odd}\\ 5\alpha_k+1, & k \text{ is even}\end{cases}\ \text{and}\ \mathcal{B}_k=\{1,2,\ldots,\beta_k\}.$$

For instance, $\mathcal{B}_1=\{1,2,3\}$, $\mathcal{B}_2=\{1,2,3,4,5,6\}$ etc.

Lemma 4.1. *With the above notations, the map*

$$\varphi_k : \mathcal{B}_k \to \mathcal{A}_k,\ \varphi_k(j) = j + \left\lceil \frac{j-3}{5} \right\rceil,$$

is bijective. Moreover, the inverse of φ_k *is the map*

$$\varphi_k^{-1} : \mathcal{A}_k \to \mathcal{B}_k,\ \varphi_k^{-1}(j) = j - \left\lceil \frac{j-3}{6} \right\rceil.$$

Proof. Let $j \in \mathcal{B}_k$ and write $j = 5i + r$ for $1 \leq r \leq 5$ and $i \geq 0$. We have that:

$$\varphi_k(5i+1) = 6i+1,\ \varphi_k(5i+2) = 6i+2,\ \varphi_k(5i+3) = 6i+3,$$
$$\varphi_k(5i+4) = 6i+5,\ \varphi_k(5i+5) = 6i+6.$$

If $k = 2p$, then $\alpha_k = p$ and

$$\varphi_k(5\alpha_k + 1) = \varphi_k(5p+1) = 6p+1 = 3k+1 = \max \mathcal{A}_k.$$

If $k = 2p+1$, then $\alpha_k = p+1$ and

$$\varphi_k(5\alpha_k - 2) = \varphi_k(5p+3) = 6p+3 = 3k = \max \mathcal{A}_k.$$

Also $\varphi_k(1) = 1$ and φ_k is increasing, hence injective.

From the above considerations, it follows that φ_k is bijective.

Let $\psi_k : \mathcal{B}_k \to \mathcal{A}_k$, $\psi_k(j) = j - \lceil \frac{j-3}{6} \rceil$. Let $j \in \mathcal{A}_k$. Then we can write $j = 6i + r$, where $1 \leq r \leq 6$ and $r \neq 4$. We have that

$$\psi_k(6i+1) = 5i+1,\ \psi_k(6i+2) = 5i+2,\ \psi_k(6i+3) = 5i+3,$$
$$\psi_k(6i+5) = 5i+4,\ \psi_k(6i+6) = 5i+5.$$

Since $\psi_k(1) = 1$ and $\psi_k(\max \mathcal{A}_k) = \max \mathcal{B}_k$, from the above identities, it follows that ψ_k is surjective and increasing. Hence, ψ_k is bijective.

The function $\psi_k \circ \varphi_k : \mathcal{A}_k \to \mathcal{A}_k$ is bijective and increasing, hence $\psi_k \circ \varphi_k$ is the identity function of $\mathcal{A}_k$. Similarly, $\varphi_k \circ \psi_k$ is the identity function of $\mathcal{B}_k$. Thus, $\psi_k = \varphi_k^{-1}$, as required. □

We consider the subset

$$\mathcal{B}'_k = \{j \in \mathcal{B}_k\ :\ j \equiv 2, 5 (\text{mod } 5) \text{ and } j \leq \varphi_k^{-1}(3\alpha_k - 1)\}.$$

We also let

$$\varepsilon_k : \mathcal{B}_k \to \{1, 2\},\ \varepsilon_k(j) = \chi_{\mathcal{B}'_k}(j) + 1,$$

where $\chi_{\mathcal{B}'_k}$ is the characteristic function of the subset $\mathcal{B}'_k$ of $\mathcal{B}_k$. We also let

$$s_k(t_1, t_2, \ldots, t_{\beta_k}) := \prod_{j \in \mathcal{B}'_k} \left(1 + \max\left\{t_j, 2\left(\frac{D[k]}{\varphi_k(j)} - 1\right) - t_j\right\}\right).$$

Theorem 4.2. *We have that:*

$$\mathcal{D}_k(n) = \frac{1}{(3k)!} \sum_{\substack{J\subset\{\alpha_k,\alpha_{k+1},\cdots,k\} \text{ and} \\ (t_1,t_2,\ldots,t_{\beta_k})\in \mathbf{A}_k(m_J)}} s_k(t_1,t_2,\ldots,t_{\beta_k}) \prod_{\ell=1}^{3k} \left(\frac{n - \sum\limits_{j=1}^{\beta_k} t_j\varphi_k(j) - m_J}{D[k]} + \ell \right),$$

where $\mathbf{A}_k(m) = \{(t_1,t_2,\ldots,t_{\beta_k}) \ : \ 0 \leq t_j \leq \varepsilon_k(j) \left(\frac{D[k]}{\varphi_k(j)} - 1\right)$ *for all* $1 \leq j \leq \beta_k$ *and* $\sum_{j=1}^{\beta_k} t_j\varphi_k(j) \equiv n - m (\bmod D[k])\}$.

Proof. First, note that for $n < m_J$ we have that:

$$\prod_{\ell=1}^{3k} \left(\frac{n - a[k]_1 j_1 - \cdots - a[k]_{3k+1} j_{3k+1} - m_J}{D[k]} + \ell \right) = 0.$$

Therefore, from Proposition 3.**1** and Theorem 2.**3** it follows that:

$$\mathcal{D}_k(n) = \frac{1}{(3k)!} \sum_{J\subset\{\alpha_k,\alpha_{k+1},\cdots,k\}} \sum_{\substack{0\leq j_1\leq \frac{D[k]}{a[k]_1}-1,\ldots,0\leq j_{3k+1}\leq \frac{D[k]}{a[k]_{3k+1}}-1 \\ a[k]_1 j_1+\cdots+a[k]_{3k+1} j_{3k+1}\equiv n-m_J (\bmod D[k])}}$$
$$\prod_{\ell=1}^{3k} \left(\frac{n - a[k]_1 j_1 - \cdots - a[k]_{3k+1} j_{3k+1} - m_J}{D[k]} + \ell \right).$$

The conclusion follows from Lemma 4.**1**. □

Example 4.3. *(MacMahon's example)* We consider

$\mathcal{D}_1(n) = \#\{(a_1,a_2,a_3,a_4) \ : \ a_1+a_2+a_3+a_4 = n, \ a_1 \geq a_2, \ a_1 \geq a_3, \ a_2 \geq a_4$ and $a_3 \geq a_4\}$.

Comparing (1.3) with (2.1), it follows that $\mathcal{D}_1(n) = p_{(1,2,2,3)}(n)$ for all $n \geq 0$. Since $\alpha_1 = 1$ and $D[1] = 6$, from Theorem 4.**2**, it follows that:

$$\mathcal{D}_1(n) = \frac{1}{6} \sum_{\substack{0\leq t_1\leq 5,\ 0\leq t_2\leq 4,\ 0\leq t_3\leq 1 \\ t_1+2t_2+3t_3\equiv n (\bmod 6)}} (\min\{t_2, 4-t_2\}+1) \prod_{\ell=1}^{6} \left(\frac{n - t_1 - 2t_2 - 3t_3}{6} + \ell \right).$$

5. **The polynomial part and Sylvester waves of $\mathcal{D}_k(n)$**

From (2.2) and Proposition 3.**1**, we can write

$$\mathcal{D}_k(n) = \sum_{j=1}^{\infty} W_j(k,n), \text{ where } W_j(k,n) = \sum_{J\subset\{\alpha_k+1,\alpha_k+2,\cdots,k\}} W_j(n-m_J, \mathbf{a}[k]), \tag{5.1}$$

$m_J = \sum_{i\in J}(3i-1)$ and $\alpha_k = \lfloor \frac{k+1}{2} \rfloor$. In particular, the *polynomial part* of $\mathcal{D}_k(n)$ is the function

$$\mathcal{P}_r(n) := \sum_{J\subset\{\alpha_k+1,\alpha_k+2,\cdots,k\}} P_{\mathbf{a}[k]}(n-m_J), \tag{5.2}$$

where $P_{\mathbf{a}[k]}(n-m_J) = W_1(n-m_J, \mathbf{a}[k])$.

Theorem 5.1. *With the above notations we have that*

$$W_j(k,n) = \frac{1}{D[k](3k)!} \sum_{J\subset\{\alpha_k+1,\alpha_k+2,\cdots,k\}} \sum_{m=1}^{3k+1}\sum_{\ell=1}^{j}\rho_j^{\ell}\sum_{t=m-1}^{3k}\begin{bmatrix}3k+1\\t+1\end{bmatrix}(-1)^{t-m+1}\binom{t}{m-1}\times$$
$$\times \sum_{\substack{0\le j_1\le \frac{D[k]_1}{a[k]_1}-1,\ldots,0\le j_{3k+1}\le \frac{D[k]_{3k+1}}{a[k]_{3k+1}}-1\\ a[k]_1 j_1+\cdots+a[k]_{3k+1}j_{3k+1}\equiv \ell (\bmod j)}} D^{-t}(a[k]_1 j_1+\cdots+a[k]_{3k+1}j_{3k+1})^{t-m+1}(n-m_J)^{m-1}.$$

Proof. The conclusion follows from Proposition 3.**1**, Proposition 2.**4** and (5.1). □

Theorem 5.2. *With the above notations, we have that $\mathcal{P}_k(n)$ equals to*

$$\frac{1}{D[k](3k)!}\sum_{\substack{J\subset\{\alpha_k,\alpha_{k+1},\cdots,k\} \text{ and}\\ (t_1,t_2,\ldots,t_{\beta_k})\in\mathbf{B}_k(m_J)}} s_k(t_1,t_2,\ldots,t_{\beta_k})\prod_{\ell=1}^{3k}\left(\frac{n-\sum\limits_{j=1}^{\beta_k}t_j\varphi_k(j)-m_J}{D[k]}+\ell\right),$$

where

$$\mathbf{B}_k(m) = \{(t_1,t_2,\ldots,t_{\beta_k}) \ : \ 0\le t_j\le \varepsilon_k(j)\left(\frac{D[k]}{\varphi_k(j)}-1\right) \text{ for all } 1\le j\le\beta_k\}.$$

Proof. The proof is similar to the proof of Theorem 4.**2**, using Proposition 3.**1**, Theorem 2.**5** and (5.2). □

Theorem 5.3. *With the above notations, we have that:*

$$\mathcal{P}_k(n) := \frac{1}{a[k]_1\cdots a[k]_{3k+1}}\sum_{J\subset\{\alpha_k+1,\alpha_k+2,\cdots,k\}}\sum_{u=0}^{3k}\frac{(-1)^u}{(3k-u)!}\times$$
$$\times\sum_{i_1+\cdots+i_{3k+1}=u}\frac{B_{i_1}\cdots B_{i_{3k+1}}}{i_1!\cdots i_{3k+1}!}a[k]_1^{i_1}\cdots a[k]_{3k+1}^{i_{3k+1}}(n-m_J)^{3k-u}.$$

Proof. The conclusion follows from Proposition 3.**1**, Theorem 2.**6** and (5.2). □

Example 5.4. *(MacMahon's example revised)*

We consider $\mathcal{D}_1(n)$; see Example 4.**3**. From Theorem 5.**2**, the polynomial part of $\mathcal{D}_1(n)$ is

$$\mathcal{P}_1(n) = \frac{1}{36}\sum_{0\le t_1\le 5,\ 0\le t_2\le 4,\ 0\le t_3\le 1}(\min\{t_2,4-t_2\}+1)\prod_{\ell=1}^{6}\left(\frac{n-t_1-2t_2-3t_3}{6}+\ell\right).$$

6. Conclusions

Let $n, k \geq 1$ be two integers. We proved new formulas for $\mathcal{D}_k(n)$, the number of plane partition diamonds of length k of n, and, also, for $\mathcal{P}_k(n)$, its polynomial part.

Our methods are can be used to study several generalizations of plane partition diamonds, like the plane broken diamonds; see [2], plane partition polygons and plane partition trees; see [9].

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